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HYERS–ULAM STABILITY OF A GENERAL QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. We obtain a general solution and solve the Hyers–Ulam stability problem for the general quadratic functional equation

$$f(x+y+z)+f(x-y)+f(x-z)=f(x-y-z)+f(x+y)+f(x+z).$$

1. Introduction

In 1940, Ulam [**16**] asked a question concerning the stability of group homomorphisms:

Let G_1 be a group and G_2 a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $h: G_1 \to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it.

It is easy to see that the quadratic function $f(x)=cx^2$ is a solution of each of the following functional equations:

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x), \tag{1.2}$$

So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x (see [**1**], [**11**]). The functional equation (1.2) was solved by Pl. Kannappan. In fact, he proved that a functional on a real vector space is

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a solution of the equation (1.2) if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x) = B(x, x) + A(x)$ for any x (see [**11**]).

A Hyers–Ulam stability theorem for the quadratic functional equation (1.1) was proved by Skof for the functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [**15**]). In [**3**], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1). Grabiec [**6**] generalized the results above. Jun and Lee [**9**] proved the Hyers–Ulam–Rassias stability of the pexiderized quadratic equation (1.1). The stability problems of several functional equations have been extensively investigated by a number of authors [**3, 8, 10, 13, 14**].

Now we introduce the following quadratic functional equation, which is somewhat different from (1.1), (1.2),

$$f(x+y+z) + f(x-y) + f(x-z) = f(x-y-z) + f(x+y) + f(x+z). \tag{1.3}$$

We will find out the general solution of the functional equation (1.3) and consider the stability problem of it in the sense of Hyers, Ulam, Rassias and Găvruta.

2. Main Results

In the following theorem, we find out the general solution of the functional equation (1.3).

THEOREM 2.1. *Let X and Y be real vector spaces. The function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if there exist a symmetric biadditive function $B : X^2 \to Y$, an additive function $A : X \to Y$ and an element $b \in Y$ such that $f(x) = B(x, x) + A(x) + b$ for all $x \in X$.*

PROOF. We first assume that f is a solution of the functional equation (1.3). If we put $g(x) = f(x) - f(0)$, then we get that g is also a solution of (1.3) and $g(0) = 0$. So we may assume, without loss of generality, that f is a solution of (1.3) and $f(0) = 0$. Let $f_e(x) = (f(x) + f(-x))/2$, $f_o(x) = (f(x) - f(-x))/2$ for all $x \in X$. Then $f_e(0) = 0 = f_o(0)$, f_e is even and f_o is odd. Since f is a solution of (1.3), f_e and f_o also satisfy (1.3). Replacing z by $-x$ and f by f_e in (1.3), we have

$$f_e(y) + f_e(x-y) + f_e(2x) = f_e(2x-y) + f_e(x+y).$$

Putting $z = x$ and f by f_e in (1.3), we obtain

$$f_e(y) + f_e(x+y) + f_e(2x) = f_e(2x+y) + f_e(x-y).$$

Summing the above two relations, we get

$$f_e(2x+y) + f_e(2x-y) = 2f_e(2x) + 2f_e(y),$$

which shows that $f_e(x) = B(x, x)$ for some symmetric biadditive function $B : X^2 \to Y$.

Replacing z by $-x$ and f by f_o in (1.3), we have

$$f_o(y) + f_o(x-y) + f_o(2x) = f_o(2x-y) + f_o(x+y).$$

Putting $z = x$ and f by f_o in (1.3), we obtain

$$-f_o(y) + f_o(x+y) + f_o(2x) = f_o(2x+y) + f_o(x-y).$$

Summing the above two relations, we get

$$f_o(2x+y) + f_o(2x-y) = 2f_o(2x),$$

which implies that f_o is a Jensen function and thus $f_o(x) = A(x)$ for some additive function $A : X \to Y$. That is, $f(x) = f_e(x) + f_o(x) = B(x,x) + A(x)$ for all $x \in X$.

Conversely, if there exist a symmetric biadditive function $B : X^2 \to Y$, an additive function $A : X \to Y$ and an element $b \in Y$ such that $f(x) = B(x,x) + A(x) + b$ for all $x \in X$, we may easily check that f satisfies the equation (1.3). □

From now on, let X be a real vector space and Y a Banach space unless stated otherwise. Let $\phi : X^3 \to \mathbb{R}^+$, $\delta : X \to \mathbb{R}^+$ be given functions and let the induced function $\Phi : X^2 \to \mathbb{R}^+$ be defined by $\Phi(x,y) := \phi(x/2, y, x/2) + \phi(x/2, y, -x/2) + \delta(y)$. In the following theorem, the Hyers–Ulam stability of (1.3) is proved under approximately even condition.

THEOREM 2.2. *Let $\phi : X^3 \to \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y, 2^i z)}{4^i} \qquad \left(\sum_{i=1}^{\infty} 4^i \phi\Big(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\Big), \textit{respectively} \right)$$

converges for all $x, y, z \in X$; let $\delta : X \to \mathbb{R}^+$ be a function satisfying:

$$\sum_{i=0}^{\infty} \frac{\delta(2^i x)}{4^i} \qquad \left(\sum_{i=1}^{\infty} 4^i \delta\Big(\frac{x}{2^i}\Big) \right)$$

converges for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies

$$\|f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) - f(x+y) - f(x+z)\| \leqslant \phi(x,y,z),$$

$$\|f(x) - f(-x)\| \leqslant \delta(x) \tag{2.1}$$

for all $x, y, z \in X - \{0\}$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying the equation (1.3) *and the inequality*

$$\|f(x) - f(0) - Q(x)\| \leqslant \frac{1}{4} \sum_{i=0}^{\infty} \frac{\Phi(2^i x, 2^i x)}{4^i} \tag{2.2}$$

$$\left(\|f(x) - f(0) - Q(x)\| \leqslant \frac{1}{4} \sum_{i=1}^{\infty} 4^i \Phi\Big(\frac{x}{2^i}, \frac{x}{2^i}\Big) \right)$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \qquad \Big(Q(x) = \lim_{n\to\infty} 4^n [f(x/2^n) - f(0)] \Big). \tag{2.3}$$

PROOF. Replacing x and z by $x/2$ in the first condition of (2.1), we get

$$\|f(x+y) + f(x/2-y) + f(0) - f(-y) - f(x/2+y) - f(x)\| \leqslant \phi(x/2, y, x/2) \tag{2.4}$$

for all $x, y \in X - \{0\}$. If we put $x/2$, $-x/2$ in (2.1) instead of x, z, respectively, we obtain

$$\|f(y) + f(x/2 - y) + f(x) - f(x - y) - f(x/2 + y) - f(0)\| \leqslant \phi(x/2, y, -x/2) \tag{2.5}$$

for all $x, y \in X - \{0\}$. By (2.4) and (2.5), we get the relation

$$\|f(x+y)+f(x-y)+2f(0)-f(y)-f(-y)-2f(x)\| \leqslant \phi(x/2, y, x/2)+\phi(x/2, y, -x/2) \tag{2.6}$$

for all $x, y \in X - \{0\}$. It then follows from the second condition of (2.1) and (2.6) that the inequality

$$\begin{aligned} &\|f(x + y) + f(x - y) + 2f(0) - 2f(x) - 2f(y)\| \\ &\leqslant \|f(x + y) + f(x - y) + 2f(0) - f(y) - f(-y) - 2f(x)\| + \|f(y) - f(-y)\| \\ &\leqslant \phi(x/2, y, x/2) + \phi(x/2, y, -x/2) + \delta(y) = \Phi(x, y) \end{aligned} \tag{2.7}$$

holds for all $x, y \in X - \{0\}$. We now define a function $F : X \to Y$ by $F(x) = f(x) - f(0)$ for all x in X. Then from (2.7) we arrive at the following inequality

$$\|F(x + y) + F(x - y) - 2F(x) - 2F(y)\| \leqslant \Phi(x, y)$$

for all $x, y \in X$. According to [**6**, Corollary 2], there exists a unique quadratic function $Q : X \to Y$ satisfying (2.2) and (2.3). To show that Q satisfies the equation (1.3), we replace x, y, and z by $2^n x$, $2^n y$ and $2^n z$, respectively, in (2.1) and divide by 4^n; then we get

$$\begin{aligned} &4^{-n}\|f(2^n(x + y + z)) + f(2^n(x - y)) + f(2^n(x - z)) \\ &\quad - f(2^n(x - y - z)) - f(2^n(x + y)) - f(2^n(x + z))\| \leqslant 4^{-n}\phi(2^n x, 2^n y, 2^n z). \end{aligned}$$

Taking the limit as $n \to \infty$, we find that Q satisfies (1.3) for all $x, y, z \in X$. This completes the proof of the theorem. □

From the main Theorem 2.2, we obtain the following corollary concerning the stability of the equation (1.3).

Corollary 2.1. *Let X and Y be a real normed space and a Banach space, respectively, and let p, q ($\neq 2$) be real numbers. Let $H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $H(tu, tv, tw) \leqslant t^p H(u, v, w)$ for all t ($\neq 0$), $u, v, w \in \mathbb{R}^+$. And let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying $E(tx) \leqslant t^q E(x)$ for all t ($\neq 0$), $x \in \mathbb{R}^+$. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} \|f(x + y + z) + f(x - y) + f(x - z) - f(x - y - z) - f(x + y) - f(x + z)\| \\ \leqslant H(\|x\|, \|y\|, \|z\|), \end{aligned}$$

$$\|f(x) - f(-x)\| \leqslant E(\|x\|)$$

for all $x, y, z \in X - \{0\}$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfying the equation* (1.3) *and the inequality*

$$\|f(x) - f(0) - Q(x)\| \leqslant \frac{2H(\|x\|/2, \|x\|, \|x\|/2)}{4 - 2^p} + \frac{E(\|x\|)}{4 - 2^q} \quad \text{if } p, q < 2$$

$$\left(\|f(x) - f(0) - Q(x)\| \leqslant \frac{2H(\|x\|/2, \|x\|, \|x\|/2)}{2^p - 4} + \frac{E(\|x\|)}{2^q - 4} \quad \text{if } p, q > 2\right)$$

for all $x \in X$.

As a consequence of the above results, we have the following.

Corollary 2.2. *Let* X *and* Y *be a real normed space and a Banach space, respectively, and let* $\varepsilon, \delta \geqslant 0$, p, q $(\neq 2)$ *be real numbers. Suppose that a function* $f : X \to Y$ *satisfies*

$$\|f(x + y + z) + f(x - y) + f(x - z) - f(x - y - z) - f(x + y) - f(x + z)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$

$$\|f(x) - f(-x)\| \leqslant \delta\|x\|^q$$

for all $x, y, z \in X - \{0\}$. *Then there exists a unique quadratic function* $Q : X \to Y$ *which satisfies the equation* (1.3) *and the inequality*

$$\|f(x) - f(0) - Q(x)\| \leqslant \frac{2\varepsilon(2 + 2^p)}{(4 - 2^p)2^p}\|x\|^p + \frac{\delta\|x\|^q}{4 - 2^q} \quad \text{if } p, q < 2$$

$$\left(\|f(x) - f(0) - Q(x)\| \leqslant \frac{2\varepsilon(2 + 2^p)}{(2^p - 4)2^p}\|x\|^p + \frac{\delta\|x\|^q}{2^q - 4} \quad \text{if } p, q > 2\right)$$

for all $x \in X$. *The function* Q *is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \quad \text{if } p, q < 2$$

$$\left(Q(x) = \lim_{n\to\infty} 4^n[f(x/2^n) - f(0)] \quad \text{if } p, q > 2\right).$$

Corollary 2.3. *Let* X *and* Y *be a real normed space and a Banach space, respectively, and let* $\varepsilon, \delta \geqslant 0$ *be real numbers. Suppose that a function* $f : X \to Y$ *satisfies*

$$\|f(x + y + z) + f(x - y) + f(x - z) - f(x - y - z) - f(x + y) - f(x + z)\| \leqslant \varepsilon,$$

$$\|f(x) - f(-x)\| \leqslant \delta$$

for all $x, y, z \in X - \{0\}$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfying the equation* (1.3) *and the inequality*

$$\|f(x) - f(0) - Q(x)\| \leqslant \frac{2\varepsilon + \delta}{2}$$

for all $x \in X$.

In the following theorem, the Hyers–Ulam stability of (1.3) is proved under approximately odd condition.

THEOREM 2.3. *Let $\phi : X^3 \to \mathbb{R}^+$ be a function such that:*

$$\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y, , 2^i z)}{2^i} \qquad \left(\sum_{i=1}^{\infty} 2^i \phi\Big(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\Big), \text{respectively} \right)$$

converges for all $x, y, z \in X$, and let $\delta : X \to \mathbb{R}^+$ be a function satisfying:

$$\sum_{i=0}^{\infty} \frac{\delta(2^i x)}{2^i} \qquad \left(\sum_{i=1}^{\infty} 2^i \delta\Big(\frac{x}{2^i}\Big) \right)$$

converges for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies

$$\|f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) - f(x+y) - f(x+z)\| \leqslant \phi(x,y,z),$$

$$\|f(x) + f(-x) - 2f(0)\| \leqslant \delta(x) \tag{2.8}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - f(0) - A(x)\| \leqslant \frac{1}{2} \sum_{i=0}^{\infty} \frac{\Phi(2^i x, 2^i x)}{2^i} \tag{2.9}$$

$$\left(\|f(x) - f(0) - A(x)\| \leqslant \frac{1}{2} \sum_{i=1}^{\infty} 2^i \Phi\Big(\frac{x}{2^i}, \frac{x}{2^i}\Big) \right)$$

for all $x \in X$. The function A is given by

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \qquad \left(A(x) = \lim_{n\to\infty} 2^n [f(x/2^n) - f(0)] \right) \tag{2.10}$$

PROOF. We now define a function $F : X \to Y$ by $F(x) = f(x) - f(0)$ for all x in X. Replacing x and z by $x/2$ in the first condition of (2.3), we get

$$\|F(x+y) + F(x/2-y) - F(-y) - F(x/2+y) - F(x)\| \leqslant \phi(x/2, y, x/2) \tag{2.11}$$

for all $x, y \in X$. If we put $x/2$, $-x/2$ in (2.3) instead of x, z, respectively, we obtain

$$\|F(y) + F(x/2-y) + F(x) - F(x-y) - F(x/2+y)\| \leqslant \phi(x/2, y, -x/2) \tag{2.12}$$

for all $x, y \in X$. By (2.11) and (2.12), we get the relation

$$\|F(x+y) + F(x-y) - F(y) - F(-y) - 2F(x)\| \leqslant \phi(x/2, y, x/2) + \phi(x/2, y, -x/2) \tag{2.13}$$

for all $x, y \in X$. It then follows from the second condition of (2.3) and (2.13) that the inequality

$$\begin{aligned} &\|F(x+y) + F(x-y) - 2F(x)\| \\ &\leqslant \|F(x+y) + F(x-y) - F(y) - F(-y) - 2F(x)\| + \|F(y) + F(-y)\| \\ &\leqslant \phi(x/2, y, x/2) + \phi(x/2, y, -x/2) + \delta(y) = \Phi(x, y) \end{aligned} \tag{2.14}$$

holds for all $x, y \in X$. The relation (2.14) for $y = x$ yields $\|F(2x) - 2F(x)\| \leqslant \Phi(x, x)$, which implies

$$\|2^{-1}F(2x) - F(x)\| \leqslant 2^{-1}\Phi(x, x).$$

Applying an induction argument to n, we obtain that

$$\|2^{-n}F(2^n x) - F(x)\| \leqslant \frac{1}{2}\sum_{i=0}^{n-1}\frac{\Phi(2^i x, 2^i x)}{2^i} \leqslant \frac{1}{2}\sum_{i=0}^{\infty}\frac{\Phi(2^i x, 2^i x)}{2^i} \tag{2.15}$$

$$\Big(\|2^n F(\frac{x}{2^n}) - F(x)\| \leqslant \frac{1}{2}\sum_{i=1}^{n} 2^i\Phi\Big(\frac{x}{2^i}, \frac{x}{2^i}\Big) \leqslant \frac{1}{2}\sum_{i=1}^{\infty} 2^i\Phi\Big(\frac{x}{2^i}, \frac{x}{2^i}\Big)\Big)$$

for any positive integer n. We have the corresponding inequality in (2.15) under the condition expressed by parentheses in the theorem. Thus by the same way as that of Theorem [**5**] there exists a unique additive function $A : X \to Y$, defined by

$$A(x) = \lim_{n\to\infty}\frac{F(2^n x)}{2^n} \qquad \Big(A(x) = \lim_{n\to\infty} 2^n F(x/2^n)\Big)$$

for all $x \in X$, satisfying (2.9) and (2.10). □

From the main Theorem 2.3, we obtain the following corollary concerning the stability of the equation (1.3).

Corollary 2.4. *Let X and Y be a real normed space and a Banach space, respectively, and let p, q $(\neq 1)$ be real numbers. Let $H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $H(tu, tv, tw) \leqslant t^p H(u, v, w)$ for all $t(\neq 0), u, v, w \in \mathbb{R}^+$. And let $O : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying $O(tx) \leqslant t^q O(x)$ for all $t(\neq 0), x \in \mathbb{R}^+$. Suppose that a function $f : X \to Y$ satisfies*

$$\|f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) - f(x+y) - f(x+z)\| \leqslant H(\|x\|, \|y\|, \|z\|),$$

$$\|f(x) + f(-x) - 2f(0)\| \leqslant O(\|x\|)$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying the equation (1.3) *and the inequality*

$$\|f(x) - f(0) - A(x)\| \leqslant \frac{2H(\|x\|/2, \|x\|, \|x\|/2)}{2 - 2^p} + \frac{O(\|x\|)}{2 - 2^q} \quad \text{if } p, q < 1$$

$$\Big(\|f(x) - f(0) - A(x)\| \leqslant \frac{2H(\|x\|/2, \|x\|, \|x\|/2)}{2^p - 2} + \frac{O(\|x\|)}{2^q - 2} \quad \text{if } p, q > 1\Big)$$

for all $x \in X$.

As a consequence of the above results, we have the following.

Corollary 2.5. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon, \delta \geqslant 0$, p, q $(\neq 1)$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\|f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) - f(x+y) - f(x+z)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$

$$\|f(x) + f(-x) - 2f(0)\| \leqslant \delta\|x\|^q$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying the equation (1.3) *and the inequality*

$$\|f(x) - f(0) - A(x)\| \leqslant \frac{2\varepsilon(2+2^p)}{(2-2^p)2^p}\|x\|^p + \frac{\delta\|x\|^q}{2-2^q} \quad \text{if } p, q < 1$$
$$\left(\|f(x) - f(0) - A(x)\| \leqslant \frac{2\varepsilon(2+2^p)}{(2^p-2)2^p}\|x\|^p + \frac{\delta\|x\|^q}{2^q-2} \quad \text{if } p, q > 1\right)$$

for all $x \in X$. The function A is given by

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{if } p, q < 1$$
$$\left(A(x) = \lim_{n\to\infty} 2^n[f(x/2^n) - f(0)] \quad \text{if } p, q > 1\right).$$

COROLLARY 2.6. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon, \delta \geqslant 0$ be real numbers. Suppose that the function $f : X \to Y$ satisfies*

$$\|f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) - f(x+y) - f(x+z)\| \leqslant \varepsilon,$$
$$\|f(x) + f(-x) - 2f(0)\| \leqslant \delta$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying the equation (1.3) *and the inequality*

$$\|f(x) - f(0) - A(x)\| \leqslant 2\varepsilon + \delta$$

for all $x \in X$.

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