

## ASYMPTOTIC ESTIMATES ON FINITE ABELIAN GROUPS

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ABSTRACT. By using Ivić's methods for general divisor problem and counting function of abelian finite groups, we obtain results related to several arithmetic functions.

### 1. Finite abelian groups and semisimple rings

Certain common algebraic structures have enumerating functions whose Dirichlet series have a simple product representation involving the Riemann zeta-function. This fact establishes an analytic approach towards the study of these functions, and provides an application of zeta-function theory.

It is a well-known fact that every finite Abelian group can be represented as a direct product of cyclic groups of prime power order, moreover, the representation is unique except for possible rearrangements of the factors.

Let the arithmetical function  $a(n)$  denote the number on nonisomorphic Abelian groups with  $n$  elements. It is well known that  $a(n)$  is a positive, integer-valued multiplicative function, with the property that  $a(p^k) = P(k)$  for every prime  $p$  and every integer  $k \geq 1$  (here and later  $p, p_1, p_2, \dots$  denote primes), where  $P(k)$  is the number of unrestricted partitions of  $k$  (see [29, pp. 7 and 204]). Thus  $a(p^k)$  does not depend on  $p$  but only on  $k$ , so that  $a(n)$  is a “prime independent” function, and moreover  $a(p) = 1$  for every prime  $p$ .

W. Schwarz and E. Wirsing [35] showed that

$$\log a(n) \leq \log 5 \cdot \pi(A) + O((\log n)^\theta), \quad \theta = (\log 121)/\log 125 < 0.994$$

with  $A \sim (1/4) \log n$ . They also show that there are infinitely many integers  $n$  for which  $\log a(n) = \log 5 \cdot \pi(A)$ . These results sharpen a result of E. Kratzel [18], who showed that  $\limsup_{n \rightarrow \infty} \{\log a(n) \cdot (\log \log n / \log n)\} = (1/4) \log 5$ .

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For  $|x| < 1$  we have  $1 + \sum_{k=1}^{\infty} P(k)x^k = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$ . Then, by the properties of Dirichlet series, it follows that

$$(1.1) \quad \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left( 1 + \sum_{k=1}^{\infty} \frac{P(k)}{p^{ks}} \right) = \prod_{k=1}^{\infty} \zeta(ks), \quad \sigma = \operatorname{Re} s > 1$$

where  $\zeta(s)$  is the Riemann zeta function, thereby revealing the important analytic connection between  $a(n)$  and  $\zeta(s)$ . From the values of  $P(k)$  we obtain

$$(1.2) \quad \begin{aligned} a(1) &= 1, & a(p) &= 1, & a(p^2) &= 2, & a(p^3) &= 3, \\ a(p^4) &= 5, & a(p^5) &= 7, & a(p^6) &= 11, & a(p^7) &= 15, \\ a(p^8) &= 22, & a(p^9) &= 30, & a(p^{10}) &= 42. \end{aligned}$$

All the necessary results on  $\zeta(s)$  are to be found in [9], [12] and [36]. Let  $A(x)$  denote the number of distinct Abelian groups of order  $\leq x$ . The problem of the estimating the asymptotic formula for  $A(x)$  was considered for the first time for P. Erdős and G. Szekeres [5], proving that

$$(1.3) \quad A(x) = \sum_{n \leq x} a(n) = C_1 x + O(x^{1/2}).$$

The first essential progress of (1.3) was made by Kendall and Rankin [16] who proved, by applying a theorem of Landau, that

$$A(x) = \sum_{n \leq x} a(n) = C_1 x + C_2 x^{1/2} + O(x^{1/3} \log x).$$

H.-E. Richert was the first to estimate the error term by sums involving the function  $\psi(x) = x - [x] - 1/2$ . He obtained a third main term of order  $x^{1/3}$  and an error term of order less than  $x^{1/3}$ , that is

$$(1.4) \quad A(x) = C_1 x + C_2 x^{1/2} + C_3 x^{1/3} + \Delta(x)$$

where  $\Delta(x) = O(x^{3/10} \log^{9/10} x)$  and  $C_j = \prod_{v=1, v \neq j}^{\infty} \zeta(v/j)$ ,  $j = 1, 2, 3$ . Latter, all the order following improvements lead to (1.4) with the error term of type

$$\Delta(x) \ll x^{\kappa} \log^{\lambda} x$$

with  $1/3 > \kappa > 1/4$ . Richert's method was latter refined by W. Schwarz.

The estimates of  $(\kappa, \lambda)$  in  $\Delta(x) \ll x^{\kappa} \log^{\lambda} x$  are as follows:

$\Delta(x) \ll x^{1/3} \log^2 x,$	[16]	Kendall–Rankin
$\Delta(x) \ll x^{3/10} \log^{9/10} x,$	[28]	H. -E. Richert
$\Delta(x) \ll x^{20/69} \log^{21/23} x,$	[34]	W. Schwarz
$\Delta(x) \ll x^{34/123+\epsilon},$	[31]	P. G. Schmidt
$\Delta(x) \ll x^{7/27} \log^2 x,$	[32]	P. G. Schmidt
$\Delta(x) \ll x^{97/381} \log^{35} x,$	[17]	G. Kolesnik
$\Delta(x) \ll x^{40/159+\epsilon},$	[20]	H. Q. Liu
$\Delta(x) \ll x^{50/199+\epsilon},$	[20]	supplement
$\Delta(x) \ll x^{55/219} \log^7 x,$	[30]	Sargos and Wu.

This value differ enough from the conjectured value  $\Delta(x) \ll x^{1/6+\epsilon}$ . In [13] A. Ivić proved that  $\int_1^X \Delta^2(x)dx = \Omega(X^{4/3} \log X)$  where

$$\Delta(x) = A(x) - \sum_{j=1}^9 \operatorname{Res}_{s=1/j} F(s)x^s s^{-1} = A(x) - \sum_{j=1}^9 C_j x^{1/j}.$$

Moreover, Ivić deduced as corollary that  $\Delta(x) = \Omega(x^{1/6} \log^{1/2} x)$ . Thus, it is known, on the one hand, that  $\Delta(x) \ll x^{55/219}(\log x)^7$ , and the other

$$(1.5) \quad A(x) = C_1 x + C_2 x^{1/2} + C_3 x^{1/3} + C_4 x^{1/4} + C_5 x^{1/5} + \Omega(x^{1/6} \log^{1/2} x)$$

(see also Balasubramanian and K. Ramachandra [1]).

So from (1.5),  $\Delta(x) \ll x^{\kappa+\epsilon}$  with  $\kappa < 1/6$  cannot hold, but W. Schwarz [34], assuming the Riemann Hypothesis, obtained the following  $\Omega$  theorem with  $\Omega(x^{1/6-\epsilon})$  for every  $\epsilon > 0$ . Sums with  $F(a(n))$  were investigated by A. Ivić in [7] and [8] for a large class of functions  $F$ , in particular the functions  $a(a(n))$ ,  $d(a(n))$ ,  $\omega(a(n))$ ,  $\Omega(a(n))$ .

As is customary,  $\epsilon$  denotes positive numbers which may be arbitrarily small, but are not necessarily the same ones at each occurrence.

Another counting function related to algebraic structures is  $S(n)$  which denotes the number of nonisomorphic semisimple rings with  $n$  elements. We know that

$$\prod_{m=1}^{\infty} \frac{1}{1-x^{m^2}} = 1 + \sum_{k=1}^{\infty} P^*(k)x^k, \quad |x| < 1$$

where  $P^*(k)$  is the number of partitions of  $k$  into parts which are square. Now, for every prime  $p$ , let  $x = p^{-rs}$ , then we can write the identity

$$(1.6) \quad \prod_{r=1}^{\infty} \left\{ 1 + \sum_{k=1}^{\infty} P^*(k)p^{-krs} \right\} = \sum_{\alpha=0}^{\infty} \frac{S(p^\alpha)}{p^{\alpha s}}$$

and the Dirichlet series of  $S(n)$  is

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^s} = \prod_{r \geq 1} \prod_{m \geq 1} \zeta(rm^2 s), \quad \sigma > 1.$$

Now, from (1.6), we can obtain the values of  $S(p^\alpha)$ . For  $\alpha = 0, 1, 2, 3$  we have  $S(p^\alpha) = a(p^\alpha) = \alpha$ , but  $S(p^4) = 6$ ,  $S(p^5) = 8$ ,  $S(p^6) = 13$ ,  $S(p^7) = 18 \dots$

**THEOREM 1.1.** *For the summatory function of  $S(n)$  we have the following estimate*

$$\sum_{n \leq x} S(n) = C_1 B_1 x + C_2 B_2 x^{1/2} + C_3 B_3 x^{1/3} + O(x^{55/219} \log^7 x)$$

where  $C_1, C_2, C_3$  are the constant of (1.4), and  $B_j = \prod_{r \geq 1} \prod_{m \geq 2} \zeta(rm^2/j)$ ,  $j = 1, 2, 3$ .

*Proof.* Is a consequence of Theorem 2 of [2] and the result of Sargos and Wu.

## 2. Direct factors and unitary factors

For positive integers  $n$ , let  $\tau(n)$  denote the number of divisors of  $n$ , and let  $t(n)$  denote the number of decompositions of  $n$  into two relative prime factors. In [4] E. Cohen has established analogues for the finite abelian groups of the classical results of Dirichlet and Mertens, that is, on the average order of  $\tau(n)$  and  $t(n)$ . It is known (see E. Cohen [4] or E. Krätzel [19]) that  $T(x) = \sum_{n \leq x} \tau_1(n)$  where  $\tau_1(n)$  is a multiplicative function defined by

$$(2.1) \quad \sum_{n=1}^{\infty} \frac{\tau_1(n)}{n^s} = \prod_{k=1}^{\infty} \zeta^2(ks), \quad \text{Re } s > 1.$$

E. Cohen, proved the representation

$$(2.2) \quad T(x) = c_1 x(\log x + 2\gamma - 1) + c_2 x + R(x)$$

$\gamma$  denotes Euler's constant, and  $R(x) \ll \sqrt{x} \log^2 x$ . In 1988, E. Krätzel [19], improved this result and show that

$$(2.3) \quad R(x) = c_3 \sqrt{x} \left( \frac{1}{2} \log x + 2\gamma - 1 \right) + c_4 \sqrt{x} + \Delta_{\tau_1}(x)$$

with the new remainder term  $\Delta_{\tau_1}(x)$  satisfying  $\Delta_{\tau_1}(x) \ll x^{5/12} \log^4 x$ . In (2.2), (2.3)  $c_i$ ,  $1 \leq i \leq 4$ , are effective constants. The result of Krätzel has been improved by many authors. A detailed history is as follows.

The exponent  $5/12$  was improved to  $83/201$ ,  $45/109$ ,  $2/5$ ,  $3/8$ ,  $7/19$ ,  $4/11$ ,  $21/58$ ,  $47/130$  by the authors:

$\Delta_{\tau_1}(x) \ll x^{83/201+\epsilon}$ ,	[24]	Menzer
$\Delta_{\tau_1}(x) \ll x^{45/109+\epsilon}$ ,	[25]	Menzer and Seibol
$\Delta_{\tau_1}(x) \ll x^{2/5+\epsilon}$ ,	[21]	Liu
$\Delta_{\tau_1}(x) \ll x^{3/8+\epsilon}$ ,	[6]	Yu Gang
$\Delta_{\tau_1}(x) \ll x^{7/19+\epsilon}$ ,	[22]	Liu
$\Delta_{\tau_1}(x) \ll x^{4/11+\epsilon}$ ,	[43]	W. Zhai and X. Cao
$\Delta_{\tau_1}(x) \ll x^{21/58+\epsilon}$ ,	[23]	Liu and Wu
$\Delta_{\tau_1}(x) \ll x^{47/130+\epsilon}$ ,	[40]	Jie Wu.

In this problem the conjecture is  $\Delta_{\tau_1}(x) \ll x^{1/4+\epsilon}$ . Ivić has proved that

$$\int_1^X \Delta_{\tau_1}^2(x) dx = \Omega(X^{3/2} \log^4 X)$$

(see [14]) which guaranteed the conjecture.

Analogously, a similar situation takes place when we consider the unitary factors of  $G$  in  $X$ . Let  $t(G)$  denote the numbers of unitary factors of  $G$  and  $T^*(x) = \sum t(G)$  where again the summation is extended over all abelian finite

groups of order not exceeding  $x$ . It is known (see Lemma 4.2 of [4] or Lemma 1 of [19]) that  $T^*(x) = \sum_{n \leq x} t_1(n)$  where  $t_1(n)$  is defined by the Dirichlet series

$$(2.4) \quad F(s) = \sum_{n=1}^{\infty} \frac{t_1(n)}{n^s} = \prod_{k=1}^{\infty} \zeta^2((2k-1)s)\zeta(2ks), \quad \operatorname{Re} s > 1.$$

Here, Cohen [4] proved that

$$T_1^*(x) = d_1 x(\log x + 2\gamma - 1) + d_2 x + R_1(x), \quad R_1(x) \ll \sqrt{x} \log x.$$

As in the previous case, E. Krätzel improved the above estimates and obtained  $R_1(x) = d_3 \sqrt{x} + \Delta_{t_1}(x)$  being  $\Delta_{t_1}(x) \ll x^{11/29} \log^2 x$ . Later, the error term was improved as follows

$$\begin{aligned} \Delta_{t_1}(x) &\ll x^{31/82+\epsilon}, & [26] & \text{H. Menzer} \\ \Delta_{t_1}(x) &\ll x^{3/8+\epsilon}, & [33] & \text{Schmidt} \\ \Delta_{t_1}(x) &\ll x^{77/208+\epsilon}, & [21] & \text{Liu} \\ \Delta_{t_1}(x) &\ll x^{9/25+\epsilon}, & [43] & \text{W. Zhai and X. Cao} \\ \Delta_{t_1}(x) &\ll x^{29/80+\epsilon}, & [22] & \text{Liu} \\ \Delta_{t_1}(x) &\ll x^{47/131+\epsilon}, & [38] & \text{Jie Wu.} \end{aligned}$$

W. Zhai [46] sharpens the exponent to  $0.354\dots$  using the method of exponent pairs. In [3] we have given bounds for the integral of an error term

$$\Delta_{t_1}^* = T_1^*(x) - \sum_{j=1}^6 \operatorname{Res}_{s=1/j} F(s)x^s s^{-1}$$

by using the Mellin inversion formula in conjunction with a certain smoothing function. Thus, for every  $\epsilon > 0$ ,

$$\int_1^x \Delta_{t_1}^*(x) dx \ll x^{1+3/20+\epsilon}.$$

### 3. Asymptotic estimates

From (1.1) and (2.1) it follows that for any integer  $n \geq 1$ ,  $\tau_1$  is the Dirichlet convolution  $\tau_1(n) = (a * a)(n)$ . Thus for any prime  $p$  and integer  $\alpha \geq 1$

$$\tau_1(p^\alpha) = \sum_{k=0}^{\alpha} a(p^k)a(p^{\alpha-k})$$

and from (1.2) we obtain the following values for  $\tau_1(p^\alpha)$ ,  $1 \leq \alpha \leq 10$

$$(3.1) \quad \begin{aligned} \tau_1(1) &= 1, & \tau_1(p) &= 2, & \tau_1(p^2) &= 5, & \tau_1(p^3) &= 10, \\ \tau_1(p^4) &= 20, & \tau_1(p^5) &= 36, & \tau_1(p^6) &= 65, & \tau_1(p^7) &= 110, \\ \tau_1(p^8) &= 185, & \tau_1(p^9) &= 300, & \tau_1(p^{10}) &= 481. \end{aligned}$$

From (2.1), (1.1) and (2.4) it follows that

$$\sum_{n=1}^{\infty} \frac{\tau_1(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^{2s}} \sum_{n=1}^{\infty} \frac{t_1(n)}{n^s}$$

then

$$(3.2) \quad \tau_1(n) = \sum_{d^2|n} a(d)t_1\left(\frac{n}{d^2}\right)$$

where the sum is over all divisors  $d|n$  such that  $d^2|n$ . From (3.2), (3.1) and (1.2) we deduce the following particular values for  $t_1(p^\alpha)$

$$(3.3) \quad \begin{aligned} t_1(1) &= 1, & t_1(p) &= 2, & t_1(p^2) &= 4, & t_1(p^3) &= 8, \\ t_1(p^4) &= 14, & t_1(p^5) &= 24, & t_1(p^6) &= 40, & t_1(p^7) &= 64, \\ t_1(p^8) &= 100, & t_1(p^9) &= 154, & t_1(p^{10}) &= 232. \end{aligned}$$

A. Ivić [13] considered the powers functions  $\tau_1^k(n)$  and obtained asymptotic formula for the summatory functions  $\sum_{n \leq x} \tau_1^k(n)$ ,  $k \geq 2$  and  $\sum_{n \leq x} \tau_1(n^2)$ .

H. Menzer [27] considered the convolutions  $w(n) = (a * a * a)(n) = (\tau_1 * a)(n)$ . Thus from (3.1) and (1.2)

$$(3.4) \quad \begin{aligned} w(1) &= 1, & w(p) &= 3, & w(p^2) &= 9, & w(p^3) &= 22, \\ w(p^4) &= 68, & w(p^5) &= 108, & w(p^6) &= 221, & w(p^7) &= 429, \\ w(p^8) &= 810, & w(p^9) &= 1479, & w(p^{10}) &= 2640. \end{aligned}$$

H. Menzer, applying results of three-dimensional exponential sums and two special divisors problems, proved that

$$W(x) = \sum_{n \leq x} w(n) = xP_2(\log x) + x^{1/2}Q_2(\log x) + O(x^{76/153} \log^6 x)$$

$P_2(\log x)$ ,  $Q_2(\log x)$  being polynomial of degree 2 in  $\log x$ , whose coefficients may be explicitly evaluated. In [45] W. Zhai, improves the error term to  $O(x^{53/116+\epsilon})$ , for any  $\epsilon > 0$ . Jie Wu remarked that using (3.13) of your paper [39], he could deduce the following result  $\Delta(1, 1, 1, 2, 2, 2; x) \ll x^{4/9} \log^7 x$ , where  $\Delta(1, 1, 1, 2, 2, 2; x)$  is the error term in the asymptotic formula for divisor problem  $D(1, 1, 1, 2, 2, 2; x)$ . This improves Zhais's exponent  $53/116$  to  $4/9$ .

Now we study some asymptotic formulas for the power functions of  $w(n)$  and  $t_1(n)$ .

A classical result of A.E. Ingham states an asymptotic formula related to fourth-moment of the zeta function for  $\sigma = 1/2$ . Ingham proved this estimation by means of the functional equation for  $\zeta^2(s)$  Ingham's results has been improved in 1979 by D.R. Heath-Brown (see [36]) to give

LEMMA 3.1. *In the critical line  $\sigma = 1/2$  the following estimate hold*

$$(3.5) \quad \int_1^T |\zeta(1/2 + it)|^4 dt = T \sum_{k=0}^4 c_k \log^k T + O(T^{7/8+\epsilon}),$$

where  $c_4 = (2\pi^2)^{-1}$  and  $c_3 = 2\{4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2}\}\pi^{-2}$ .

The proof requires an asymptotic formula for  $\sum_{n \leq x} \tau(n)\tau(n+r)$  with a good error term, uniform in  $r$ . These estimates are obtained by Heath-Brown applying Weil's bound for the Kloosterman sum. But in 1986, N.I. Zavorotnyi [42] improved Heath-Brown's exponent  $7/8$  to  $2/3$ .

For the following theorems, all the necessary results on  $\zeta(s)$  are to be found in [9] and [36].

THEOREM 3.1. *For any given  $\epsilon > 0$*

$$(3.6) \quad \sum_{n \leq x} w^2(n) = xP_8(\log x) + O(x^{2/3+\epsilon}).$$

*Proof.* As  $a(n) \ll n^\epsilon$ , and  $\tau_1(n) \ll n^\epsilon$ , their convolution is  $w(n) \ll n^\epsilon$ . By the properties of Dirichlet series, the multiplicativity of  $w(n)$  and using (3.4) we have in  $\text{Re } s = \sigma > 1$

$$(3.7) \quad \begin{aligned} \sum_{n=1}^{\infty} \frac{w^2(n)}{n^s} &= \prod_p \left( 1 + \sum_{k=1}^{\infty} \frac{w^2(p^k)}{p^{ks}} \right) \\ &= \zeta^9(s) \prod_p \{1 - p^{-s}\}^9 \{1 + 9p^{-s} + 81p^{-2s} + \dots\} \\ &= \zeta^9(s) \zeta^{36}(2s) \zeta^{-5}(3s) H_1(s) \end{aligned}$$

where  $H_1(s)$  represents a Dirichlet series which converges absolutely for  $\sigma > 1/4$ .

By the truncated Perron's inversion formula (see Appendix of [9] or [36]) with  $b = 1 + 1/\log x$ , ( $x \geq x_0 > 1$ ),  $\alpha = 9$  and  $\Psi(n) = n^\epsilon$ , we obtain

$$(3.8) \quad \begin{aligned} \sum_{n \leq x} w^2(n) &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta^9(s) \zeta^{36}(2s) \zeta^{-5}(3s) H_1(s) \frac{x^s}{s} ds \\ &+ O\left(\frac{x \log^9 x}{T}\right) + O\left(\frac{x^{1+\epsilon}}{T}\right). \end{aligned}$$

In  $s = 1$  the subintegral function has a pole of ninth-order and the residue is

$$\text{Res}_{s=1} \zeta^9(s) \zeta^{36}(2s) \zeta^{-5}(3s) H_1(s) \frac{x^s}{s} = xP_8(\log x).$$

Moving the line of integration to  $\sigma = 35/54$  and by the residue theorem, we obtain

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta^9(s) \zeta^{36}(2s) \zeta^{-5}(3s) H_1(s) \frac{x^s}{s} ds = I_1 + I_2 + I_3 + xP_8(\log x)$$

being

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \int_{(35/54)-iT}^{35/54+iT} \zeta^9(s) \zeta^{36}(2s) \zeta^{-5}(3s) H_1(s) \frac{x^s}{s} ds \\ I_2 &= \frac{1}{2\pi i} \int_{(35/54)+iT}^{b+iT} \zeta^9(s) \zeta^{36}(2s) \zeta^{-5}(3s) H_1(s) \frac{x^s}{s} ds \\ I_3 &= \frac{1}{2\pi i} \int_{b-iT}^{(35/54)-iT} \zeta^9(s) \zeta^{36}(2s) \zeta^{-5}(3s) H_1(s) \frac{x^s}{s} ds. \end{aligned}$$

For the integrals  $I_2, I_3$  we have

$$I_2 + I_3 \ll \int_{(35/54)}^b |\zeta(\sigma + iT)|^9 |\zeta(2\sigma + i2T)|^{36} |\zeta^{-5}(3(\sigma + iT))| |H_1(\sigma + iT)| \frac{x^\sigma}{T} d\sigma.$$

We know that  $\zeta(\sigma + it) \ll (|t|^{(1-\sigma)/3} + 1) \log |t|$  if  $1/2 \leq \sigma \leq 2$ , and we also know that

$$(3.9) \quad \zeta(1 + it) \ll \log^{2/3} |t|, \quad t \geq t_0$$

(see Theorem 6.3 of [9]). Then

$$\begin{aligned} (3.10) \quad I_2 + I_3 &\ll \frac{1}{T} \int_{(35/54)}^b |\zeta(\sigma + iT)|^9 x^\sigma d\sigma \\ &\ll \frac{1}{T} \int_{(35/54)}^b (T^{(1-\sigma)/3} + 1)^9 \log^9 T x^\sigma d\sigma \\ &\ll \frac{\log^9 T}{T} \left( \frac{x^{35/54} T^{19/18} + x}{\log(x/T^3)} + x^b \right). \end{aligned}$$

For the integral  $I_1$ , we use Theorem 8.4 [9] with  $m = 9$ , that is

$$(3.11) \quad \int_1^T \left| \zeta\left(\frac{35}{54} + it\right) \right|^9 dt \ll T^{1+\epsilon}$$

then we obtain

$$I_1 \ll x^{35/54} + \int_1^T \left| \zeta\left(\frac{35}{54} + it\right) \right|^9 \left| \zeta\left(\frac{35}{27} + i2t\right) \right|^{36} \frac{x^{35/54}}{|35/54 + it|} dt.$$



From (3.11) and using an integration by parts, it follows that

$$\int_1^T \left| \zeta\left(\frac{35}{54} + it\right) \right|^9 \frac{dt}{t} = \int_1^T \frac{\Phi'(t)}{t} dt \ll T^\epsilon$$

where  $\Phi(t) = \int_1^t |\zeta(35/54 + iu)|^9 du$ . From this estimation we can deduce

$$(3.12) \quad I_1 \ll x^{35/54} \cdot T^\epsilon.$$

By (3.11) and (3.12) obtain that (3.8) is

$$\begin{aligned} \sum_{n \leq x} w^2(n) &= xP_8(\log x) + O(x^{1+\epsilon}T^{-1}) \\ &\quad + O\left(\frac{\log^9 T}{T} \left(\frac{x^{35/54}T^{19/18}}{\log(x/T^3)} + x^b\right)\right) + O(x^{35/54}T^\epsilon). \end{aligned}$$

Choosing  $T = (x/e)^{1/3}$  we obtain (3.6).

**THEOREM 3.2.** *We have the estimation*

$$(3.13) \quad \sum_{n \leq x} t_1^2(n) = xP_3(\log x) + O(x^{1/2} \log^9 x).$$

*Proof.* By using the multiplicativity of  $t_1^2(n)$  and (3.3), we have for  $\operatorname{Re} s = \sigma > 1$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{t_1^2(n)}{n^s} &= \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{t_1^2(p^k)}{p^{ks}}\right) \\ (3.14) \quad &= \zeta^4(s) \prod_p \{1 - p^{-s}\}^4 \{1 + 4p^{-s} + 16p^{-2s} + \dots\} \\ &= \zeta^4(s) \zeta^6(2s) \zeta^{20}(3s) H_2(s), \end{aligned}$$

where  $H_2(s)$  represents a Dirichlet series which converges absolutely for  $\sigma > 1/4$ .

By the inversion formula for Dirichlet series (see Appendix of [9] or [36]) with  $b = 1 + 1/\log x$ , and  $t_1(n) \ll n^\epsilon$ , we obtain

$$\sum_{n \leq x} t_1^2(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta^4(s) \zeta^6(2s) \zeta^{20}(3s) H_2(s) \frac{x^s}{s} ds + O(x^{1+\epsilon}T^{-1}).$$

By the residue theorem, moving the integrating line to  $\sigma = 1/2$  and using (3.5), we have that

$$\sum_{n \leq x} t_1^2(n) = xP_3(\log x) + O(x^{1+\epsilon}T^{-1}) + I_1 + I_2 + I_3$$

with  $I_1 = I'_1 + I''_1 + I_\rho$  where  $I'_1 = \int_{1/2+i\rho}^{1/2+iT} G(s) x^s ds$ ,  $I''_1 = \int_{1/2-iT}^{1/2-i\rho} G(s) x^s ds$  and  $I_\rho$  surrounded the pole  $s = 1/2$ . So, the integral  $I_\rho \ll x^{1/2} P_5(\log x)$ . In fact, we denote  $G(s) = \zeta^4(s) \zeta^{20}(3s) H_2(s) s^{-1}$ . Then  $I_\rho = \int_{C_\rho} \zeta^6(2s) G(s) x^s ds$  and let  $C_\rho = \{s = 1/2 + \rho e^{i\alpha} : \alpha \in [-\pi/2, \pi/2], 0 < \rho < 1/2\}$ , then  $G(s)$  is an analytic function on  $C_\rho$  for all  $0 < \rho < 1/2$ .

Consider  $\rho = 1/\log x$  and  $x > e^2$ , for  $\zeta^6(2s)$  we can write

$$\zeta^6(2s) = \frac{c_6}{(s-1/2)^6} + \cdots + \frac{c_1}{(s-1/2)} + \sum_{n=0}^{\infty} c_n (s-1/2)^n$$

and for the integral  $I_\rho$  we have

$$\begin{aligned} |I_\rho| &\leq \sum_{j=0}^6 \left| \int_{C_\rho} \frac{c_{6-j}}{(s-1/2)^{6-j}} G(s) x^s ds \right| + \cdots + \left| \int_{C_\rho} \sum_{n=0}^{\infty} c_n (s-1/2)^n G(s) x^s ds \right| \\ &\leq M_6 |c_6| \pi \frac{x^{1/2}}{\rho^5} + M_5 \pi |c_5| \frac{x^{1/2}}{\rho^4} + \cdots + M_1 |c_1| \pi x^{1/2} + M_0 \pi x^{1/2} \rho \\ &\leq M_6 |c_6| \pi x^{1/2} \log^5 x + \cdots + M_1 |c_1| \pi x^{1/2} + M_0 \pi \frac{x^{1/2}}{\log x} \ll x^{1/2} \log^5 x. \end{aligned}$$

The others integrals are

$$I'_1 \ll \int_{\rho}^T |\zeta(1/2+it)|^4 |\zeta(1+2it)|^6 |\zeta(3(1/2+it))|^{20} |H_2(1/2+it)| \frac{x^{1/2}}{|1/2+it|} dt$$

as  $t \geq \rho > 0$

$$|\zeta(1+2it)|^6 \ll (\log^{2/3} t)^6 = \log^4 t$$

we obtain

$$\begin{aligned} I'_1 &\ll x^{1/2} \log^4 T \int_{\rho}^T \left| \zeta\left(\frac{1}{2}+it\right) \right|^4 \frac{dt}{t} \\ &= x^{1/2} \log^4 T \left( \int_{\rho}^1 \left| \zeta\left(\frac{1}{2}+it\right) \right|^4 \frac{dt}{t} + \int_1^T \left| \zeta\left(\frac{1}{2}+it\right) \right|^4 \frac{dt}{t} \right) \ll x^{1/2} \log^9 T \end{aligned}$$

since

$$\int_{\rho}^1 \left| \zeta\left(\frac{1}{2}+it\right) \right|^4 \frac{dt}{t} \ll 1$$

and the same estimate holds for  $I''_1$ . Thus  $I_1 \ll x^{1/2} P_5(\log x) + x^{1/2} \log^9 T$ .

For the integrals  $I_2, I_3$  we have

$$I_2 + I_3 \ll \int_{1/2}^b \left| \zeta^4(\sigma+iT) \zeta^6(2(\sigma+iT)) \zeta^{20}(3(\sigma+iT)) H_2(\sigma+iT) \right| \frac{x^\sigma}{|\sigma+iT|} d\sigma$$

By (3.9), as  $2\sigma \geq 1$  we have  $|\zeta^6(2\sigma + 2iT)| \ll \log^4 T$ , then

$$\begin{aligned} I_2 + I_3 &\ll \int_{1/2}^b |\zeta(\sigma + iT)|^4 \log^4 T \frac{x^\sigma}{|\sigma + iT|} d\sigma \\ &\ll \frac{\log^4 T}{T} \int_{1/2}^b (T^{(1-\sigma)/3} + 1)^4 \log^4 T x^\sigma d\sigma \\ &\ll \frac{\log^8 T}{T \log(x/T^{4/3})} (x + x^{1/2}|T|^{2/3}). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n \leq x} t_1^2(n) &= xP_3(\log x) + O(x^{1/2}P_5(\log x)) + O(x^{1+\epsilon}T^{-1}) \\ &\quad + O(x^{1/2} \log^9 x) + O\left(\frac{\log^8 T}{T \log(x/T^{4/3})} (x + x^{1/2}T^{2/3})\right). \end{aligned}$$

Choosing  $T = (x/e)^{3/4}$  we deduce the formula (3.13).

**THEOREM 3.3.** *For any given  $\epsilon > 0$*

$$(3.15) \quad \sum_{n \leq x} t_1^3(n) = xP_7(\log x) + O(x^{5/8+\epsilon}).$$

*Proof.* For  $\text{Re } s = \sigma > 1$  holds

$$(3.16) \quad \sum_{n=1}^{\infty} \frac{t_1^3(n)}{n^s} = \prod_p \left( 1 + \sum_{k=1}^{\infty} \frac{t_1^3(p^k)}{p^{ks}} \right) = \zeta^8(s) \zeta^{28}(2s) H_3(s),$$

where  $H_3(s)$  represents a Dirichlet series which converges absolutely for  $\sigma > 1/3$ . By the inversion formula for Dirichlet series (see Appendix of [9] or [36]), with  $a(n) \ll n^\epsilon$ ,  $\alpha = 8$ ,  $b = 1 + \epsilon$  we deduce

$$(3.17) \quad \sum_{n \leq x} t_1^3(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta^8(s) \zeta^{28}(2s) H_3(s) \frac{x^s}{s} ds + O(x^{1+\epsilon}T^{-1}).$$

Moving the line of integration to  $\sigma = 5/8$  and by Theorem 8.4 [9], we deduce

$$\int_1^T |\zeta(5/8 + it)|^8 dt \ll T^{1+\epsilon},$$

hence

$$(3.18) \quad I_1 \ll x^{5/8} T^\epsilon.$$

Moreover, the integrals  $I_2, I_3$  satisfy

$$(3.19) \quad \begin{aligned} I_2 + I_3 &\ll \int_{5/8}^b |\zeta(\sigma + iT)|^8 |\zeta(2\sigma + 2iT)|^{28} |H_3(\sigma + iT)| \frac{x^\sigma}{|\sigma + iT|} d\sigma \\ &\ll \int_{5/8}^b |\zeta(\sigma + iT)|^8 \frac{x^\sigma}{|\sigma + iT|} d\sigma \ll \frac{\log^8 T}{T} \left( \frac{x + x^{5/8}T}{\log(x/T^{8/3})} + x^{1+\epsilon} \right). \end{aligned}$$

Now from (3.17), (3.18), (3.19) we have

$$\begin{aligned} \sum_{n \leq x} t_1^3(n) &= xP_7(\log x) + O(x^{1+\epsilon}T^{-1}) + O(x^{5/8}T^\epsilon) \\ &\quad + O\left(\frac{\log^8 T}{T} \left(\frac{x + x^{5/8}T}{\log(x/T^{8/3})} + x^{1+\epsilon}\right)\right) \end{aligned}$$

thus by choosing  $T = (x/e)^{3/8}$  we deduce the estimation (3.15).

In the asymptotic formulas (3.6), (3.13) and (3.15), we observe that in their product representation (3.7), (3.14) and (3.16) respectively, the Dirichlet series have a power of  $\zeta(2s)$  as factor. This hints at the existence of a second main term in each case of the form  $x^{1/2}P_{35}(\log x)$ ,  $x^{1/2}P_5(\log x)$ ,  $x^{1/2}P_{27}(\log x)$  respectively, and we expect to obtain an error term of order  $o(x^{1/2})$ , as  $x \rightarrow \infty$ . Also, it is possible that the error terms  $\Delta_{w^2}(x) \ll x^{\alpha+\epsilon}$ ,  $\Delta_{t_1^2}(x) \ll x^{\beta+\epsilon}$ ,  $\Delta_{t_1^3}(x) \ll x^{\gamma+\epsilon}$ , with  $\alpha < 4/9$ ,  $\beta < 3/8$ ,  $\gamma < 7/16$  respectively, cannot hold.

In the next theorem we obtain  $\Omega$ -estimates for the mean square of the error terms.

**THEOREM 3.4.** *The following  $\Omega$ -estimates hold*

$$(3.20) \quad \int_1^X \Delta_{w^2}^2(x) dx = \Omega(X^{1+8/9}),$$

$$(3.21) \quad \int_1^X \Delta_{t_1^2}^2(x) dx = \Omega(X^{1+3/4}),$$

$$(3.22) \quad \int_1^X \Delta_{t_1^3}^2(x) dx = \Omega(X^{1+7/8}).$$

*Proof.* It is a consequence of Theorem 3 of [11]. For the function  $w^2(n)$ , the Theorem may be applied with  $a_1 = a_2 = \dots = a_9 = 1$ ,  $a_{10} = 2$ ,  $r = 9$  then

$$g = \frac{r-1}{2(a_1 + a_2 + \dots + a_r)} = \frac{4}{9}, \quad a_r g_r = \frac{4}{9} < \frac{1}{2}, \quad A = 0.$$

In the case of function  $t_1^2(n)$ , the Theorem may be applied with  $a_1 = a_2 = a_3 = a_4 = 1$ ,  $a_5 = 2$ ,  $r = 4$  and

$$g = \frac{r-1}{2(a_1 + a_2 + \dots + a_r)} = \frac{3}{8}, \quad a_r g_r = \frac{3}{8} < \frac{1}{2}, \quad A = 0.$$

If the function is  $t_1^3(n)$ , we take  $a_1 = a_2 = \dots = a_8 = 1$ ,  $a_9 = 2$ ,  $r = 8$ , and then

$$g = \frac{r-1}{2(a_1 + a_2 + \dots + a_r)} = \frac{7}{16}, \quad a_r g_r = \frac{7}{16} < \frac{1}{2}, \quad A = 0.$$

Then (3.20), (3.21) and (3.22) holds.

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