PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 72(86) (2002), 95–106

SEMI-RIEMANNIAN MANIFOLDS WHOSE WEYL TENSOR IS A KULKARNI–NOMIZU SQUARE

Małgorzata Głogowska

Communicated by Mileva Prvanović

Abstract. We investigate curvature properties of semi-Riemannian manifolds (M, g), $n \geqslant 4$, whose Weyl curvature tensor C can be expressed by a Kulkarni-Nomizu square of the tensor $S - \frac{\kappa}{n-1} g$. We investigate also the problem of isometric immersion of such manifolds into space forms.

1. Introduction

We investigate semi-Riemannian manifolds (M, g), $n \geqslant 4$, whose Weyl curvature tensor C is expressed by the square, in the sense of the product of Kulkarni–Nomizu, of the tensor $S - \frac{\kappa}{n-1} g$. More precisely, we investigate curvature properties of semi-Riemannian manifolds (M, g), $n \geqslant 4$, satisfying on $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ the condition

$$C = \frac{L}{2}\Big(S - \frac{\kappa}{n-1} g\Big) \wedge \Big(S - \frac{\kappa}{n-1} g\Big). \tag{1}$$

For the definitions of the symbols used, we refer to Section 2. Clearly, if the scalar curvature κ of a manifold (M, g) satisfying (1) is zero, then (1) reduces to

$$C = \frac{L}{2} S \wedge S. \tag{2}$$

Some essentially conformally symmetric manifolds, as well as manifolds admitting some Akivis-Goldberg metrics, satisfy (2). In Section 2 we present curvature properties of these classes of manifolds. There are also manifolds with nonzero scalar

2000 *Mathematics Subject Classification.* Primary 53B20; Secondary 53B25, 53A40, 53A60, 53C25, 53C40.

Key words and phrases. semisymmetric manifold; essentially conformally symmetric manifold; hypersurface; almost Grassmann structure.

The results of the paper were presented by the author at the workshop *Harmonic Maps and Curvature Properties of Submanifolds, 2* held in Leeds (England) in April 2000. The author is supported by the grant 207/GW/99 of the Agricultural University of Wrocław (Poland).

curvature κ satisfying (1) (see Section 3). In Section 3 among other things we state (Proposition 3.1) that (1) is equivalent on $\mathcal{U}$ to the following four relations:

$$R \cdot R = 0, \tag{3}$$

$$(i) \quad C \cdot C = 0, \qquad (ii) \quad S^2 = \frac{\kappa}{n-1} S, \qquad (iii) \quad \operatorname{rank}\Big(S - \frac{\kappa}{n-1} g\Big) > 1. \tag{4}$$

Using this we prove (see Theorem 3.1) that if $n = 4$, then on $\mathcal{U}$ we have (2) and

$$(i) \quad \kappa = 0, \quad (ii) \quad S^2 = 0, \quad (iii) \quad \operatorname{rank} S = 2. \tag{5}$$

In Section 4 we consider the problem of isometric immersion of manifolds satisfying (1) in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with constant curvature $c = \frac{\widetilde{\kappa}}{n(n+1)}$ and signature $(n+1-s, s)$, $n \geqslant 4$. In Proposition 4.1 we present necessary and sufficient conditions for realization of such immersion into a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. A consequence of Proposition 4.1 is that manifolds satisfying (2), i.e. (1) with $\kappa = 0$, cannot be immersed isometrically in $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, (Theorem 4.1). In particular, this means that essentially conformally symmetric manifolds satisfying (2) cannot be realized as a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, (Corollary 4.1(i)). From Theorem 4.1 it follows also that the metrics g_1 and g_2 defined in Examples 3.14 and 3.16 of [**2**] cannot be realized on a hypersurface M in $\mathbb{E}_s^5$ as a metric induced on M from the metric of the ambient space (Corollary 4.1(ii)). In Proposition 4.2 we present necessary and sufficient conditions for realiazation of the isometric immersion of manifolds satisfying (1) into a semi-Riemannian space $N_s^{n+1}(c)$, $n \geqslant 4$.

The author would like to express her thanks to Professor Dr. Ryszard Deszcz for his help during the preparation of this paper, as well as to the referee for hints and comments.

2. Curvature properties of Akivis-Goldberg metrics

Let (M, g) be a connected n-dimensional, $n \geqslant 4$, semi-Riemannian manifold of class C^∞ and let ∇ be its Levi-Civita connection. We define on M the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ by

$$\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y, \\
\mathcal{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\
\mathcal{C}(X,Y) &= \mathcal{R}(X,Y) - \frac{1}{n-2}\Big(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\Big),
\end{aligned}$$

respectively, where S, $\mathcal{S}$, and κ are the Ricci tensor, the Ricci operator and the scalar curvature of (M, g), respectively, $g(\mathcal{S}X, Y) = S(X,Y)$, $X, Y, Z \in \Xi(M)$ and $\Xi(M)$ is the Lie algebra of vector fields of M. The tensors: S^2, G, the Riemann-Christoffel curvature tensor R and the Weyl conformal tensor C of (M, g) are defined by

$$\begin{aligned} S^2(X,Y) &= S(\mathcal{S}X,Y),\\ G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4),\\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4),\\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4), \end{aligned}$$

respectively. For $(0,2)$-tensors A and B their Kulkarni–Nomizu product $A \wedge B$ is given by

$$\begin{aligned} (A\wedge B)(X_1,X_2;X,Y) = {}& A(X_1,Y)B(X_2,X) + A(X_2,X)B(X_1,Y)\\ &- A(X_1,X)B(X_2,Y) - A(X_2,Y)B(X_1,X). \end{aligned}$$

We note that $G = \frac{1}{2} g \wedge g$.

We define the subsets $\mathcal{U}_R$, $\mathcal{U}_S$ and $\mathcal{U}_C$ of M by

$$\begin{aligned} \mathcal{U}_R &= \Big\{x \in M \mid R - \frac{\kappa}{n(n-1)}G \neq 0 \text{ at } x\Big\},\\ \mathcal{U}_S &= \Big\{x \in M \mid S - \frac{\kappa}{n}g \neq 0 \text{ at } x\Big\},\\ \mathcal{U}_C &= \big\{x \in M \mid C \neq 0 \text{ at } x\big\}, \end{aligned}$$

respectively. We note that $\mathcal{U}_S \subset \mathcal{U}_R$ and $\mathcal{U}_C \subset \mathcal{U}_R$ on M. In this paper we restrict our considerations to the set $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$. Evidently, we will asume that $\mathcal{U}$ is a nonempty set.

Further, for a symmetric $(0,2)$-tensor A and a $(0,k)$-tensor T, $k \geqslant 1$, we define the $(0,k)$-tensor $A \cdot T$ and the $(0,k+2)$-tensors $R \cdot T$ and $Q(A,T)$ by

$$\begin{aligned} (A\cdot T)(X_1,\ldots,X_k) &= -T(\mathcal{A}X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{A}X_k),\\ (R\cdot T)(X_1,\ldots,X_k;X,Y) &= (\mathcal{R}(X,Y)\cdot T)(X_1,\ldots,X_k)\\ &= -T(\mathcal{R}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{R}(X,Y)X_k),\\ Q(A,T)(X_1,\ldots,X_k;X,Y) &= ((X\wedge_A Y)\cdot T)(X_1,\ldots,X_k)\\ &= -T((X\wedge_A Y)X_1,X_2,\ldots,X_k) - \ldots - T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k), \end{aligned}$$

respectively, where $\mathcal{A}$ is the endomorphism of $\Xi(M)$ defined by $g(\mathcal{A}X,Y) = A(X,Y)$. Setting $T = R$, $T = S$, $T = C$, $A = g$ or $A = S$ in the above formulas we obtain the tensors: S^2, $R\cdot R$, $R\cdot S$, $R\cdot C$, $Q(g,R)$, $Q(g,S)$, $Q(g,C)$, $Q(S,R)$, $Q(S,C)$ and $S\cdot C$. The tensor $C\cdot C$ is defined in the same way as the tensor $R\cdot R$.

A semi-Riemannian manifold (M,g), $n \geqslant 3$, is *locally symmetric* if $\nabla R = 0$ on M. There exist many various possibilities to obtain curvature conditions weaker than the last one. A semi-Riemannian manifold (M,g), $n \geqslant 3$, is said to be *semisymmetric* if it satisfies $\mathcal{R}(X,Y)\cdot R = 0$, for all vector fields X,Y. The last relation will be shortly denoted by (3). It is well known that any locally symmetric manifold is semisymmetric. The converse statement is not true. A semi-Riemannian manifold (M,g), $n \geqslant 3$, is said to be *pseudosymmetric* [**6**, Section 3.1] if at every point of M the following condition is satisfied:

(*) the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent.

Thus (M,g) is pseudosymmetric if and only if $R\cdot R = L_R Q(g,R)$ on $\mathcal{U}_R$, where L_R is a function on $\mathcal{U}_R$. It is clear that every semisymmetric manifold is pseudosymmetric. The converse statement is not true (see e.g., [**6**, Section 3.6, Example 3.1]. It is known [**6**, Section 5.3] that every 3-dimensional semi-Riemannian manifold satisfies

$$R\cdot R = Q(S,R). \tag{6}$$

Semi-Riemannian conformally flat manifolds of dimension $\geqslant 4$ fulfilling (6) are pseudosymmetric [**6**, Section 6.3]. We mention also that every hypersurface M of a $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geqslant 3$, fulfils (6) [**6**, Section 5.4]. It is also known [**6**, Section 5.5] that on $\mathcal{U}_C \subset M$ of a hypersurface M of $N^{n+1}_s(c)$, $n \geqslant 4$, we have

(**) the tensors $R\cdot R - Q(S,R)$ and $Q(g,C)$ are linearly dependent.

Precisely, the following condition is fulfilled on $\mathcal{U}_C$ [**6**, Section 5.5]

$$R\cdot R - Q(S,R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}Q(g,C) \tag{7}$$

where $\widetilde{\kappa}$ is the scalar curvature of $N^{n+1}_s(c)$. Pseudosymmetric hypersurfaces M of $N^{n+1}_s(c)$, $n \geqslant 4$, were investigated in [**7**]. We note that (**) is on $\mathcal{U}_C$ equivalent to

$$R\cdot R - Q(S,R) = LQ(g,C), \tag{8}$$

where L is a function on $\mathcal{U}_C$. Manifolds satisfying (*) and (**) were studied in [**9**].

Let M be a manifold of dimension $n = pq$, and let $SC(p,q)$ be a differentiable field of Segre cones $SC_x(p,q) \subset T_x(M)$, $x \in M$. The pair $(M, SC(p,q))$ is called an *almost Grassmann structure* and is denoted by $AG(p-1, p+q-1)$. The manifold M endowed with such a structure is said to be an *almost Grassmann manifold* [**2**, Definition 1.1]. Certain additional conditions lead to so-called *semiintegrable almost Grassmann structures* [**2**, Definition 1.2]. The latter were studied in [**2**] and examples of such structures, mainly 4-dimensional, are presented there. For more details about almost Grassmannian structures see [**1**]. Some 4-dimensional semi-Riemannian metrics are related to these structures (see Examples 3.5–3.16 of [**2**]). These metrics are named *Akivis–Goldberg*, in short *AG-metrics* [**11**]. Curvature properties and, in particular, curvature properties of pseudosymmetry type of *AG*-metrics are presented in [**11**], where it is shown that if a 4-dimensional manifold M admits an *AG*-metric, then on $\mathcal{U} \subset M$ we have: (5), (8) and

$$S\cdot C = 0. \tag{9}$$

We consider the *AG*-metrics g_r, $r = 1,2$, defined in the Examples 3.14 and 3.16 of [**2**], respectively. We assume that these metrics are defined on an open, connected and nonempty set $U \subset \mathbb{R}^4$. The metrics g_r satisfy (5) and (9) and do not satisfy (6) [**11**, Remark 4.8]. In addition, their Weyl tensors are nonzero at every point of U, i.e. for both metrics we have $\mathcal{U}_C = U$. Further, g_1 and g_2 are semisymmetric metrics fulfilling (8) and $\nabla C = C \otimes \psi_r$, where ψ_r are some 1-forms on U. The last relation means that both metrics are conformally recurrent. From Theorem

3.1 it follows that g_r are non-warped product metrics. Moreover, g_r satisfy on U the relation $C = \frac{1}{2} L_r S \wedge S$, where L_r are some nonconstant functions on U.

A semi-Riemannian manifold (M, g), $n \geqslant 4$, is said to be *essentially conformally symmetric manifold* [4], [5], in short e.c.s. manifold, if it is a non-conformally flat $(C \neq 0)$ and non-locally symmetric $(\nabla R \neq 0)$ manifold with parallel Weyl tensor $(\nabla C = 0)$. We refer to [4] and [5] for a review of results on these manifolds. Every e.c.s. manifold is semisymmetric. We mention also that such manifolds satisfy (8) [9]. E.c.s. manifolds and manifolds admitting AG-metrics form disjoint classes of semi-Riemannian manifolds [11, Remark 4.9].

3. Preliminary results

Let (M, g) be a semi-Riemannian manifold covered by a system of charts $\{U; x^h\}$. We denote by g_{ij}, Γ^h_{ij}, R_{hijk}, S_{ij}, $S_i^{\ j} = g^{ri} S_{rj}$, $S^2_{ij} = S_{ir} S_r^{\ j}$, $G_{hijk} = g_{hk} g_{ij} - g_{hj} g_{ik}$ and

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{hj}S_{ik}) + \frac{\kappa}{(n-2)(n-1)} G_{hijk},$$

the local components of the metric tensor g, the Levi-Civita connection ∇ and the tensors R, S, $\mathcal{S}$, S^2, G and C of (M, g), respectively, where $h, i, j, k, l, m, r, s \in \{1, 2, \ldots, n\}$. The local components of the tensors $R \cdot T$ and $Q(A, T)$ are given by

$$\begin{aligned}(R \cdot T)_{hijklm} &= g^{rs}(T_{rijk} R_{shlm} + T_{hrjk} R_{silm} + T_{hirk} R_{sjlm} + T_{hijr} R_{sklm}),\\ Q(A, T)_{hijklm} &= A_{hl} T_{mijk} + A_{il} T_{hmjk} + A_{jl} T_{himk} + A_{kl} T_{hijm}\\ &\quad - A_{hm} T_{lijk} - A_{im} T_{hljk} - A_{jm} T_{hilk} - A_{km} T_{hijl},\end{aligned}$$

respectively, where T_{hijk} and A_{hk} are the local components of a $(0, 4)$-tensor T and a symmetric $(0, 2)$-tensor A, respectively.

LEMMA 3.1. *Let B be a symmetric $(0, 2)$-tensor on a semi-Riemannian manifold (M, g), $n \geqslant 3$, and let $\mathcal{U}_B$ be the set of all points of M at which B is not proportional to g. If at $x \in \mathcal{U}_B$ we have*

$$B \wedge B = 2\alpha g \wedge B + 2\beta G, \qquad \alpha, \beta \in \mathbb{R}, \tag{10}$$

then $\alpha^2 = -\beta$ and $\operatorname{rank}(B - \alpha g) = 1$ *at x.*

PROOF. We set $A = B - \alpha g$. Thus from (10) we obtain

$$\frac{1}{2} A \wedge A = (\alpha^2 + \beta) G, \tag{11}$$

whence $\frac{1}{2} Q(A, A \wedge A) = (\alpha^2 + \beta) Q(A, G)$. This, by $Q(A, A \wedge A) = 0$, reduces to

$$(\alpha^2 + \beta) Q(A, G) = 0. \tag{12}$$

We suppose that $\alpha^2 + \beta \neq 0$. Now (12) turns into $Q(A, G) = 0$. From this, by suitable contractions, we obtain $A = \frac{\operatorname{tr}(A)}{n} g$. Thus $B = \frac{\operatorname{tr}(B)}{n} g$ at x, i.e., $x \in M - \mathcal{U}_B$, a contradiction. Therefore $\alpha^2 + \beta = 0$. Now (11) reduces to $A \wedge A = 0$, whence $\operatorname{rank} A = 1$, which completes the proof. □

LEMMA 3.2. *Let B be a symmetric $(0,2)$-tensor on a semi-Riemannian manifold (M,g), $n \geqslant 3$, and let $\mathcal{U}_B$ be the set of all points of M at which B is not proportional to g. Let T be a generalized curvature tensor satisfying at $x \in \mathcal{U}_B$ the following relation:*

$$T = \frac{\alpha_1}{2} B \wedge B + \beta_1 g \wedge B + \gamma_1 G, \qquad \alpha_1, \beta_1, \gamma_1 \in \mathbb{R}. \tag{13}$$

(i) *If, in addition, we have at x the following decomposition of the tensor T*

$$T = \frac{\alpha_2}{2} B \wedge B + \beta_2 g \wedge B + \gamma_2 G, \qquad \alpha_1 - \alpha_2 \neq 0, \quad \alpha_2, \beta_2, \gamma_2 \in \mathbb{R}, \tag{14}$$

then at x we have: $(\beta_2 - \beta_1)^2 = (\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)$ and $\operatorname{rank}\big(B - \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} g\big) = 1$.

(ii) If the conditions $\operatorname{rank}\big(B - \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} g\big) = 1$ and $(\beta_2 - \beta_1)^2 = (\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)$ are satisfied at x, then (14) holds at x.

PROOF. (i) From (13) and (14) we get

$$B \wedge B = 2 \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} g \wedge B + 2 \frac{\gamma_2 - \gamma_1}{\alpha_1 - \alpha_2} G. \tag{15}$$

Applying Lemma 3.1 to this we obtain easily our assertion.

(ii) From our assumptions we have $\big(B - \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} g\big) \wedge \big(B - \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} g\big) = 0$, which is equivalent to (15). But this, together with (13), leads to (14). Our lemma is thus proved. □

PROPOSITION 3.1. *Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold. If (1) is satisfied on $\mathcal{U} \subset M$, then (3) and (4) hold on $\mathcal{U}$. Conversely, (3) and (4) imply (1) on $\mathcal{U}$.*

PROOF. First of all, we present (1) in the following form

$$C_{hijk} = L\Big(\frac{1}{2}(S \wedge S)_{hijk} - \frac{\kappa}{n-1}(g \wedge S)_{hijk} + \frac{\kappa^2}{(n-1)^2} G_{hijk}\Big), \tag{16}$$

whence we obtain

$$R_{hijk} = \frac{L}{2}(S \wedge S)_{hijk} + \Big(\frac{L\kappa}{n-1} - \frac{1}{n-2}\Big)\Big(-(g \wedge S)_{hijk} + \frac{\kappa}{n-1} G_{hijk}\Big). \tag{17}$$

We assume that (1) holds on $\mathcal{U}$. From our assumptions it follows that (4)(iii) is satisfied at every point of $\mathcal{U}_C$. Further, contracting (16) with g^{ij} we obtain (4)(ii). Next, from (1) we have $Q\big(S - \frac{\kappa}{n-1} g, C\big) = 0$. This, in view of Lemma 3.4 of [**9**], implies (4)(i). Finally, from (17), in view of Theorem 4.2 of [**9**], it follows that (3) holds on $\mathcal{U}$. Now we prove that the converse statement is also true. By making use of Theorem 3.1 of [**10**] and (3) and (4)(i) we obtain on $\mathcal{U}$

$$Q\Big(S - \frac{\kappa}{n-1} g, C - \frac{\mu}{n(n-2)} G\Big) = 0, \tag{18}$$

where the function μ satisfies $A = \mu\big(S - \frac{\kappa}{n-1} g\big)$, A is the $(0,2)$-tensor with the local components $A_{ij} = S^{rs} C_{rijs}$, $S^{rs} = g^{rj} S_j{}^s$ and $S_j{}^q = g^{iq} S_{ij}$. We note that at every point of $\mathcal{U}$ we have $\operatorname{rank}\big(S - \frac{\kappa}{n-1} g\big) > 1$. Now from (18), in view of Lemma 3.4 of [**9**], it follows that

$$\tau\Big(S - \frac{\kappa}{n-1} g\Big) \wedge \Big(S - \frac{\kappa}{n-1} g\Big) = C - \frac{\mu}{n(n-2)} G, \qquad \tau \in \mathbb{R} - \{0\}, \tag{19}$$

at every point of $\mathcal{U}$. From the last relation, by contraction and making use of (4)(ii), we get easily $\mu = 0$. Now (19) reduces to (1), which completes the proof. □

REMARK 3.1. From (1), by making use of (4)(ii), it follows that

$$C(\mathcal{S}X_1, X_2, X_3, X_4) = 0 \tag{20}$$

on $\mathcal{U}$. Thus we see that (9) holds on $\mathcal{U}$. Further, combining Proposition 3.1 with Theorem 4.2 of [**9**], we obtain on $\mathcal{U}$

$$Q(S, R) = \Big(\frac{\kappa}{n-1} - \frac{1}{(n-2)L}\Big) Q(g, C). \tag{21}$$

From (20), by (4)(ii), we get

$$R(\mathcal{S}X_1, X_2, X_3, X_4) = \frac{1}{2(n-2)} (S \wedge S)(X_1, X_2, X_3, X_4).$$

Using this, (3) and the identities $\frac{1}{2} Q(g, S \wedge S) = -Q(S, g \wedge S)$ and $Q(S, G) = -Q(g, g \wedge S)$ we can check that on $\mathcal{U}$ we have $C \cdot R = 0$.

REMARK 3.2. Let (M, g), $n \geqslant 4$, be a semi-Riemannian manifold satisfying (1) on the set $\mathcal{U} \subset M$. Thus the curvature tensor R of (M, g) satisfies (17) on $\mathcal{U}$. This decomposition of R in the tensors $S \wedge S$, $g \wedge S$ and G is unique. This statement is a consequence of Lemma 3.2 and the fact that the Ricci tensor S of (M, g) cannot be decomposed on $\mathcal{U}$ into a sum of a metrical term and a term of rank one.

We present now some results on 4-dimensional semi-Riemannian manifolds.

THEOREM 3.1. *Let (M, g), $n = 4$, be a semi-Riemannian manifold satisfying* (1) *on $\mathcal{U} \subset M$. Then* (2) *and* (5) *hold on $\mathcal{U}$.*

PROOF. From Lemma 2.1(ii) of [**11**] and Remark 3.1 it follows that $\kappa = 0$ on $\mathcal{U}$. Therefore (4)(ii) and (1) reduce on $\mathcal{U}$ to $S^2 = 0$ and (2), respectively. Further, it is known that on every 4-dimensional semi-Riemannian manifold (M, g) we have the following identity of E. M. Patterson

$$\begin{aligned} &g_{ri} C_{jklm} + g_{ji} C_{krlm} + g_{ki} C_{rjlm} + g_{rl} C_{jkmi} + g_{jl} C_{krmi} \\ &\quad + g_{kl} C_{rjmi} + g_{rm} C_{jkil} + g_{jm} C_{kril} + g_{km} C_{rjil} = 0. \end{aligned} \tag{22}$$

Transvecting (22) with S^r_h and using (20) we get $S_{hi} C_{jklm} + S_{hl} C_{jkmi} + S_{hm} C_{jkil} = 0$. This, by (2), turns into

$$S_{hi}(S_{jm} S_{kl} - S_{jl} S_{km}) + S_{hl}(S_{ji} S_{km} - S_{jm} S_{ki}) + S_{hm}(S_{jl} S_{ki} - S_{ji} S_{kl}) = 0,$$

which means that $\operatorname{rank} S = 2$. Finally, (9) is a consequence of Remark 3.1. Our theorem is thus proved. □

THEOREM 3.2. *Let* (M, g), $n = 4$, *be a semi-Riemannian manifold satisfying* (1) *on* $\mathcal{U} \subset M$. *Then for every* $x \in \mathcal{U}$ *there exists no chart* $V \subset \mathcal{U}$ *around* x *such that* g *is expressed as a warped product metric on* V.

PROOF. The assertion is a consequence of our Theorem 3.1 and Theorems 3.1 and 3.3 of [**11**]. □

In [**9**, Theorem 4.2] it was shown that if the curvature tensor R of a manifold (M, g), $n \geqslant 4$, has on $\mathcal{U} \subset M$ the form

$$R = \frac{L}{2} S \wedge S + \mu g \wedge S + \eta G, \tag{23}$$

then

$$R \cdot R = \Big(\frac{\mu}{L}((n-2)\mu - 1) - (n-2)\eta\Big)\, Q(g, R) \tag{24}$$

on $\mathcal{U}$, where L, μ and η are some functions on $\mathcal{U}$. We note that (23) is equivalent on $\mathcal{U}$ to

$$\begin{aligned} C = {} & \frac{L}{2}\left(S + \frac{1}{L}\Big(\mu - \frac{1}{n-2}\Big)g\right) \wedge \left(S + \frac{1}{L}\Big(\mu - \frac{1}{n-2}\Big)g\right) \\ & + \left(\eta - \frac{1}{L}\Big(\mu - \frac{1}{n-2}\Big)^2 + \frac{\kappa}{(n-2)(n-1)}\right) G. \end{aligned}$$

Evidently, from (24) it follows that if (M, g) is a manifold satisfying (23) on $\mathcal{U} \subset M$, then $R \cdot R = 0$ on $\mathcal{U}$ if and only if $\frac{\mu}{L}((n-2)\mu - 1) = (n-2)\eta$ on $\mathcal{U}$. Thus we see that if (M, g) is a manifold satisfying (23) on $\mathcal{U} \subset M$, then $R \cdot R = 0$ on $\mathcal{U}$ if and only if

$$\begin{aligned} C = {} & \frac{L}{2}\left(S + \frac{1}{L}\Big(\mu - \frac{1}{n-2}\Big)g\right) \wedge \left(S + \frac{1}{L}\Big(\mu - \frac{1}{n-2}\Big)g\right) \\ & + \frac{1}{n-2}\left(\frac{1}{L}\Big(\mu - \frac{1}{n-2}\Big) + \frac{\kappa}{n-1}\right) G \end{aligned} \tag{25}$$

on $\mathcal{U}$. In addition, if on $\mathcal{U}$ we have

$$\frac{1}{L}\Big(\frac{1}{n-2} - \mu\Big) = \frac{\kappa}{n-1},$$

then (25) turns into (1). In particular, if $\mu = 0$ on $\mathcal{U}$, then (1) is equivalent on $\mathcal{U}$ to

$$R = \frac{n-1}{2(n-2)\kappa} S \wedge S. \tag{26}$$

An example of a manifold fulfilling (26) is given in [**13**, Example 3.1]. Manifolds satisfying the condition $R = \frac{L}{2} S \wedge S$ were investigated in [**12**].

4. Hypersurfaces

Let M, $n = \dim M \geqslant 3$, be a connected hypersurface isometrically immersed into a semi-Riemannian manifold $(N, \widetilde{g})$. We denote by g the metric tensor induced on M from $\widetilde{g}$. Further, we denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections corresponding to the metric tensors $\widetilde{g}$ and g, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = \widetilde{g}(\xi, \xi) = \pm 1$. We can write the Gauss formula and the Weingarten formula of M in N by: $\widetilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi$ and $\widetilde{\nabla}_X \xi = -\mathcal{A}(X)$, respectively, where X, Y are vector fields tangent to M, H is the second fundamental tensor of M in N, $\mathcal{A}$ is the shape operator of M in N and $H^k(X, Y) = g(\mathcal{A}^k(X), Y)$ and $\operatorname{tr}(H^k) = \operatorname{tr}(\mathcal{A}^k)$, $k = 1, 2$.

We assume that the ambient space is $N_s^{n+1}(c)$, $n \geqslant 4$. Let $x^r = x^r(y^h)$ be the local parametric expression of M in $N_s^{n+1}(c)$, where y^h and x^r are local coordinates of M and $N_s^{n+1}(c)$, respectively, and $h, i, j, k \in \{1, 2, \ldots, n\}$ and $r \in \{1, 2, \ldots, n+1\}$. The Gauss equation of M in $N_s^{n+1}(c)$ is given by

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\widetilde{\kappa}}{n(n+1)} G_{hijk}, \tag{27}$$

where $\widetilde{\kappa}$ is the scalar curvature of $N_s^{n+1}(c)$, R_{hijk}, S_{hk} and H_{hk} are the local components of the curvature tensor R, the Ricci tensor S and the second fundamental tensor H of M, respectively. Contracting (27) with g^{ij} we obtain

$$S_{hk} = \varepsilon(\operatorname{tr}(H)H_{hk} - H^2_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)} g_{hk}, \qquad H^2_{hk} = g^{ij}H_{ih}H_{jk}. \tag{28}$$

Let M be a hypersurface of $N_s^{n+1}(c)$, $n \geqslant 4$, and let $\mathcal{U}_1$ be a connected component of the set $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$. From now on we restrict our considerations to the hypersurface $\mathcal{U}_1$ of $N_s^{n+1}(c)$, $n \geqslant 4$, with the metric g induced on M from the metric of $N_s^{n+1}(c)$.

Proposition 4.1. *Let (M, g), $n \geqslant 4$, be a semi-Riemannian manifold and let $\mathcal{U}_1$ be a connected component of the set $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ and assume that $(\mathcal{U}_1, g)$ can be realized as a hypersurface of $\mathbb{E}_s^{n+1}$. Then (1) is fulfilled on $(\mathcal{U}_1, g)$ if and only if on $(\mathcal{U}_1, g)$ we have*

$$(i)\quad R = \frac{L}{2} S \wedge S \qquad \text{and} \qquad (ii)\quad L = \frac{n-1}{(n-2)\kappa}. \tag{29}$$

Proof. It is easy to see that (29)(i) and (29)(ii) imply (17), which is equivalent to (1). We assume that (1) is fulfilled on $(\mathcal{U}_1, g)$. Thus, in particular, (3) holds on $(\mathcal{U}_1, g)$. Evidently, now (6) turns into $Q(S, R) = 0$. Applying this in (21) we obtain $\left(\frac{1}{(n-2)L} - \frac{\kappa}{n-1}\right)Q(g, C) = 0$. If the tensor $Q(g, C)$ vanishes at $x \in \mathcal{U}_1$, then C also vanishes at x [**6**, Section 2.3], i.e., $x \in M - \mathcal{U}_C$, a contradiction. Thus we see that (29)(ii) must be satisfied. Further, by (29)(ii), (17) reduces to (29)(i). Our proposition is thus proved. □

Remark 4.1. (i) An example of a semi-Riemannian manifold $(\mathcal{U}_1, g)$ satisfying (29)(i) and (29)(ii), which can be realized as a hypersurface of $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, was given in [**13**] (see Examples 3.1 and 4.2).

(ii) It is clear that $R\cdot R = 0$ implies $R\cdot C = 0$. Using this fact and Remark 3.1 we can state that on every hypersurface M of $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, satisfying (1) the following relations: $R\cdot C = 0$ and $C\cdot R = 0$ are fulfilled on the subset $\mathcal{U}_1 \subset \mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$. Hypersurfaces fulfilling the last two conditions were investigated in [**3**] and [**8**].

As an immediate consequence of Proposition 4.1 we have the following

THEOREM 4.1. *A nonconformally flat semi-Riemannian manifold* (M, g) *satisfying* (2), *i.e.,* (1) *with* $\kappa = 0$, *cannot be realized as a hypersurface of* $\mathbb{E}_s^{n+1}$, $n \geqslant 4$.

COROLLARY 4.1. (*i*) *Every e.c.s. metric satisfying* (2) *cannot be realized on a hypersurface* M *of* $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, *as the metric induced on* M *from the metric of* $\mathbb{E}_s^{n+1}$.

(*ii*) *The AG-metrics* g_1 *and* g_2, *defined in Examples* 3.14 *and* 3.16 *of* [**2**], *cannot be realized on a hypersurface* M *of* $\mathbb{E}_s^5$ *as the metric induced on* M *from the metric of* $\mathbb{E}_s^5$.

We present now some results related to the problem of immersions of semi-Riemannian manifolds satisfying (1) into $N_s^{n+1}(c)$, $c \neq 0$, $n \geqslant 4$.

PROPOSITION 4.2. *Let* (M, g), $n \geqslant 4$, *be a semi-Riemannian manifold and let* $\mathcal{U}_1$ *be a connected component of* $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$. *Let the manifold* $(\mathcal{U}_1, g)$ *be realized as a hypersurface of* $N_s^{n+1}(c)$, $c \neq 0$, $n \geqslant 4$. *If* (1) *is satisfied on* $(\mathcal{U}_1, g)$, *then on* $\mathcal{U}_1$ *we have:*

$$\frac{(n-2)\widetilde{\kappa}}{n(n+1)} = \frac{\kappa}{n-1} - \frac{1}{(n-2)L}, \tag{30}$$

$$R = \frac{L}{2} S \wedge S - L\frac{(n-2)\widetilde{\kappa}}{n(n+1)} g \wedge S + \frac{L\kappa}{n-1}\frac{(n-2)\widetilde{\kappa}}{n(n+1)} G, \tag{31}$$

$$(a) \quad H^2 = \alpha H + \frac{\varepsilon\widetilde{\kappa}}{n(n+1)} g, \qquad (b) \quad S = \rho H + \frac{(n-2)\widetilde{\kappa}}{n(n+1)} g, \tag{32}$$

$$(a) \quad \rho = \varepsilon(\mathrm{tr}(H) - \alpha), \qquad (b) \quad L = \frac{\varepsilon}{\rho^2}, \tag{33}$$

$$(a) \quad \mathrm{tr}(H) = \frac{(n-1)\varepsilon\rho}{n-2} - \frac{(n-2)\widetilde{\kappa}}{n(n+1)\rho}, \qquad (b) \quad \alpha = \frac{\varepsilon\rho}{n-2} - \frac{(n-2)\widetilde{\kappa}}{n(n+1)\rho}. \tag{34}$$

PROOF. We assume that (1) holds on $\mathcal{U}_1$. Now (7), by making use of (3) and (21), turns into

$$\Big(\frac{(n-2)\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} + \frac{1}{(n-2)L}\Big) Q(g, C) = 0. \tag{35}$$

If the tensor $Q(g, C)$ vanishes at $x \in \mathcal{U}_1$, then $C = 0$ at x, i.e., $x \in M - \mathcal{U}_C$, a contradiction. Thus the tensor $Q(g, C)$ is nonzero at every point of $\mathcal{U}_1$. Now (35) implies (30). Further, since (3) is fulfilled on $\mathcal{U}_1$, Theorem 5.1 of [**7**] states that at every $x \in \mathcal{U}$ we have: $\mathrm{rank}(H) = 2$ or

$$H^2 = \alpha H + \beta g, \qquad \alpha, \beta \in \mathbb{R}. \tag{36}$$

If $\operatorname{rank}(H)=2$ is fulfilled at x, then, in view of Theorem 3.1(i) of [**7**], at x we have $R\cdot R=\frac{\widetilde{\kappa}}{n(n+1)}Q(g,R)$, whence, by (3), $\widetilde{\kappa}Q(g,R)=0$ and $Q(g,R)=0$. The last relation implies $R=\frac{\kappa}{(n-1)n}G$ [**6**, Section 2.3], i.e., $x\in M-\mathcal{U}_R$, a contradiction. Thus at every point of $\mathcal{U}$ we have (36). Now (36), by Theorem 3.1(ii) of [**7**], yields $R\cdot R=\big(\frac{\widetilde{\kappa}}{n(n+1)}-\varepsilon\beta\big)Q(g,R)$, which, by (3), yields $\big(\frac{\widetilde{\kappa}}{n(n+1)}-\varepsilon\beta\big)Q(g,R)=0$. Since $Q(g,R)$ is nonzero at x, the last relation reduces to $\frac{\widetilde{\kappa}}{n(n+1)}=\varepsilon\beta$. Therefore (36) turns into (32)(a). From (32)(a) we obtain

$$\operatorname{tr}(H^2)=\alpha\operatorname{tr}(H)+\frac{\varepsilon\widetilde{\kappa}}{n+1}.$$

Applying (32)(a) to (28) we get (32)(b), where ρ is defined by (33)(a). From (32)(b), by our assumptions, it follows that ρ is nonzero at every point of $\mathcal{U}_1$. Further, applying (32)(b) to (27) we find

$$R=\frac{\varepsilon}{\rho^2}\frac{1}{2}S\wedge S-\frac{\varepsilon}{\rho^2}\frac{(n-2)\widetilde{\kappa}}{n(n+1)}g\wedge S+\Big(\frac{\varepsilon}{\rho^2}\frac{(n-2)^2\widetilde{\kappa}^2}{n^2(n+1)^2}+\frac{\widetilde{\kappa}}{n(n+1)}\Big)G. \tag{37}$$

Comparing the right-hand sides of (37) and (17) we obtain

$$\Big(\frac{\varepsilon}{\rho^2}-L\Big)\frac{1}{2}S\wedge S=\left(\frac{\varepsilon}{\rho^2}\frac{(n-2)\widetilde{\kappa}}{n(n+1)}-L\Big(\frac{\kappa}{n-1}-\frac{1}{(n-2)L}\Big)\right)g\wedge S$$
$$-\left(\frac{\varepsilon}{\rho^2}\frac{(n-2)^2\widetilde{\kappa}^2}{n^2(n+1)^2}+\frac{\widetilde{\kappa}}{n(n+1)}-\frac{\kappa L}{n-1}\Big(\frac{\kappa}{n-1}-\frac{1}{(n-2)L}\Big)\right)G.$$

If $\varepsilon/\rho^2-L\neq 0$ at $x\in\mathcal{U}$, then, in view of Lemma 3.1, we have $\operatorname{rank}(S-\mu g)=1$, for some $\mu\in\mathbb{R}$. The last relation, by (32)(b), yields $H=\mu_1 g+\mu_2 w\otimes w$, $w\in T^*_xM$, $\mu_1,\mu_2\in\mathbb{R}$, which means that M is quasi-umbilical at x. So, the Weyl tensor C must vanish at x, a contradiction. Thus (33)(b) holds at x. Now from (32)(b), by contraction, we get

$$\kappa-\rho\operatorname{tr}(H)=\frac{(n-2)\widetilde{\kappa}}{n+1}. \tag{38}$$

Further, (30), by (33)(b), yields

$$\varepsilon\rho^2=\frac{n-2}{n-1}\kappa-\frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}.$$

Applying (33)(a) and (38) we obtain

$$\alpha\rho=\frac{\kappa}{n-1}-\frac{2(n-2)\widetilde{\kappa}}{n(n+1)}.$$

From the last two equations, by making use of (33)(a), we get (34)(a). Finally, (34)(b) is an immediate consequence of (34)(a) and (33)(a). Our proposition is thus proved. □

As an immediate consequence of Proposition 4.2 we have the following

COROLLARY 4.2. *Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold and let $\mathcal{U}_1$ be a connected component of the set $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$. If* (1) *is satisfied on the manifold $(\mathcal{U}_1, g)$ and the function L is nonconstant on $\mathcal{U}_1$, then the manifold $(\mathcal{U}_1, g)$ cannot be realized as a hypersurface of $N_s^{n+1}(c)$, $c \neq 0$, $n \geqslant 4$. In particular, every e.c.s. metric satisfying* (2), *with nonconstant function L, as well as the AG-metrics g_1 and g_2, defined in Examples* 3.14 *and* 3.16 *of* [**2**], *cannot be realized on a hypersurface of $N_s^{n+1}(c)$, $c \neq 0$, $n \geqslant 4$, and $N^5(c)$, $c \neq 0$, respectively.*

References

[1] M. A. Akivis and V. V. Goldberg, *Conformal Differential Geometry and Its Generalizations*, Wiley-Interscience Publication, New York, 1996.
[2] M. A. Akivis and V. V. Goldberg, *Semiintegrable almost Grassmann structures*, Differential Geom. Appl. **10** (1999), 257–293.
[3] D. E. Blair, P. Verheyen, and L. Verstraelen, *Hypersurfaces satisfaisant à $R \cdot C = 0$ ou $C \cdot R = 0$*, C.R. Acad. Bulgare Sci. **37** (1984), 459–462.
[4] A. Derdziński and W. Roter, *Some theorems on conformally symmetric manifolds*, Tensor (N.S.) **32** (1978), 11–23.
[5] A. Derdziński and W. Roter, *Some properties of conformally symmetric manifolds which are not Ricci-recurrent*, Tensor (N.S.) **34** (1980), 11–20.
[6] R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A **44** (1992), 1–34.
[7] R. Deszcz, *Pseudosymmetric hypersurfaces in spaces of constant curvature*, Tensor (N.S.) **58** (1997), 253–269.
[8] R. Deszcz, M. Głogowska, M. Hotloś, and Z. Şentürk, *On certain quasi-Einstein semisymmetric hypersurfaces*, Annales Univ. Sci. Budapest. **41** (1998), 151–164.
[9] R. Deszcz and M. Hotloś, *On a certain subclass of pseudosymmetric manifolds*, Publ. Math. Debrecen **53** (1998), 29–48.
[10] R. Deszcz and Ş. Yaprak, *Curvature properties of certain pseudosymmetric manifolds*, Publ. Math. Debrecen **45** (1994), 333–345.
[11] M. Głogowska, *Curvature properties of some four-dimensional manifolds*, Demonstratio Math. **34** (2001), 900–918.
[12] D. Kowalczyk, *On some subclass of semisymmetric manifolds*, Soochow J. Math. **27** (2001), 445–461.
[13] C. Murathan, K. Arslan, R. Deszcz, R. Ezentaş, and C. Özgür, *On a certain class of hypersurfaces of semi-Euclidean spaces*, Publ. Math. Debrecen **58** (2001), 587–604.

Department of Mathematics
Agricultural University of Wrocław
Grunwaldzka 53, 50-357 Wrocław
Poland

mglog@ozi.ar.wroc.pl

(Received 18 12 2000)
(Revised 10 09 2002)