PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 74(88) (2003), 111–114

MONOTONE IMAGES OF W-SETS AND HEREDITARILY WEAKLY CONFLUENT IMAGES OF CONTINUA

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Communicated by Rade Živaljević

Abstract. A proper subcontinuum H of a continuum X is said to be a W-set provided for each continuous surjective function f from a continuum Y onto X, there exists a subcontinuum C of Y that maps entirely onto H. Hereditarily weakly confluent (HWC) mappings are those with the property that each restriction to a subcontinuum of the domain is weakly confluent. In this paper, we show that the monotone image of a W-set is a W-set and that there exists a continuum which is not in class W but which is the HWC image of a class W continuum.

1. Introduction

In what follows, a continuum is a compact, connected metric space, and the term map is used to denote a continuous function. It is known that monotone images of class W continua are in class W, as shown in [**1**]. In the summer of 2000, two questions arose related to this result. First, in personal communication, W. J. Charatonik asked whether HWC maps preserve membership in class W. We answer his question in the negative in Section 3. Second, while discussing approaching continuum theory from an analytical viewpoint and attempting to characterize continua which are not intrinsic W-sets, the idea of examining the preimages of such continua under certain types of maps arose. We give a related theorem in Section 4.

2. Definitions

A proper subcontinuum H of a continuum X is said to be a W-set provided for each continuous surjective function f from a continuum Y onto X, there exists

2000 *Mathematics Subject Classification*: Primary 54F15, 54C50.
Key words and phrases: W-sets, Class W, Monotone maps, HWC maps.
*The author would like to thank Dr Dragan Blagojević for subtile editorial touchings that improve appearance of the paper.

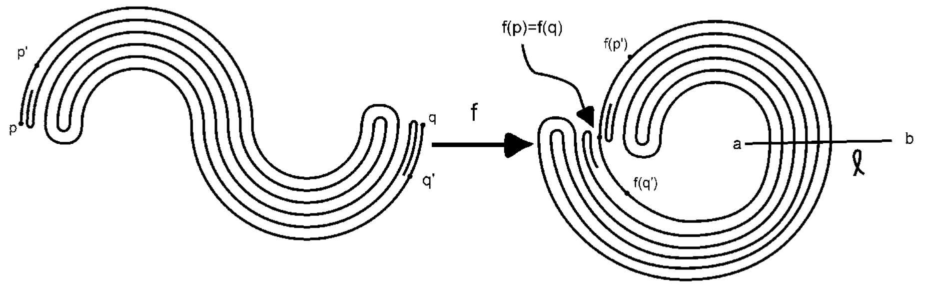

Figure 1. A two endpoint Knaster-type continuum and an appropriate quotient space. This provides the basis of Theorem 1.

a subcontinuum C of Y that maps entirely onto H. A continuum each proper subcontinuum of which is a W-set is said to be in class W. A weakly confluent map $f : X \to Y$ is one so that, for each subcontinuum K of the range, there is at least one component C of $f^{-1}(K)$ so that $f(C) = K$. A hereditarily weakly confluent (HWC) map $f : X \to Y$ is one so that for each subcontinuum K of X, $f|_K$ is weakly confluent. A monotone map $f : X \to Y$ is one so that $f^{-1}(y)$ is connected for each $y \in Y$. The Hausdorff distance between two compact sets A and B is defined to be

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subseteq \mathcal{B}_\epsilon(B) \text{ and } B \subseteq \mathcal{B}_\epsilon(A)\}.$$

3. HWC maps and class W

W. J. Charatonik asked whether the HWC image of a class W continuum is necessarily in class W. The answer is no.

Theorem 1. *There exist a a continuum M in class W, a continuum X, and a surjective HWC map $f : M \to X$ where X is a continuum not in class W.*

Proof. First, consider the continuum formed when one takes a two endpoint Knaster-type continuum, which can be realized as an inverse limit on arcs with a three-pass bonding map, and joins the two endpoints. The result, which we will denote by X, is an indecomposable continuum homeomorphic to the one illustrated on the right in Figure 1. It is clear that X is not in class W, since for the quotient map itself, there is no continuum in the domain which is mapped onto the arc from $f(p')$ to $f(q')$.

Denote by C the composant of X containing $f(p)$, the joining point. In the strip $\mathbb{R} \times [0, 1]$, consider the collection of straight line segments of the following form: C_0 is the straight line segment from $(0, 1)$ to $(-1, 1/2)$. Then, for each positive integer n, let C_n be the straight line segment from $\big((-1)^n \cdot n, 1/(n+1)\big)$ to $\big((-1)^{n+1} \cdot (n+1), 1/(n+2)\big)$. Let $\hat{Y} = \bigcup_{n \geqslant 0} C_n$, and observe that $\hat{Y}$ is a connected set containing all of $\mathbb{R} \times \{0\}$ in its closure.

To construct M, which will be a subset of $X \times [0,1]$, first consider the straight line ℓ in the plane connecting the points a and b. There is a natural surjective and injective map $\hat{g}$ from $\mathbb{R}$ to C with the following properties: first, that $\hat{g}(0) = f(p)$ and second, that for each integer n in $\mathbb{R} \setminus \{0\}$, $\hat{g}(n) \in \ell$. Extend $\hat{g}$ to $g : \mathbb{R} \times [0,1] \to C \times [0,1]$ by setting $g(x,t) = (\hat{g}(x), t)$. Let $Y = g(Y)$ and define $M = X \cup Y$.

Observe that $\overline{Y}$ contains C, and since C is dense in X, $\overline{Y} = M$. It is easily verified that M is in class W (for example by using Theorem 67.1 of [**3**] and Proposition 4 of [**4**]). Let $\pi : X \times [0,1] \to X$ is be simple projection map, and we will now show that $\pi|_M$ is HWC. Let K be any subcontinuum of M. If $K \subseteq X$ or $X \subseteq K$, then since $\pi|_X$ is essentially the identity, $\pi|_K$ is clearly weakly confluent. If $K \not\subseteq X$ and $X \not\subseteq K$, then $K \subset Y$, since X is a C-set in M. For $K \subset Y$, K is an arc, and since arcs are in class W, $\pi|_K$ is weakly confluent. Hence $\pi|_M$ is HWC. Thus X is the HWC image of a class W continuum, and X is not in class W. □

4. Monotone maps and W-sets

In investigating what properties of a subcontinuum imply that it is not an intrinsic W-set, the idea arose that perhaps considering the preimages of such a subcontinuum under various types of maps might be informative. One of the questions that was generated by that discussion was about the monotone preimage of a continuum which was not an intrinsic W-set. After answering this question, we realized that there was a different statement of the same result which might be more useful.

The proof of our Theorem 2 depends on the following theorem that will appear in [**2**]

THEOREM. [**2**, Theorem 7] *A subcontinuum H of a continuum X is a W-set in X if and only if for each $\epsilon > 0$ there is a pair H_1, H_2 of compact subsets of X so that any continuum C intersecting both H_1 and H_2 which is not separated by $H_1 \cup H_2$ has Hausdorff distance from H less than ϵ.*

THEOREM 2. *Let X be a continuum with W-set H, and let $f : X \to Y$ be a map of X to a continuum Y. If f is monotone and $f(H)$ a proper subcontinuum of $f(X)$, then $f(H)$ is a W-set in $f(X)$.*

PROOF. Let H be a W-set in continuum X, and let $f : X \to Y$ be a monotone map so that $f(H)$ is nondegenerate. Without loss of generality, assume that f is surjective. Given any $\epsilon > 0$ so that $\epsilon < \frac{1}{4}\mathrm{diam}(f(H))$, there is a $\delta > 0$ so that if $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \epsilon$. Since H is a W-set, there exist two compact subsets, H_1 and H_2, so that for any continuum C from H_1 to H_2 which is not separated by $H_1 \cup H_2$, $d_H(c, H) < \delta$.

Assume, for the sake of contradiction, that there is a point $p \in f(H_1) \cap f(H_2)$. Then $f^{-1}(p)$ is a continuum intersecting both H_1 and H_2. $f^{-1}(p)$ must therefore contain a continuum C irreducible between H_1 and H_2, which by thus must have $d_H(C, H) < \delta$. Therefore $d_H(\{p\}, f(H)) < \epsilon$, which implies that $f(H) \subset \mathcal{B}_\epsilon(p)$. This contradicts our choice of p, so $f(H_1)$ and $f(H_2)$ must be disjoint.

If C is a continuum from $f(H_1)$ to $f(H_2)$ not separated by their union, then $f^{-1}(C)$ is a continuum intersecting H_1 and H_2. Define M and N as follows:

$$M = f^{-1}\left(\overline{C \setminus (f(H_1) \cup f(H_2))}\right), \quad N = \overline{f^{-1}\left(C \setminus (f(H_1) \cup f(H_2))\right)}.$$

Observe that N is a subcontinuum of M. Let P and Q be subcontinua of M so that P is irreducible between $H_1 \cap M$ and N and Q is irreducible between $H_2 \cap M$ and N. The continuum $N \cup P \cup Q$ intersects H_1 and H_2 but is not separated by their union, so $d_H(P \cup N \cup Q, H) < \delta$. Hence each point of $f(H)$ is within ϵ of $f(P \cup N \cup Q) \subseteq C$, and each point of C is either in $f(P \cup N \cup Q)$, and hence within ϵ of $f(H)$, or in $f(H_1) \cup f(H_2)$, which must be within ϵ of $F(H)$, since H_1 and H_2 must both be within δ of H. Thus $d_H(C, H) < \epsilon$.

This satisfies the conditions in Theorem [**2**, Theorem 7] for $f(H)$ to be a W-set in $f(X)$. $\square$

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(Received 05 05 2003)

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