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# INTEGRAL KERNELS WITH REGULAR VARIATION PROPERTY

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Abstract. We give a necessary and sufficient condition for a positive measurable kernel $\mathbf{C}(\cdot)$ to satisfy

$$\int_1^x f(t)\mathbf{C}(t)dt \sim f(x)\int_1^x \mathbf{C}(t)dt \qquad (x \to \infty)$$

whenever $f(\cdot)$ is from the class of Karamata's regularly varying functions.

## Introduction

We deal with the class of regularly varying functions introduced by Karamata [2]. A positive measurable function $f$, defined on some neighborhood of infinity, is said to be regularly varying with index $\rho \in R$ if it can be represented in the form $f(x) = x^\rho \ell(x)$, where the slowly varying function $\ell(x)$ satisfies

$$\ell(sx) \sim \ell(x) \qquad (x \to \infty)$$

for each $s > 0$. It is supposed throughout, without loss of generality, that $\ell(x)$ is defined for $x \geqslant 1$.

An excellent survey on regular variation is given in [1] and [3].

Slowly varying functions have, among many others, a remarkable property of an easy asymptotic calculation of integrals involving them. Namely, under some conditions imposed on the kernel $W(\cdot)$, we have [1, pp. 198–201], [4]

$$\int_A^B \ell(xt)W(t)dt \sim \ell(x)\int_A^B W(t)dt \qquad (x \to \infty).$$

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Analogously with those results, in this paper we shall characterize the class $K$ of measurable kernels satisfying the following definition.

A positive measurable kernel $\mathbf{C}(\cdot)$ is said to belong to the class $K$ if the asymptotic relation

$$(1) \qquad \int_1^x f(t)\mathbf{C}(t)dt \sim f(x)\int_1^x \mathbf{C}(t)dt \qquad (x \to \infty)$$

holds for every regularly varying function $f(\cdot)$ of arbitrary index.

Therefore, the class $K$ provides an easy calculation of integrals involving *regularly varying* functions.

For the purpose of characterization, we have to introduce the class $\Theta$ of positive measurable functions $p(\cdot)$ satisfying

$$(2) \qquad \int_1^x p(t)dt/xp(x) \to 0 \qquad (x \to \infty).$$

This class consists of rapidly growing functions and its connection with classes $MR_\infty$, and $R_\infty$ is given in [1]. However, the structure of the class $\Theta$ is very ambiguous. For example, it is not closed under multiplication (see Proposition 4, below). Our recent communication with Professors N. H. Bingham and C. M. Goldie, the authors of [1], shows that there is no representation for this class of functions (and $MR_\infty$, $R_\infty$ as well).

In the first part we shall give some propositions related to the class $\Theta$ including the characterization of the largest possible subclass of $\Theta$ which is closed under multiplication (Theorem 1). Finally we shall prove our main result.

THEOREM 2. *A positive measurable function $C(\cdot)$ belongs to the class $K$ i.e., the asymptotic relation* (1) *holds for any regularly varying function $f$, if and only if*

$$\int_1^x C(t)dt \in \Theta.$$

## Results

We prove first some assertions concerning the class $\Theta$.

PROPOSITION 1. *The following are equivalent:*
*(i)* $p(x) \in \Theta$; *(ii)* $x^a p(x) \in \Theta$, *for some/any real $a$.*

PROOF. For the proof we need two lemmas.

LEMMA 1. *If $p(x) \in \Theta$, then $x^a p(x) \to \infty$ for any fixed $a \in R$.*

PROOF. Since $\frac{xp(x)}{\int_1^x p(t)dt} \to \infty$ $(x \to \infty)$, we can find $x_0 > 1$ such that

$$\frac{xp(x)}{\int_1^x p(t)dt} > |a| + 2, \qquad x > x_0;$$

i.e.,

$$D\left(\log\int_1^x p(t)dt\right) > (|a|+2)/x, \qquad x > x_0.$$

Integrating (1.2) over $[x_0, x]$, we get

$$\int_1^x p(t)dt > C(x_0,a)x^{|a|+2}, \qquad x > x_0,$$

i.e., taking into account (1.1),

$$x^a p(x) > C'(x_0,a)x^{a+|a|+1}, \qquad x > x_0,$$

and the assertion of Lemma 1 follows.

LEMMA 2. *If $p(x) \in \Theta$, then for every fixed $a \in R$,*

$$\int_1^x t^a p(t)dt \sim x^a \int_1^x p(t)dt \qquad (x \to \infty).$$

PROOF. According to the preceding lemma and definition (2), we have

$$\frac{D(\int_1^x t^a p(t)dt)}{D(x^a \int_1^x p(t)dt)} = \frac{1}{1 + a\int_1^x p(t)dt/xp(x)} \to 1 \qquad (x \to \infty).$$

Hence, the result follows from L'Hospital's rule.

Now the proof of Proposition 1 follows easily.
If $p(x) \in \Theta$, then, according to Lemma 2,

$$\frac{\int_1^x t^a p(t)dt}{x^{a+1}p(x)} = \frac{\int_1^x p(t)dt}{xp(x)} \frac{\int_1^x t^a p(t)dt}{x^a \int_1^x p(t)dt} \to 0 \qquad (x \to \infty),$$

i.e., $x^a p(x) \in \Theta$ for every $a \in R$.
Conversely, if $x^a p(x) \in \Theta$ for some $a \in R$, then

$$\frac{\int_1^x p(t)dt}{xp(x)} = \frac{\int_1^x t^{-a}(t^a p(t))dt}{xp(x)} \sim \frac{x^{-a}\int_1^x t^a p(t)dt}{xp(x)} = \frac{\int_1^x t^a p(t)dt}{x^{a+1}p(x)} \to 0 \quad (x \to \infty),$$

i.e., $p(x) \in \Theta$.

As we already said, the class $\Theta$ is not closed under multiplication. But we can assert the following

PROPOSITION 2. *(i) If* $p, q \in \Theta$*, then* $p + q \in \Theta$*;*
*(ii) if* $p$ *is nondecreasing and* $q \in \Theta$*, then* $p \cdot q \in \Theta$*.*

PROOF. To prove (i) note that by definition (2), for any $A > 0$ we can find $x_1$, $x_2$ such that

$$\frac{xp(x)}{\int_1^x p(t)dt} > A, \quad x > x_1; \quad \frac{xq(x)}{\int_1^x q(t)dt} > A, \quad x > x_2.$$

But then, for $x > \max(x_1, x_2)$, we get

$$\frac{x(p(x) + q(x))}{\int_1^x (p(t) + q(t))dt} > A,$$

and, since $A$ can be taken arbitrary large, (i) follows.

For a nondecreasing positive $p(\cdot)$ (not necessarily from $\Theta$), (ii) follows at once

$$\frac{xp(x)q(x)}{\int_1^x p(t)q(t)dt} > \frac{xp(x)q(x)}{p(x)\int_1^x q(t)dt} = \frac{xq(x)}{\int_1^x q(t)dt} \to \infty \quad (x \to \infty),$$

i.e., $p \cdot q \in \Theta$.

There is another assertion of this type.

PROPOSITION 3. *If* $p, q \in \Theta$*, then*
*(i)* $\int_1^x p(t)dt \in \Theta$*; (ii)* $\int_1^x p(t)dt \cdot \int_1^x q(t)dt \in \Theta$*.*

PROOF. By Lemma 1, we have

$$\frac{D(\int_1^x(\int_1^t p(u)du)dt)}{D(x\int_1^x p(t)dt)} = \frac{1}{1 + xp(x)/\int_1^x p(t)dt} \to 0 \quad (x \to \infty).$$

Hence

$$\frac{\int_1^x(\int_1^t p(u)du)dt}{x\int_1^x p(t)dt} \to 0 \quad (x \to \infty),$$

i.e., $\int_1^x p(t)dt \in \Theta$.

Note that the converse statement is not true. Namely, it is not difficult to find a positive measurable function $p$ such that $p \notin \Theta$; $\int_1^x p(t)dt \in \Theta$ (see the example from Proposition 5).

The assertion (ii) is a consequence of the second part of Proposition 2.

The question of multiplication in the class $\Theta$ is not exhausted by the two preceding propositions. Our main result concerning this problem is contained in the next theorem.

THEOREM 1. *Let the class $\Sigma$ consist of all positive measurable functions $s(\cdot)$ such that $s^2 \in \Theta$. Then $\Sigma$ is the largest proper subclass of $\Theta$ closed under multiplication.*

PROOF. We show first that $\Sigma \subset \Theta$. Indeed, if $s \in \Sigma$, using Cauchy's inequality we get

$$\left(\frac{\int_1^x s(t)dt}{xs(x)}\right)^2 \leqslant \frac{\int_1^x s^2(t)dt}{xs^2(x)} \to 0 \qquad (x \to \infty),$$

hence $s \in \Theta$.

The class $\Sigma$ is closed under multiplication because if $r, s \in \Sigma$ then also $r, s \in \Theta$ and, using Cauchy's inequality again, we obtain

$$\left(\frac{\int_1^x r(t)s(t)dt}{xr(x)s(x)}\right)^2 \leqslant \frac{\int_1^x r^2(t)dt}{xr^2(x)} \frac{\int_1^x s^2(t)dt}{xs^2(x)} \to 0 \qquad (x \to \infty),$$

i.e., $r \cdot s \in \Theta$.

The next statement shows that $\Sigma$ is a proper subclass of $\Theta$.

PROPOSITION 4. *There is a positive measurable function $h \in \Theta$ such that $h^2 \notin \Theta$.*

PROOF. We shall construct $h$ in the following way. Let $h(1) := 1$, $h(x) := 2\frac{\log x}{x} \exp(\log^2 x)$ for $x > 1$, except at the points $x = e^n$, $n \in N$ where we put $h(e^n) := \sqrt{n} \exp(n^2 - n)$. Then $\int_1^x h(t)dt = \exp(\log^2 x) - 1$ and it is easy to check that $h \in \Theta$. But

$$\begin{aligned}\int_1^{\exp n} h^2(t)dt > \int_{\exp(n-1/n)}^{\exp n} h^2(t)dt > \frac{4}{n}(n-1/n)^2 \exp(2(n-1/n)^2 - (n-1/n)) \\ = 4(n + O(1/n)) \exp(2n^2 - n - 4 + O(1/n)).\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \frac{e^n h^2(e^n)}{\int_1^{\exp n} h^2(t)dt} \leqslant e^4/4,$$

i.e., $h^2 \notin \Theta$. Therefore $\Sigma$ is a proper subclass of $\Theta$.

REMARK 1. In view of Proposition 4, it will be difficult to find a representation form for the classes $\Theta$ and $R_\infty$.

REMARK 2. As the referee notes, it is easy to give a continuous counterexample $h^*$.

Namely, let $h^* = h$ except on the intervals $I_n := (\exp(n - c_n), \exp n)$ and $J_n := (\exp n, \exp(n + c_n))$, where the sequence $c$ is tending to zero sufficiently fast, and $h^*$ is linear on the closure of each of the intervals $I$ and $J$. It is obvious that $\Sigma$ has to be the largest subclass of $\Theta$ which is closed under multiplication. If $\Pi \subset \Theta$ is another subclass with this property and $s \in \Pi$, then $s \cdot s$ has to be in $\Sigma$; hence $\Pi \subset \Sigma$.

Now we will turn to the question of characterization of the class $K$ and prove Theorem 2 above.

PROOF OF THEOREM 2. The condition $\int_1^x C(t)dt \in \Theta$ is necessary for (1) to hold for every regularly varying $f(\cdot)$. Indeed, for $f(x) = x$, from (1) it follows that

$$\frac{\int_1^x tC(t)dt}{x\int_1^x C(t)dt} = 1 + o(1) \quad (x \to \infty),$$

i.e.,

$$o(1) = \frac{\int_1^x tC(t)dt - x\int_1^x C(t)dt}{x\int_1^x C(t)dt} = -\frac{\int_1^x (\int_1^t C(u)du)dt}{x\int_1^x C(t)\,dt} \quad (x \to \infty),$$

i.e., $\int_1^x C(t)dt \in \Theta$.

Suppose now that $\int_1^x C(t)dt \in \Theta$. We shall prove first that (1) is satisfied for $\ell(x) = 1$ and for arbitrary index $a \in R$.

According to Lemma 2 and our assumption, we have

$$\int_1^x t^a C(t)dt - x^a \int_1^x C(t)dt = -a\int_1^x t^{a-1}\left(\int_1^t C(u)du\right)dt$$
$$\sim -ax^{a-1}\int_1^x \left(\int_1^t C(u)du\right)dt = o(1)x^a\int_1^x C(t)dt \qquad (x \to \infty),$$

since $\int_1^x C(t)dt \in \Theta$ implies $\int_1^x(\int_1^t C(u)du)dt = o(1)x\int_1^x C(t)dt \qquad (x \to \infty)$.

Therefore, Theorem 2 is proved for $\ell(x) = 1$.

To prove it in the general case we need the following lemma.

LEMMA 3. *For some $\alpha > 0$ and for every slowly varying $\ell(\cdot)$ we have*

$$(i)\ \sup_{t\leqslant x}(t^\alpha \ell(t)) \sim x^\alpha \ell(x); \quad (ii)\ \inf_{t\leqslant x}(t^{-\alpha}\ell(t)) \sim x^{-\alpha}\ell(x) \quad (x \to \infty).$$

This lemma is proved in [1, p. 23].

Now, using the first part of Theorem 2 and Lemma 3, with $f(x) = x^a\ell(x)$, $a \in R$, we get

$$\int_1^x t^a\ell(t)C(t)dt \leqslant \sup_{t\leqslant x}(t^\alpha\ell(t))\int_1^x t^{a-\alpha}C(t)dt \sim x^\alpha\ell(x)\int_1^x t^{a-\alpha}C(t)dt$$
$$\sim x^a\ell(x)\int_1^x C(t)dt;$$

and, analogously,

$$\int_1^x t^a\ell(t)C(t)dt \geqslant \inf_{t\leqslant x}(t^{-\alpha}\ell(t))\int_1^x t^{a+\alpha}C(t)dt \sim x^a\ell(x)\int_1^x C(t)dt \quad (x \to \infty).$$

Therefore

$$1 \leqslant \liminf_{x\to\infty}\frac{\int_1^x t^a\ell(t)C(t)dt}{x^a\ell(x)\int_1^x C(t)dt} \leqslant \limsup_{x\to\infty}\frac{\int_1^x t^a\ell(t)C(t)dt}{x^a\ell(x)\int_1^x C(t)dt} \leqslant 1.$$

Hence, Theorem 2 is proved.

At this place it will be useful, as the referee suggests, to formulate an equivalent theorem.

THEOREM 2′. *If the asymptotic relation* (1) *holds for* $f(t) = t$, *then it holds for any regularly varying function* $f$.

It is evident that the proof of Theorem 2′ follows from the proof of Theorem 2.

We shall conclude with the following proposition.

PROPOSITION 5. *The class* $\Theta$ *is a proper subclass of* $K$.

PROOF. For any $p \in \Theta$, from Proposition 3(i), it follows that $\int_1^x p(t)dt \in \Theta$, i.e., by Theorem 2, $p \in K$. Hence, $\Theta$ is a subclass of $K$.

That it is a proper subclass we shall show by the following example.

Let $p(x) := e^x$, $x > 1$, except at the points $x_n = n$, where we put $p(n) := 1$. Then $\liminf_{x\to\infty} \frac{xp(x)}{\int_1^x p(t)dt} = 0$, i.e., $p \notin \Theta$.

But, Theorem 2′ shows that $p \in K$, i.e., the class $\Theta$ is a proper subclass of $K$. Therefore $\Sigma \subset \Theta \subset K$.

## References

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