

ON SOME AKIVIS–GOLDBERG TYPE METRICS

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ABSTRACT. We investigate Akivis–Goldberg type metrics satisfying some additional assumptions.

1. Introduction

Let M be a manifold of dimension $n = pq$, and let $SC(p, q)$ be a differentiable field of Segre cones $SC_x(p, q) \subset T_xM$, $x \in M$. The pair $(M, SC(p, q))$ is called an *almost Grassmann structure* and is denoted by $AG(p - 1, p + q - 1)$. The manifold M endowed with such structure is said to be an *almost Grassmann manifold* (e.g., see [1, Definition 1.1]). Some additional conditions lead to so-called semiintegrable almost Grassmann structures [1, Definition 1.2]. The latter were studied in [1] and examples of such structures, mainly 4-dimensional, are presented there. Certain semi-Riemannian metrics are related to these structures (see Examples 3.5–3.16 of [1]). These metrics are called *Akivis–Goldberg*, in short *AG-metrics* [20]. Manifolds admitting *AG-metrics* will be called *AG-manifolds*. Curvature properties and, in particular, curvature properties of pseudosymmetry type of *AG-manifolds* were obtained in [20]. For instance, on such manifolds we have [20]

$$(1.1) \quad \text{rank } S \leq 2,$$

$$(1.2) \quad (i) \quad S^2 = 0, \quad (ii) \quad \kappa = 0, \quad (iii) \quad S \cdot C = 0.$$

For precise definitions of the symbols used, we refer to Section 2 of this paper. We note that (1.2)(iii), by making use of (1.2)(i), (1.2)(ii) and the identity

$$(1.3) \quad S \cdot C = S \cdot R + \frac{4}{n-2} \bar{S} + \frac{2}{n-2} g \wedge S^2 - \frac{2\kappa}{(n-2)(n-1)} g \wedge S,$$

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turns into $S \cdot R = -\frac{4}{n-2}\bar{S}$. Moreover, on every AG -manifold (M, g) the following condition of pseudosymmetry type is satisfied [20]

$$(1.4) \quad R \cdot R - Q(S, R) = L_C Q(g, C),$$

where L_C is some function on $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$. With respect to the above presentation of curvature properties of AG -manifolds we can define the following extension of this class of manifolds.

Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold such that $U_C \cap U_S \subset M$ is a nonempty set, where $U_S = \{x \in M \mid S - \frac{\kappa}{n}g \neq 0 \text{ at } x\}$. The metric g will be called an *Akivis–Goldberg type metric*, in short an *AG type metric* if on $U_C \cap U_S$ the following three conditions are fulfilled: (1.4),

$$(1.5) \quad S \cdot R = L_1 \bar{S} + L_2 g \wedge S + L_3 G,$$

$$(1.6) \quad S^2 = L_4 S + L_5 g,$$

where L_1, \dots, L_5 are some functions on $U_C \cap U_S$. A manifold admitting an AG type metric will be called an *Akivis–Goldberg type manifold*, in short an *AG type manifold*. Evidently, every AG manifold is an AG type manifold. The converse statement is not true. In Section 3 we present examples of AG type manifolds. In particular, we state that every semi-Riemannian manifold satisfying the Roter type equation [9] is an AG type manifold. Some AG type manifolds satisfy also (1.1). In Section 2 we prove (see Corollary 2.1) that if an AG type manifold (M, g) satisfies on $U_C \cap U_S \subset M$ the condition

$$(1.7) \quad \text{rank } S = 2$$

then (1.6) reduces on $U_C \cap U_S$ to

$$(1.8) \quad S^2 = \frac{\kappa}{2}S.$$

In Remark 3.1 (v) and (vi) we present examples of AG type manifolds satisfying (1.7). These manifolds can be locally realized as hypersurfaces of semi-Euclidean spaces. In the last section we consider hypersurfaces M in semi-Riemannian spaces of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geq 4$, or in particular, in semi-Euclidean spaces \mathbb{E}_s^{n+1} , with nonempty set $U_C \cap U_S \subset M$, satisfying on this set (1.4), (1.5) and (1.6). It means that the metric g induced on M from the metric of the ambient space is an AG type metric. Hypersurfaces M , with nonempty set $U_C \cap U_S \subset M$, satisfying on this set (1.4), (1.5) and (1.6) will be called *Akivis–Goldberg type hypersurfaces*, in short *AG type hypersurfaces*.

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. We denote by U_H the set of all points of M at which the tensor H^2 is not a linear combination of H and g . Using (2.18) and Theorem 4.1 of [19] we can deduce that $U_H \subset U_C \cap U_S \subset M$. AG type hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, are also investigated in [22] and [23]. Among others things in [22] it was shown that (1.4), (1.5) and (1.6) hold on $U_C \cap U_S - U_H$. Therefore we restrict our considerations on AG type hypersurfaces M in $N_s^{n+1}(c)$ to the set $U_H \subset M$. We mention that an extension of the class of AG type manifolds was introduced in [22] (see also [23]).

Our main result states (see Theorem 4.1) that if M is an AG type hypersurface in \mathbb{E}_s^{n+1} , $n \geq 5$, the set $U_H \subset M$ is nonempty, and (1.7) holds on U_H , then the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent at all points of U_H at which $\kappa \neq 0$. An example of a semisymmetric AG type hypersurface, with $\kappa \neq 0$, is given in Section 3 (see Remark 3.1(v)). That hypersurface satisfies

$$(1.9) \quad R = \frac{2}{\kappa} \bar{S}.$$

2. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞ . Let (M, g) be an n -dimensional, $n \geq 3$, semi-Riemannian manifold. We denote by ∇ , R , C , S and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M, g) , respectively. The Ricci operator \mathcal{S} is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M . We define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)$ of $\Xi(M)$ by

$$\begin{aligned} (X \wedge_A Y)Z &= A(Y, Z)X - A(X, Z)Y \\ \mathcal{R}(X, Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \\ \mathcal{C}(X, Y) &= \mathcal{R}(X, Y) - \frac{1}{n-2} \left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y \right), \end{aligned}$$

respectively, where $X, Y, Z \in \Xi(M)$ and A is a symmetric $(0, 2)$ -tensor. Now the Riemann-Christoffel curvature tensor R , the Weyl conformal curvature tensor C and the $(0, 4)$ -tensor G of (M, g) are defined by

$$\begin{aligned} R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4), \\ C(X_1, X_2, X_3, X_4) &= g(\mathcal{C}(X_1, X_2)X_3, X_4), \\ G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge_g X_2)X_3, X_4), \end{aligned}$$

respectively, where $X, Y, Z, X_1, X_2, \dots \in \Xi(M)$. Let $\mathcal{B}(X, Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be a $(0, 4)$ -tensor associated with $\mathcal{B}(X, Y)$ by

$$(2.1) \quad B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4).$$

The tensor B is said to be a *generalized curvature tensor* if

$$\begin{aligned} B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) &= 0, \\ B(X_1, X_2, X_3, X_4) &= B(X_3, X_4, X_1, X_2). \end{aligned}$$

For a generalized curvature tensor B we denote by $\text{Ric}(B)$, $\text{Weyl}(B)$ and $\kappa(B)$ the Ricci tensor, the Weyl tensor and the scalar curvature of B , respectively. The subsets U_B , $U_{\text{Ric}(B)}$ and $U_{\text{Weyl}(B)}$ are defined in the same way as the subsets U_R , U_S and U_C , respectively. Clearly, the tensors R , C and G are generalized curvature tensors. For symmetric $(0, 2)$ -tensors E and F we denote by $E \wedge F$ their Kulkarni-Nomizu product. The tensor $E \wedge F$ is also a generalized curvature tensor. For a

symmetric $(0, 2)$ -tensor E we define the $(0, 4)$ -tensor \bar{E} by $\bar{E} = \frac{1}{2}E \wedge E$. In particular, we have $\bar{g} = G = \frac{1}{2}g \wedge g$. Let $\mathcal{B}(X, Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the tensor defined by (2.1). We extend the endomorphism $\mathcal{B}(X, Y)$ to derivation $\mathcal{B}(X, Y) \cdot$ of the algebra of tensor fields on M , assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$ for any smooth function on M . Now for a $(0, k)$ -tensor field T , $k \geq 1$, and a symmetric $(0, 2)$ -tensor A we can define the $(0, k+2)$ -tensors $B \cdot T$ and $Q(A, T)$ and the $(0, k)$ -tensor $A \cdot T$. For the definition of these tensors we refer, for instance, to [2] or [13]. Setting $T = R$, $T = C$ or $T = S$ and $A = g$ or $A = S$ we obtain the tensors: $S \cdot R$, $S \cdot C$, $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $C \cdot S$, $Q(g, R)$, $Q(g, C)$, $Q(g, S)$, $Q(S, R)$, and $Q(S, C)$. The tensors $C \cdot R$, $C \cdot C$ and $C \cdot S$ are defined in the same manner as the tensors $R \cdot R$ and $R \cdot S$, respectively.

A semi-Riemannian manifold (M, g) , $n \geq 3$, is called a *quasi-Einstein manifold* if its Ricci tensor S has the form

$$(2.2) \quad S = \alpha g + \epsilon w \otimes w, \quad \epsilon = \pm 1,$$

for some function α and 1-form w on M . We refer to [2] for a review of results on quasi-Einstein manifolds. AG type quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature are investigated in [23].

A semi-Riemannian manifold (M, g) , $n \geq 3$, is said to be *pseudosymmetric* if at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. This is equivalent to

$$(2.3) \quad R \cdot R = L_R Q(g, R)$$

on $U_R = \{x \in M \mid R - \frac{\kappa}{(n-1)n}G \neq 0 \text{ at } x\}$, where L_R is some function on U_R . We note that $U_C \subset U_R$ and $U_S \subset M$. The class of pseudosymmetric manifolds is an extension of the class of *semisymmetric manifolds* ($R \cdot R = 0$). A semi-Riemannian manifold (M, g) , $n \geq 3$, is said to be *Ricci-pseudosymmetric* if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$(2.4) \quad R \cdot S = L_S Q(g, S)$$

on U_S , where L_S is some function on U_S . We say that (2.3) and (2.4) are certain *conditions of pseudosymmetry type* [2], [12]. The class of Ricci-pseudosymmetric manifolds is an extension of the class of *Ricci-semisymmetric* manifolds ($R \cdot S = 0$) as well as of the class of pseudosymmetric manifolds. Some geometrical considerations show that (2.3), resp., (2.4), is a more natural curvature condition than the condition $R \cdot R = 0$, resp. $R \cdot S = 0$. For a presentation of facts related to these statements and, in general, on pseudosymmetry type conditions we refer to [2] and [12].

LEMMA 2.1. *Let (M, g) , $n \geq 3$, be a semi-Riemannian manifold and let A be a nonzero symmetric $(0, 2)$ -tensor at $x \in M$.*

(i) *If*

$$(2.5) \quad \text{rank } A = 2$$

at x , then at x we have

$$(2.6) \quad A^3 = \operatorname{tr}(A)A^2 + \frac{\operatorname{tr}(A^2) - (\operatorname{tr}(A))^2}{2}A.$$

Moreover, if

$$(2.7) \quad A^2 = \alpha A + \beta g, \quad \alpha, \beta \in \mathbb{R},$$

at x , then at x we have

$$(2.8) \quad A^2 = \frac{\operatorname{tr}(A)}{2}A.$$

(ii) If $\operatorname{rank} A \leq 2$ and

$$(2.9) \quad A = \alpha g + \epsilon w \otimes w, \quad \alpha \in \mathbb{R}, \quad \epsilon = \pm 1, \quad w \in T_x^*M,$$

at x and w is nonzero, then at x we have $\operatorname{rank} A = 1$.

PROOF. (i) It is clear that (2.5) is equivalent to

$$A_{il}(A_{hk}A_{jm} - A_{hm}A_{jk}) + A_{jl}(A_{ik}A_{hm} - A_{im}A_{hk}) + A_{hl}(A_{jk}A_{im} - A_{ik}A_{jm}) = 0.$$

Contracting this with g^{hk} and g^{jl} we obtain

$$(2.10) \quad \operatorname{tr}(A)(A_{il}A_{jm} - A_{im}A_{jl}) + A_{jl}A_{im}^2 + A_{im}A_{jl}^2 - A_{il}A_{jm}^2 - A_{jm}A_{il}^2 = 0$$

and (2.6), respectively. Further, substituting (2.7) into (2.10) we get

$$(2.11) \quad (\operatorname{tr}(A) - 2\alpha)A \wedge A = 2\beta g \wedge A.$$

We suppose that $\operatorname{tr}(A) - 2\alpha \neq 0$ at x . Now (2.11) yields

$$(2.12) \quad A \wedge A = \frac{2\beta}{\operatorname{tr}(A) - 2\alpha}g \wedge A.$$

We note that from (2.5) it follows that A is not proportional to g . Thus (2.12), in view of Lemma 3.1 of [21], implies $\beta = 0$ and, in a consequence, $\operatorname{rank} A = 1$, a contradiction. Therefore $2\alpha = \operatorname{tr}(A)$. Now (2.11) reduces to $\beta g \wedge A = 0$ whence $\beta(A - \frac{\operatorname{tr}(A)}{n}g) = 0$, and in a consequence, $\beta = 0$, completing the proof of (i).

(ii) We suppose that (2.5) holds at x . From (2.9) we have

$$(2.13) \quad A_{ij} = \alpha g_{ij} + \epsilon w_i w_j,$$

$$(2.14) \quad A_{ij}^2 = \alpha A_{ij} + \epsilon w^r A_{ri} w_j, \quad w^r = g^{rs} w_s.$$

(2.14) yields $w^r A_{ri} w_j = w^r A_{rj} w_i$ whence

$$(2.15) \quad w^r A_{ri} = \lambda w_i, \quad \lambda \in \mathbb{R}.$$

Now (2.14) turns into $A_{ij}^2 = \alpha A_{ij} + \epsilon \lambda w_i w_j$, which by making use of (2.8) and (2.9) gives $(\alpha + \lambda - \frac{\operatorname{tr}(A)}{2})A = \alpha \lambda g$. This implies $\alpha + \lambda = \frac{\operatorname{tr}(A)}{2}$ and $\alpha \lambda = 0$. We suppose that $\alpha \neq 0$. Now the last two relations yield

$$(2.16) \quad (a) \quad \lambda = 0, \quad (b) \quad \alpha = \operatorname{tr}(A)/2.$$

Evidently, (2.15) by (2.16)(a) reduces to $w^r A_{ri} = 0$. Now, contracting (2.13) with g^{ij} and transvecting with w^j , respectively, and using (2.16)(b) we obtain $\frac{n-2}{2} \operatorname{tr}(A) + \epsilon w^r w_r = 0$ and $\operatorname{tr}(A) + \epsilon w^r w_r = 0$, respectively. These relations imply

$\text{tr}(A) = 0$, which by (2.16)(b) yields $\alpha = 0$, a contradiction. Since $\alpha = 0$, (2.9) reduces to $A = \epsilon w \otimes w$, completing the proof. \square

COROLLARY 2.1. *Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold.*

- (i) *If (1.6) and (1.7) are satisfied on $U_S \subset M$, then (1.8) holds on this set.*
(ii) *If (1.1) and (2.2) are satisfied at every point of $U_S \subset M$, then $\text{rank } S = 1$ on this set.*

Let M , $n \geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold (N, g^N) . We denote by g the metric tensor induced on M from the metric tensor g^N . Further, we denote by ∇ and ∇^N the Levi-Civita connections corresponding to the metric tensors g and g^N , respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = g^N(\xi, \xi) = \pm 1$. We can present the Gauss formula and the Weingarten formula of (M, g) in (N, g^N) in the form: $\nabla_X^N Y = \nabla_X Y + \varepsilon H(X, Y)\xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where X, Y are vector fields tangent to M , H is the second fundamental tensor of (M, g) in (N, g^N) , \mathcal{A} is the shape operator and $H^k(X, Y) = g(\mathcal{A}^k X, Y)$, $k \geq 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and R^N the Riemann–Christoffel curvature tensors of (M, g) and (N, g^N) , respectively. The Gauss equation of (M, g) in (N, g^N) has the form $R(X_1, \dots, X_4) = R^N(X_1, \dots, X_4) + \varepsilon \overline{H}(X_1, \dots, X_4)$, where $\overline{H} = \frac{1}{2} H \wedge H$ and X_1, \dots, X_4 are vector fields tangent to M . Let the equations $x^r = x^r(y^k)$ be the local parametric expression of (M, g) in (N, g^N) , where y^k and x^r are the local coordinates of M and N , respectively, and $a, b, h, i, j, k, l, m \in \{1, 2, \dots, n\}$ and $p, r, t, u \in \{1, 2, \dots, n+1\}$.

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, $c = \frac{\tau}{n(n+1)}$, where τ denote the scalar curvature of the ambient space. Now the Gauss reads (see e.g. [14])

$$(2.17) \quad R_{hijk} = \varepsilon \overline{H}_{hijk} + \frac{\tau}{n(n+1)} G_{hijk},$$

where R_{hijk} , G_{hijk} , H_{hk} and $\overline{H}_{hijk} = H_{hk}H_{ij} - H_{hj}H_{ik}$ denote the local components of the tensors R , G , H and \overline{H} , respectively. Contracting (2.17) with g^{ij} we obtain

$$(2.18) \quad S_{hk} = \varepsilon (\text{tr}(H)H_{hk} - H_{hk}^2) + \frac{(n-1)\tau}{n(n+1)} g_{hk},$$

where $\text{tr}(H) = g^{hk}H_{hk}$ and S_{hk} are the local components of the Ricci tensor S of M . From (2.18) we easily get

$$(2.19) \quad \begin{aligned} S_{hk}^2 = g^{ij} S_{hi} S_{kj} = H_{hk}^4 - 2 \text{tr}(H) H_{hk}^3 + ((\text{tr}(H))^2 - \frac{2(n-1)\varepsilon\tau}{n(n+1)}) H_{hk}^2 \\ + \frac{2\varepsilon(n-1)\tau \text{tr}(H)}{n(n+1)} H + \left(\frac{(n-1)\tau}{n(n+1)} \right)^2 g_{hk}. \end{aligned}$$

Further, on every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, we have [19]

$$(2.20) \quad R \cdot R - Q(S, R) = -\frac{(n-2)\tau}{n(n+1)} Q(g, C).$$

Thus (1.4) is satisfied on every hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. Evidently, if $x \in U_R - U_H$, then at x we have $H^2 = \alpha H + \beta g$, $\alpha, \beta \in \mathbb{R}$. The last relation leads to (cf. [17, Proposition 3.1(ii)])

$$(2.21) \quad R \cdot R = \left(\frac{\tau}{n(n+1)} - \varepsilon\beta \right) Q(g, R).$$

Thus (2.3) holds on $U_R - U_H$. Further, if M is a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 3$, then on $U_H \subset M$ we have [8, Theorem 3.1]

$$(2.22) \quad R \cdot R = \frac{\tau}{n(n+1)} Q(g, R).$$

It is also known [7, eq. (3.8)] that if M is a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 3$, then on $U_S \subset M$ we have

$$(2.23) \quad Q\left(S - \left(L_R + \frac{(n-2)\tau}{n(n+1)}\right)g, R - \frac{\tau}{n(n+1)}G\right) = 0.$$

In particular, applying (2.22) into (2.23) we get on $U_H \subset U_S$

$$Q\left(S - \frac{(n-1)\tau}{n(n+1)}g, R - \frac{\tau}{n(n+1)}G\right) = 0.$$

From this, in view of Lemma 3.4 of [15] it follows that

$$R - \frac{\tau}{n(n+1)}G = \frac{\phi}{2} \left(S - \frac{(n-1)\tau}{n(n+1)}g \right) \wedge \left(S - \frac{(n-1)\tau}{n(n+1)}g \right),$$

on the set V of all points of U_H at which S has no a decomposition of the form (2.2) and ϕ is some function on V .

3. Examples

Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold, with nonempty set $U_C \cap U_S \subset M$, and let its curvature tensor R satisfies on $U_C \cap U_S$

$$(3.1) \quad R = \phi \bar{S} + \mu g \wedge S + \eta G,$$

where ϕ , μ and η are some functions on $U_C \cap U_S$. According to [9], (3.1) is called the *Roter type equation*. We mention that above decomposition of R on $U_C \cap U_S$ is unique [16, Lemma 3.2]. From (3.1) we have [15, Theorem 4.2]: (2.3), with $L_R = (n-2)\left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right)$,

$$\begin{aligned} R \cdot R - Q(S, R) &= \left(L_R + \frac{\mu}{\phi} \right) Q(g, C), \\ S^2 &= \left(\kappa + \frac{(n-2)\mu - 1}{\phi} \right) S + \frac{\mu\kappa + (n-1)\eta}{\phi} g. \end{aligned}$$

Further, as it was shown in [15], (3.1) implies

$$(3.2) \quad \begin{aligned} S_m^r R_{rijk} &= (\alpha + \mu)(S_{mk}S_{ij} - S_{mj}S_{ik}) + \left(\frac{\alpha\mu}{\phi} + \eta \right) (g_{ij}S_{mk} - g_{ik}S_{mj}) \\ &+ \beta(g_{mk}S_{ij} - g_{mj}S_{ik}) + \frac{\beta\mu}{\phi} G_{mijk}, \end{aligned}$$

where $\alpha = \phi\kappa - 1 + (n - 2)\mu$, $\beta = \mu\kappa + (n - 1)\eta$. Now (3.2) leads to (1.5), where

$$L_1 = -4(\alpha + \mu), \quad L_2 = -2\left(\frac{\alpha\mu}{\phi} + \eta + \beta\right), \quad L_3 = -\frac{4\beta\mu}{\phi}.$$

Thus we have

THEOREM 3.1. *Every semi-Riemannian manifold (M, g) , $n \geq 4$, satisfying the Roter type equation is an AG type manifold.*

REMARK 3.1. (i) Semi-Riemannian manifolds satisfying $R = \phi\bar{S}$, i.e. the special case of (3.1), were investigated in [24] (see also references therein).

(ii) Examples of warped products satisfying (3.1) are given in [18]. In Example 5.1 of that paper a warped product fulfilling (3.1) is given. That warped product can be locally realized on a hypersurface in a semi-Riemannian space of constant curvature.

(iii) Applying Lemma 3.4 of [15] to (2.23) we conclude that the curvature tensor R of a pseudosymmetric hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, is of the form (3.1) at all points of $U_S \cap U_C \subset M$ at which its Ricci tensor is not of the form (2.2).

(iv) Let $M_1 \times_F M_2$, $p = n - 1 = \dim M_1 \geq 3$, $\dim M_2 = 1$, be the warped product defined in [13, Example 4.1]. This manifold satisfies (1.2) and $\text{rank } S = 1$. Furthermore, applying the two last relations to (1.3) we get $S \cdot R = 0$. The manifold $M_1 \times_F M_2$, satisfies $R \cdot R = Q(S, R)$, i.e. (1.4) with $L_C = 0$. Thus we see that the warped product $M_1 \times_F M_2$ is an AG type manifold. This manifold is locally isometric to a hypersurface in a semi-Euclidean space [13, Example 5.1]. We mention that warped products satisfying (1.4) were investigated in [5]. For instance, in [5] it was shown that any warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = 3$, satisfies (1.4).

(v) Let $M_1 \times_F M_2$, $p = \dim M_1 \geq 3$, $n - p = \dim M_2 \geq 1$, be the warped product defined in Section 4 of [4]. This manifold satisfies $R \cdot R = Q(S, R)$, i.e. (1.4) with $L_C = 0$, and $\text{rank } S \geq n - p + 1$. Further, if we assume that $n - p = 1$ and the constant $\xi^f \xi_f$, defined in Section 4 of [4], is nonzero, then $\text{rank } S = 2$. Moreover, from (44) of [4] it follows that in this case the scalar curvature κ of $M_1 \times_F M_2$ is a nonzero constant and (1.7) and (1.8) are satisfied. On such manifolds we also have (1.9) [26, Example 3.1]. Thus, in view of Theorem 3.1, $M_1 \times_F M_2$ is an AG type manifold. In addition, this warped product is locally isometric to a hypersurface in a semi-Euclidean space ([4]; see also [26, Example 4.2]).

(vi) Let (\bar{M}, \bar{g}) be a non-flat 2-dimensional Riemannian manifold. It is easy to check that the product manifold $\bar{M} \times \mathbb{E}^{n-2}$, $n \geq 4$, satisfies (1.7), (1.8) and (1.9). Moreover, the manifold $\bar{M} \times \mathbb{E}^{n-2}$, $n \geq 4$, can be realized as a hypersurface in \mathbb{E}^{n+1} .

Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold. We define on $U_C \cap U_S \subset M$ the tensor $W(R)$ by

$$W(R) = R - \phi\bar{S} - \mu g \wedge S - \eta G,$$

where ϕ , μ and η are some functions on $U_C \cap U_S$. The tensor $W(R)$ will be called the *Roter type tensor*. Manifolds satisfying pseudosymmetry type curvature conditions related to the Roter type tensor will be investigated in subsequent papers.

We present now an extension of the above definition. Namely, for a generalized curvature tensors B and symmetric $(0, 2)$ -tensors A and D we define on $U_{\text{Ric}(B)} \cap U_{\text{Weyl}(B)} \subset M$ the $(0, 4)$ -tensor $W(B, A, D)$ by

$$W(B, A, D) = B - \phi\bar{A} - \mu A \wedge D - \eta\bar{D},$$

where ϕ , μ and η are some functions on $U_{\text{Ric}(B)} \cap U_{\text{Weyl}(B)}$. The tensor $W(B, A, D)$ will be also called a Roter type tensor. For instance, we have the following Roter type tensors

$$\begin{aligned} W(B, A, g) &= B - \phi\bar{A} - \mu g \wedge A - \eta G, \\ W(B) &= W(B, \text{Ric}(B), g) = B - \phi\overline{\text{Ric}(B)} - \mu g \wedge \text{Ric}(B) - \eta G. \end{aligned}$$

Some results on Roter type tensors $W(B, A, g)$ and $W(B, \text{Ric}(B), g)$ are given in [12] and [25]. For instance, we have

PROPOSITION 3.1. [25] *Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor B satisfying $W(B, A, g) = 0$ on $U_{\text{Ric}(B)} \cap U_{\text{Weyl}(B)} \subset M$. Then on this set we have*

$$B \cdot B - Q(\text{Ric}(B), B) = LQ(g, \text{Weyl}(B)), \quad L = (n-2)\left(\frac{\mu^2}{\phi} - \eta\right).$$

Moreover, if $A = \text{Ric}(B)$ on $U_{\text{Ric}(B)} \cap U_{\text{Weyl}(B)}$, then on this set we have

$$B \cdot B = L_B Q(g, B), \quad L_B = (n-2)\left(\frac{\mu^2}{\phi} - \eta\right) - \frac{\mu}{\phi}.$$

PROPOSITION 3.2. [12] *Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor B and let the conditions $B \cdot B = Q(\text{Ric}(B), B) + LQ(g, \text{Weyl}(B))$ and $B \cdot B = L_B Q(g, B)$ be satisfied on $U_{\text{Ric}(B)} \cap U_{\text{Weyl}(B)} \subset M$. Then on this set we have*

$$Q\left(\text{Ric}(B) - (L_B - L)g, B - \frac{L}{n-2}G\right) = 0.$$

PROPOSITION 3.3. [2, Corollary 6.1] *Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor B and let*

$$Q(\text{Ric}(B) - L_2 g, B - L_1 G) = 0$$

be satisfied on $U = U_{\text{Ric}(B)} \cap U_{\text{Weyl}(B)} \subset M$. Then $W(B) = \phi\overline{\text{Ric}(B)} + \mu g \wedge \text{Ric}(B) + \eta G$ on the subset $V \subset U$ of all points at which the tensor $\text{Ric}(B)$ has no a decomposition in a metrical term and in a term of rank one, where ϕ , μ and η are some functions on V .

4. AG type hypersurfaces satisfying $\text{rank } S = 2$

Let now M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, We set [14, eq. (13)]

$$(4.1) \quad A = H^3 - \text{tr}(H)H^2 + \frac{\varepsilon\kappa}{n-1}H.$$

Further, let B be a $(0, 2)$ -tensor with the local components B_{hk} defined by $B_{hk} = g^{ij}H_{hi}S_{kj}$. Using (2.17), (2.18) and (4.1) we obtain

$$(4.2) \quad B = -\varepsilon A + \left(\frac{(n-1)\tau}{n(n+1)} + \frac{\kappa}{n-1} \right) H,$$

$$(4.3) \quad S \cdot R = -2\varepsilon H \wedge B - \frac{2\tau}{n(n+1)} g \wedge S,$$

respectively. Substituting (4.2) into (4.3) and using (2.17) we get

$$(4.4) \quad S \cdot R = 2H \wedge A - 4 \left(\frac{(n-1)\tau}{n(n+1)} + \frac{\kappa}{n-1} \right) \left(R - \frac{\tau}{n(n+1)} G \right) - \frac{2\tau}{n(n+1)} g \wedge S.$$

Let now M be a Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. On $U_H \subset M$ we have [3, Theorem 3.1 and Proposition 3.2]

$$(4.5) \quad R \cdot S = \frac{\tau}{n(n+1)} Q(g, S).$$

It is known (see Proposition 3.2 and Theorem 3.1 of [3]) that (4.5) is equivalent on U_H to

$$(4.6) \quad H^3 = \text{tr}(H)H^2 + \lambda H,$$

where λ is some function on U_H . Now (4.1) turns into

$$(4.7) \quad A = \left(\lambda + \frac{\varepsilon\kappa}{n-1} \right) H.$$

Applying (2.17) and (4.7) in (4.4) we obtain (cf. [11, Theorem 3.1])

$$(4.8) \quad S \cdot R = 4 \left(\varepsilon\lambda - \frac{(n-1)\tau}{n(n+1)} \right) \left(R - \frac{\tau}{n(n+1)} G \right) - \frac{2\tau}{n(n+1)} g \wedge S.$$

If the ambient space is \mathbb{E}_s^{n+1} , then (2.20) reduces to

$$(4.9) \quad R \cdot R = Q(S, R).$$

Similarly, in this case, (2.17) reduces to

$$(4.10) \quad R_{hijk} = \varepsilon \bar{H}_{hijk}.$$

PROPOSITION 4.1. *Let M be a hypersurface in \mathbb{E}_s^{n+1} , $n \geq 4$. If at $x \in U_C \cap U_S - U_H \subset M$ we have $R \cdot S = 0$, then $R \cdot R = 0$ at x .*

PROOF. Evidently, (2.21) reduces to $R \cdot R = -\varepsilon\beta Q(g, R)$, which implies $R \cdot S = -\varepsilon\beta Q(g, S)$, and in a consequence, $\beta = 0$ at x . This completes the proof. \square

It is clear that every semisymmetric manifold is Ricci-semisymmetric. The converse statement is not true. Under some additional assumptions both conditions are equivalent to each other. This problem, named the *problem of P.J. Ryan*, was considered by several authors, see [6], [10] and [11] and references therein. Among other things, in [6] it was proved that the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on hypersurfaces in $N_s^5(c)$.

PROPOSITION 4.2. *Let M be a Ricci-semisymmetric an AG type hypersurface in \mathbb{E}_s^{n+1} , $n \geq 5$, and let the set $U_H \subset M$ be nonempty. In addition, let (1.7) be satisfied on U_H .*

(i) *The condition $R \cdot R = 0$ is satisfied at all points of U_H at which $\kappa \neq 0$. Moreover, (1.9) holds at such points.*

(ii) *The condition $R \cdot R \neq 0$ is satisfied at all points of U_H at which $\kappa = 0$.*

PROOF. Let $x \in U_H$. From (2.19), in view of Corollary 2.1(i) and (4.6), we get

$$(4.11) \quad H^4 = \left(2(\operatorname{tr}(H))^2 - \frac{\varepsilon\kappa}{2}\right)H^2 + \left(2\lambda + \frac{\varepsilon\kappa}{2}\right)\operatorname{tr}(H)H.$$

Furthermore, from (4.6) we get

$$(4.12) \quad H^4 = ((\operatorname{tr}(H))^2 + \lambda)H^2 + \lambda\operatorname{tr}(H)H.$$

Comparing the right-hand sides of (4.11) and (4.12) we obtain

$$\left(\lambda + \frac{\varepsilon\kappa}{2} - \operatorname{tr}(H)\right)H^2 + \left(\lambda + \frac{\varepsilon\kappa}{2}\right)\operatorname{tr}(H)H = 0,$$

whence $\lambda + \frac{\varepsilon\kappa}{2} = \operatorname{tr}(H)$ and $(\lambda + \frac{\varepsilon\kappa}{2})\operatorname{tr}(H) = 0$. These relations yield

$$(4.13) \quad (a) \quad \lambda = -\frac{\varepsilon\kappa}{2}, \quad (b) \quad \operatorname{tr}(H) = 0.$$

Now (4.6) and (4.8) turn into

$$(4.14) \quad A = -\frac{\varepsilon(n-3)\kappa}{2(n-1)}H,$$

$$(4.15) \quad S \cdot R = -\frac{\kappa}{2}R.$$

respectively. Since M is an AG type manifold, (1.5) holds on U_H . Now (4.15), by (1.5), leads to

$$(4.16) \quad -\frac{\kappa}{2}R_{hijk} = L_1(S_{hk}S_{ij} - S_{hj}S_{ik}) + L_3(g_{hk}g_{ij} - g_{hj}g_{ik}) \\ + L_2(g_{ij}S_{hk} + g_{hk}S_{ij} - g_{hj}S_{ik} - g_{ik}S_{hj}).$$

If $\kappa \neq 0$ at x , then from (4.16), in view of Theorem 4.2 of [15], it follows that (2.3) holds at x . Evidently, (2.3) implies (2.4), and in a consequence, we obtain $L_R = 0$ and $R \cdot R = 0$ at x . Further, contracting (4.16) with S_h^l and using (1.8) we obtain

$$-\frac{\kappa}{2}S_h^l R_{hijk} = \left(L_2 + \frac{\kappa L_1}{2}\right)\bar{S}_{lijk} + \left(L_3 + \frac{\kappa L_2}{2}\right)(g_{ij}S_{lk} - g_{ik}S_{lj}).$$

Symmetrizing this in l and i and using the relation $R \cdot S = 0$ we get $(L_3 + \frac{\kappa L_2}{2})Q(g, S) = 0$, whence

$$(4.17) \quad L_3 = -\frac{\kappa L_2}{2}.$$

On the other hand, contracting (4.16) with g^{ij} and using (1.8) we find

$$\left(\frac{\kappa}{2} + \frac{\kappa L_1}{2} + (n-2)L_2\right)S = -(\kappa L_2 + (n-1)L_3)g,$$

whence

$$(4.18) \quad (a) \quad \kappa L_2 = -(n-1)L_3, \quad (b) \quad \frac{\kappa}{2} + \frac{\kappa L_1}{2} + (n-1)L_2 = 0.$$

From (4.17) and (4.18)(a) we get $L_3 = 0$. Now (4.17) reduces to $L_2 = 0$. Applying this to (4.18)(b) we obtain $\kappa(L_1 + 1) = 0$, whence $L_1 = -1$. Now (4.16) reduces to (1.9). But this completes the proof of (i).

Let now $\kappa = 0$ at $x \in U_H$. Thus (4.14) turns into $A = 0$. This, together with (4.13)(b), reduces (2.18) and (4.2) to

$$(4.19) \quad S_{jk} = -\varepsilon H_{jk}^2, \quad B_{hk} = H_h^j S_{jk} = H_{jk}^3 = 0,$$

respectively. We suppose that $R \cdot R = 0$ at x . Now (4.9) yields

$$\begin{aligned} S_{hl}R_{mijk} + S_{il}R_{hmjk} + S_{jl}R_{himk} + S_{kl}R_{hijm} \\ - S_{hm}R_{lijk} - S_{im}R_{hljk} - S_{jm}R_{hilk} - S_{km}R_{hijl} = 0. \end{aligned}$$

This, by transvection with H_a^l and H_b^h and making use of (4.10) and (4.19), leads

$$(4.20) \quad S_{im}(S_{bj}S_{lk} - S_{bk}S_{lj}) + S_{il}(S_{bj}S_{km} - S_{bk}S_{jm}) = 0.$$

We set $Y_k = X^j S_{jk}$, where X^j and Y^j are the local components of vectors $X, Y \in T_x M$ such that $Y_1^2 + \dots + Y_n^2 > 0$, where $Y_k = g_{jk} Y^j$. Transvecting now (4.20) with X^l and X^m we obtain $Y_i(Y_k S_{bj} - Y_j S_{bk}) = 0$, whence it follows that $\text{rank } S = 1$ at x , a contradiction. Thus if $\kappa = 0$ at $x \in U_H$, then $R \cdot R \neq 0$ at x . Our proposition is thus proved. \square

The last proposition implies

THEOREM 4.1. *Let M be an AG type hypersurface in \mathbb{E}_s^{n+1} , $n \geq 5$, satisfying (1.7) on nonempty $U_H \subset M$. The conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on the subset of U_H of all points at which $\kappa \neq 0$.*

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