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SUBPARACOMPACT INVERSE IMAGES OF 2-SUBPARACOMPACT SPACES

Ying Ge

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Abstract. We prove that subparacompact mappings inversely preserve 2-subparacompactness. As some applications of the above result, we obtain that both perfect mappings and closed Lindelof regular mappings inversely preserve 2-subparacompactness, which answer a question on 2-subparacompactness posed by Qu and Yasui affirmatively. Also we give a counterexample to show that closed Lindelof mappings do not inversely preserve 2-subparacompactness.

1. Introduction

In [**6**], Qu and Yasui discussed relative subparacompactness, and gave some beautiful characterizations of 1-subparacompactness [**6**]. By this characterizations, they obtained that 1-subparacompactness is inversely preserved under perfect mappings [**6**]. Unfortunately, we do not know whether analogous characterizations of 2-subparacompactness are true, so authors of [**6**] raised the following question.

Question 1.1. [**6**] Is 2-subparacompactness inversely preserved under perfect mappings?

Notice that subparacompactness is inversely preserved under both perfect mappings (need not with regular domain) and closed Lindelof mappings with regular domain [**3**]. We are even more interested in the following question.

Question 1.2. (1) Is 2-subparacompactness inversely preserved under closed Lindelof mappings with regular domain?

(2) Furthermore, can regularity in the above (1) be omitted?

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We use subparacompact mappings, which were introduced by Buhagiar and Miwa in [**2**], to investigate the above Questions 1.2. We prove that subparacompact mappings inversely preserve 2-subparacompactness. As some applications of this result, we obtain that both perfect mapping and closed Lindelof regular mappings inversely preserve 2-subparacompactness. Especially, closed Lindelof mappings with regular domain inversely preserve 2-subparacompactness. Also we give a counterexample to show that closed Lindelof mappings do not inversely preserve 2-subparacompactness.

Throughout this paper, all spaces are T_1 and all maps are continuous and onto. ω denotes the first infinite ordinal. Let $x \in X$, A be a subset of a space X and $\mathcal{U}$ be a collection of subsets of X. $\bigcup\mathcal{U}=\bigcup\{U : U \in \mathcal{U}\}$, $\mathcal{U} \wedge A = \{U \cap A : U \in \mathcal{U}\}$ and $\mathrm{ord}(x,\mathcal{U})$ denotes the cardinal of the family $\{U \in \mathcal{U} : x \in U\}$. Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of a space X. We say that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$, if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$; moreover, we say that $\mathcal{V}$ is a refinement of $\mathcal{U}$, if in addition $\bigcup\mathcal{V} = \bigcup\mathcal{U}$ is also satisfied. Let X_0 be a subspace of a space X, and $\mathcal{U}$ be a family of subsets of X. We say that $\mathcal{U}$ is discrete at X_0 in X, if for every $x \in X_0$ there is an open in X neighborhood of x that intersects at most one member of $\mathcal{U}$. Having the above definition, we define σ-discreteness at X_0 in X in a natural way. If $f : X \longrightarrow Y$ is a mapping, then $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$. The sequence $\{\mathcal{P}_n : n < \omega\}$ of collections of subsets of a space are abbreviated to $\{\mathcal{P}_n\}$. One may refer to [**3**] and [**5**] for undefined notations and terminology.

Definition 1.3. A space X is called subparacompact if every open cover of X has a σ-discrete closed refinement.

Definition 1.4. [**6**] A subspace X_0 of a space X is called 2-subparacompact in X, if for any open cover $\mathcal{U}$ of X, there exists a partial refinement $\mathcal{F}$ of $\mathcal{U}$ such that $\mathcal{F}$ is σ-discrete at X_0 in X closed in X and $\bigcup\mathcal{F} \supset X_0$.

Remark 1.5. [**6**] In the above Definition 1.4, 2-subparacompactness coincide with the subparacompactness if $X_0 = X$.

Definition 1.6. [**2**] A mapping $f : X \longrightarrow Y$ is called T_2 (T_1), if for every $y \in Y$ and all $x, x' \in f^{-1}(y)$, $x \neq x'$, the points x and x' have disjoint neighborhoods in X (every of the points x, x' has a neighborhood in X not containing the other point); is called regular, if for every $x \in X$ and every closed set F in X such that $x \notin F$, there exists a neighborhood G of $f(x)$ such that x and $F \cap f^{-1}(G)$ have disjoint neighborhoods in $f^{-1}(G)$.

Remark 1.7. (1) A mapping is T_1, T_2 and regular respectively if the domain is T_1, T_2 and regular respectively.

(2) Since all spaces are assumed to be T_1, all mappings are T_1 from [**2**].

Definition 1.8. A closed mapping $f : X \longrightarrow Y$ is called perfect (closed Lindelof), if for every $y \in Y$, $f^{-1}(y)$ is a compact subset (Lindelof subset) of X.

Definition 1.9. [**2**] A mapping $f : X \longrightarrow Y$ is called paracompact, if for every $y \in Y$ and every open (in X) cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of $f^{-1}(y)$ (i.e., $f^{-1}(y) \subset \bigcup\{U_\alpha : \alpha \in A\}$), there exists a neighborhood G_y of y such that $f^{-1}(G_y)$

is covered by $\mathcal{U}$ and $\mathcal{U} \wedge f^{-1}(G_y)$ has a y-locally finite open refinement $\mathcal{F}$, that is the open refinement $\mathcal{F}$ of $\mathcal{U} \wedge f^{-1}(G_y)$ has the following property: for every $x \in f^{-1}(y)$, there exists a neighborhood O_x of x such that O_x meets finitely many elements of $\mathcal{F}$.

DEFINITION 1.10. [**2**] A mapping $f : X \to Y$ is called subparacompact, if for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}(y)$, there exists a neighborhood G_y of y such that $f^{-1}(G_y)$ is covered by $\mathcal{U}$ and $\mathcal{U} \wedge f^{-1}(G_y)$ has a σ-discrete closed refinement $\mathcal{F}$ in $f^{-1}(G_y)$, that is the refinement $\mathcal{F}$ of $\mathcal{U} \wedge f^{-1}(G_y)$ is of the form $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$, where $\mathcal{F}_n$ is closed and discrete in $f^{-1}(G_y)$ for every $n < \omega$.

REMARK 1.11. [**2**] Perfect mapping $\Longrightarrow$ paracompact mapping.

2. The main results

LEMMA 2.1. [**4**] *A mapping $f : X \to Y$ is closed if and only if for every $y \in Y$ and every open subset U in X which contains $f^{-1}(y)$, there exists an open neighborhood V of y such that $f^{-1}(V) \subset U$.*

LEMMA 2.2. [**2**] *For a mapping $f : X \to Y$ the following are equivalent.*

(1) *f is paracompact T_2;*

(2) *f is regular and for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}(y)$, there exists a neighborhood G_y of y such that $f^{-1}(G_y)$ is covered by $\mathcal{U}$ and $\mathcal{U} \wedge f^{-1}(G_y)$ has a $y - \sigma$-discrete open refinement $\mathcal{V}$. This is equivalent to saying that the refinement $\mathcal{V}$ of $\mathcal{U} \wedge f^{-1}(G_y)$ is of the form $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$, where for every $n < \omega$ there exists a neighborhood $G_n(y)$ of y such that $G_n(y) \subset G_y$ and $\mathcal{V}_n$ is discrete in $f^{-1}(G_n(y))$;*

(3) *f is T_2 and for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}(y)$, there exists a neighborhood G_y of y such that $f^{-1}(G_y)$ is covered by $\mathcal{U}$ and $\mathcal{U} \wedge f^{-1}(G_y)$ has a closed locally finite refinement in $f^{-1}(G_y)$.*

LEMMA 2.3. [**2**] *A mapping $f : X \to Y$ is subparacompact if and only if for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}(y)$, there exists a neighborhood G_y of y such that $f^{-1}(G_y)$ is covered by $\mathcal{U}$ and $\mathcal{U} \wedge f^{-1}(G_y)$ has a σ-locally finite closed refinement $\mathcal{F}$ in $f^{-1}(G_y)$, that is the refinement $\mathcal{F}$ of $\mathcal{U} \wedge f^{-1}(G_y)$ is of the form $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$, where $\mathcal{F}_n$ is closed and locally finite in $f^{-1}(G_y)$ for every $n < \omega$.*

REMARK 2.4. Paracompact T_2 mapping $\Longrightarrow$ subparacompact mapping from Lemma 2.2 and Lemma 2.3.

PROPOSITION 2.5. *Let $f : X \to Y$ be a closed Lindelof regular mapping. Then f is paracompact, and so is subparacompact.*

PROOF. Let $y \in Y$ and $\mathcal{U}$ be an open (in X) cover of $f^{-1}(y)$. Then there exists a countable $\{U_n : n < \omega\} \subset \mathcal{U}$ such that $f^{-1}(y) \subset \bigcup\{U_n : n < \omega\}$. Since f is closed, there exists an open neighborhood G_y of y such that $f^{-1}(G_y) \subset \bigcup\{U_n : n < \omega\}$ from Lemma 2.1. Put $\mathcal{V}_n = \{U_n \cap f^{-1}(G_y)\}$ for every $n < \omega$ and $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$. Then $\mathcal{V}$ is a $y - \sigma$-discrete open refinement of $\mathcal{U} \wedge f^{-1}(G_y)$. So f is paracompact from Lemma 2.2. $\square$

LEMMA 2.6. *Let $f : X \to Y$ be a mapping, Y_0 be a subspace of Y and $X_0 = f^{-1}(Y_0)$. If $\mathcal{F}$ is discrete at Y_0 in Y closed in Y, then $f^{-1}(\mathcal{F})$ is discrete at X_0 in X closed in X.*

PROOF. The proof of this lemma is routine, so we omit it. $\square$

THEOREM 2.7. *Let $f : X \to Y$ be a subparacompact mapping. If Y_0 is 2-subparacompact in Y, then $X_0 = f^{-1}(Y_0)$ is 2-subparacompact in X.*

PROOF. Let $\mathcal{U}$ be an open cover of X. Since f is subparacompact, there exists a neighborhood G_y of y for every $y \in Y_0$ such that $f^{-1}(G_y)$ is covered by $\mathcal{U}$ and $\mathcal{U} \wedge f^{-1}(G_y)$ has a σ-discrete closed refinement $\mathcal{F}_y = \bigcup_{m<\omega} \mathcal{F}_{m,y}$ in $f^{-1}(G_y)$. Let $\mathcal{G} = \{G_y : y \in Y_0\}$. Since Y_0 is 2-subparacompact in Y, $\mathcal{G}$ has a partial refinement $\mathcal{W} = \bigcup\{\mathcal{W}_n : n < \omega\}$ such that every $\mathcal{W}_n$ is discrete at Y_0 in Y closed in Y and $\bigcup \mathcal{W} \supset Y_0$. For every $n < \omega$, we can assume $\mathcal{W}_n = \{W_{n,y} : y \in Y_0\}$ and $W_{n,y} \subset G(y)$. Put $\mathcal{P}_{n,m} = \bigcup\{\mathcal{F}_{m,y} \wedge f^{-1}(W_{n,y}) : y \in Y_0\}$ for every $n < \omega$ and $m < \omega$, and $\mathcal{P} = \bigcup_{n,m<\omega} \mathcal{P}_{n,m}$. It is easy to see that $\mathcal{P}$ is a partial refinement of $\mathcal{U}$ and $\bigcup \mathcal{P} \supset X_0$. To complete the proof, it suffices to show the following two claims for every $n < \omega$ and $m < \omega$.

Claim A: $\mathcal{P}_{n,m}$ is discrete at X_0 in X.

Let $x \in X_0$. $f^{-1}(\mathcal{W}_n) = \{f^{-1}(W_{n,y}) : y \in Y_0\}$ is discrete at X_0 in X closed in X from Lemma 2.6, there exists an open neighborhood U_x of x in X and $y' \in Y_0$ such that $U_x \cap f^{-1}(W_{n,y}) = \emptyset$ for all $y \in Y_0 \setminus \{y'\}$. If $x \notin f^{-1}(W_{n,y'})$, notice that $f^{-1}(W_{n,y'})$ is closed in X, then there exists an open neighborhood V_x of x in X such that $V_x \cap f^{-1}(W_{n,y'}) = \emptyset$. Put $W_x = U_x \cap V_x$. Then W_x is a neighborhood of x in X and $W_x \cap P = \emptyset$ for all $P \in \mathcal{P}_{n,m}$. If $x \in f^{-1}(W_{n,y'})$, notice that $\mathcal{F}_{m,y'}$ is discrete in $f^{-1}(G_{y'})$ and $f^{-1}(W_{n,y'}) \subset f^{-1}(G_{y'})$, then there exists an open neighborhood V_x of x in $f^{-1}(G_{y'})$ (and so V_x open in X) such that V_x intersects at most one member of $\mathcal{F}_{m,y'}$. Put $W_x = U_x \cap V_x$. Then W_x is an open neighborhood of x in X and W_x intersects at most one member of $\mathcal{P}_{n,m}$. This shows that $\mathcal{P}_{n,m}$ is discrete at X_0 in X.

Claim B: $\mathcal{P}_{n,m}$ is closed in X.

Let $f^{-1}(W_{n,y}) \cap F \in \mathcal{P}_{n,m}$, where $F \in \mathcal{F}_{m,y}$ and $y \in Y_0$. For whenever $x \notin f^{-1}(W_{n,y}) \cap F$, if $x \notin f^{-1}(W_{n,y})$, put $U_x = X \setminus f^{-1}(W_{n,y})$; If $x \in f^{-1}(W_{n,y})$ (so $x \notin F$), put $U_x = f^{-1}(G_y) \setminus F$. Then U_x is a neighborhood of x in X and $U_x \cap (f^{-1}(W_{n,y}) \cap F) = \emptyset$. This shows that $f^{-1}(W_{n,y}) \cap F$ is closed in X consequently, $\mathcal{P}_{n,m}$ is closed in X. $\square$

We have the following corollaries from Remark 1.7(1), Remark 1.11 and Proposition 2.5.

COROLLARY 2.8. *Both perfect mappings and closed Lindelof regular mappings inversely preserve 2-subparacompactness.*

COROLLARY 2.9. *Closed Lindelof mappings with regular domain inversely preserve 2-subparacompactness.*

3. The counterexample

Now we give a counterexample to show that both regularity of closed Lindelof mapping in Corollary 2.8 and regularity of the domain in Corollary 2.9 can not be omitted, even can not be replaced by T_2. By Remark 1.5, It suffices to give a counterexample to show that closed Lindelof T_2 mappings do not inversely preserve subparacompactness, even if the domain is T_2. Recall a space X is said to be strongly paracompact [**3**] if every open cover has star-finite open refinement; is said to be (countable) θ-refinable [**3**], if for every (countable) open cover of X, there exists a sequence $\{\mathcal{U}_n\}$ of open refinements such that for every $x \in X$, there exists some $n \in N$ with $\text{ord}(x,\mathcal{U}_n) < \infty$. It is well known that strong paracompactness $\Longrightarrow$ subparacompactness $\Longrightarrow$ θ-refinability for T_2-spaces.

Example 3.1. There exists a closed Lindelof T_2 mapping $f : X \to Y$, such that X is T_2, but not θ-refinable, and Y is T_2 strongly paracompact.

Let X, Q and I be the set of all real numbers, the set of all rational numbers and the set of all irrational numbers respectively. Define a base $\mathcal{B}$ of X by $\mathcal{B} = \{\{x\} : x \in I\} \cup \{G(x,n) : x \in Q, n \in N\}$; here $G(x,n) = \{y \in I : -1/n < y - x < 1/n\} \cup \{x\}$. So, X is a Bennett and Lutzer's space [**1**]. Define an equivalence relation R on X as follows: $x\,R\,y$ if and only if $x, y \in Q$ or $x = y$. Let Y be the quotient space X/R and let f be a natural mapping from X onto Y. Then

(1) X is T_2, it is neigher regular nor θ-refinable [**1**].

(2) Y is T_2 strongly paracompact: The fact that Y is T_2 is clear. Let $\mathcal{U}$ be any open cover. Pick $x_0 \in Q$. Put $y_0 = f(x_0)$. Pick $U \in \mathcal{U}$ such that $y_0 \in U$. Then $\{U\} \cup \{\{y\} : y \in Y - U\}$ is a discrete (hence star-finite) open refinement of $\mathcal{U}$, so Y is strongly paracompact.

(3) f is a closed Lindelof mapping: It is clear.

(4) f is a T_2 mapping: It is clear from that X is a T_2-space.

Remark 3.2. (1) In fact, X is not countably θ-refinable: Assume X is countably θ-refinable. Let $\mathcal{U}$ be any open cover of X. Then there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ which cover Q. Put $W = \bigcup \mathcal{V}$. Then W is clopen in X and $\mathcal{V}$ is a countable open cover of W. Notice that countable θ-refinability is hereditary to closed subspace, W is countably θ-refinable, so there exists a sequence of open refinements $\{\mathcal{V}_n : n \in N\}$ of $\mathcal{V}$, such that for every $x \in W$ there exists $n \in N$ such that $\text{ord}(x, \mathcal{V}_n) < \infty$. Put $\mathcal{U}_n = \mathcal{V}_n \cup \{\{x\} : x \in X - W\}$ for every $n < \omega$. Then $\{\mathcal{U}_n\}$ is a sequence of open refinements of $\mathcal{U}$. For every $x \in X$, if $x \in W$, then there exists $n < \omega$ such that $\text{ord}(x, \mathcal{V}_n) < \infty$, hence $\text{ord}(x,\mathcal{U}_n) = \text{ord}(x, \mathcal{V}_n) < \infty$; if $x \in X - W$, then $\text{ord}(x,\mathcal{U}_n) = 1 < \infty$ for every $n \in N$. Thus X is θ-refinable. This is a contradiction, as X is not θ-refinable [**1**].

(2) The above Example 2.1 show that all covering property which are between strong paracompactness and countable θ-refinability need not be inversely preserved under closed Lindelof T_2 mappings even if the domain is T_2.

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Department of Mathematics
Suzhou University
Suzhou 215006
P. R. China

geying@pub.sz.jsinfo.net

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