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TAUBERIAN RETRIEVAL THEORY

Časlav V. Stanojević and Vera B. Stanojević

ABSTRACT. Karamata's Hauptsatz [11] and its corollary is the main tool for convergence recovery from Abel's necessary conditions and the control of oscillatory behavior of limiting processes. By modifying the basic discovery from [11], relaxing Abel's necessary conditions and lightening the control of oscillatory behavior an extended Tauberian theory is outlined. This theory goes beyond convergence recovery. It retrieves various kinds of moderate divergence.

1. Abel's discovery of manageable divergence

Although Abel denounced divergence as devil's invention, he discovered [**1**] a device that controls divergence of sequences $u = \{u_n\}$ by requiring the existence of

$$\lim_{z\to 1-}(1-z)\lim_n \sum_{k=0}^{n} u_k z^k = A(u). \tag{1.1}$$

Hence the class A of all sequences for which (1.1) exist is a class of sequences whose divergence is manageable. That is, those sequences range from convergent ones up to not so badly divergent ones. This situation opens a new area of Analysis, namely Analysis of Divergence. Indeed, using some devices that restrict the oscillatory behavior of sequences or their modes of boundedness one could retrieve convergent sequences out of A. For instance, $n\Delta u_n = n(u_n - u_{n-1}) = o(1)$, $n\to\infty$ is such a device, or

$$V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k = o(1), \quad n\to\infty \tag{1.2}$$

a less obvious one, found by Tauber [**2**]. Thereafter there was an open season on finding more subtle devices that would retrieve convergent sequences out of A. Littlewood [**3**] found a special manner of bounding (1.2). To this end he proved that

$$n\Delta u_n = O(1), \quad n\to\infty \tag{1.3}$$

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retrieves convergence of $u = \{u_n\}$ out of (1.1). From there on it was harder to design such devices and even harder to prove that they work. For instance, Hardy and Littlewood [**4**], thought of

$$\frac{1}{n+1}\sum_{k=0}^{n} k^p|\Delta u_k|^p = O(1), \quad n \to \infty, \quad p > 1, \tag{1.4}$$

but Szasz [**5**] proved that (1.4) is a suitable device for the convergence recovery of $u = \{u_n\}$ out of (1.1). It turns out that all these devices (1.2)–(1.4) are special cases of a truly new concept introduced by Landau [**6**]. His original definition of slow oscillation as well as the later one of Schmidt [**7**] were cumbersome to use in the proofs, and moreover they did not provide enough of insight into the nature of the concept of slow oscillation. For that reason definition given in [**8**] and extensively used in [**9,10**] is given below.

DEFINITION 1.1. A sequence $u = \{u_n\}$ in a normed linear space with norm $\|\cdot\|$ is slowly oscillating if

$$\lim_{\lambda\to 1+} \overline{\lim_{n}} \max_{n+1\leqslant k\leqslant[\lambda n]} \left\|\sum_{j=n+1}^{k} \Delta u_j\right\| = 0 \tag{1.5}$$

There is a generalization of slow oscillation.

DEFINITION 1.2. A sequence $u = \{u_n\}$ in a normed linear space with norm $\|\cdot\|$ is moderately oscillatory if for $\lambda > 1$

$$\overline{\lim_{n}} \max_{n+1\leqslant k\leqslant[\lambda n]} \left\|\sum_{j=n+1}^{k} \Delta u_j\right\| < \infty \tag{1.6}$$

A few lines below demonstrate how Definition 1.1 is easy to use. For $u = \{u_n\}$ assume (1.4). Then

$$\begin{aligned}
\lim_{\lambda\to 1+} \overline{\lim_{n}} \max_{n+1\leqslant k\leqslant[\lambda n]} \left|\sum_{j=n+1}^{k} \Delta u_j\right| &\leqslant \lim_{\lambda\to 1+} (\lambda-1)^{1/q}\, \overline{\lim_{n}} \left(\frac{1}{n+1}\sum_{j=1}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{1/p} \\
&\leqslant \lim_{\lambda\to 1+} (\lambda-1)^{1/q}\lambda^{1/p}\, \overline{\lim_{n}} \left(\frac{1}{[\lambda n]+1}\sum_{j=1}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{1/p} \\
&\leqslant C \lim_{\lambda\to 1+} (\lambda-1)^{1/q}\lambda^{1/p} = 0, \qquad 1/p + 1/q = 1
\end{aligned}$$

where C comes from (1.4). This is a sufficient proof that sequences with property (1.4) are slowly oscillating. In closing this section we shall define another important concept of Landau [**6**].

DEFINITION 1.3. A real sequence $u = \{u_n\}$ is left onesidedly bounded if for some $C \geqslant 0$ and all nonnegative integers n

$$u_n \geqslant -C.$$

Landau [**6**] proved that if $\{n\Delta u_n\}$ is left onesidedly bounded and $\lim_n \sigma_n^{(1)}(u)$ exists, then $\lim_n u_n = \lim_n \sigma_n^{(1)}(u)$.

Hardy and Littlewood [**4**] improved Landau's theorem by proving that if (1.1) exists and $\{n\Delta u_n\}$ is left onesidedly bounded, then $\{u_n\}$ converges to the limit (1.1), i.e., $A(u)$.

2. Karamata's Hauptsatz and its ramifications

As seen in the Section 1, the research after Tauber's discovery has been focused on finding more wide classes of sequences, via devices that controlled oscillatory behavior. Very little attention was directed to the nature of (1.1). Until a young mathematician Jovan Karamata from Beograd, made a fundamental discovery regarding (1.1).

KARAMATA'S HAUPTSATZ. [**11**] *For the real sequence* $u = \{u_n\}$ *such that*

$$\lim_{x\to 1-}(1-x)\lim_n \sum_{k=0}^{n} u_k x^k = A(u) \tag{2.1}$$

exists, let

$$u_n \geqslant -C \tag{2.2}$$

for some $C \geqslant 0$ *and all nonnegative integers* n. *Then for every Riemann integrable function* g *on* $(0,1)$

$$\lim_{x\to 1-}(1-x)\lim_n \sum_{k=0}^{n} u_k g(x^k) x^k = A(u)\int_0^1 g. \tag{2.3}$$

PROOF. Since for $\alpha \geqslant 0$, replacing x by $x^{1+\alpha}$, we have

$$\lim_{x\to 1-}(1-x)\lim_n \sum_{k=0}^{n} u_k x^{\alpha k} x^k = \frac{A(u)}{1+\alpha};$$

consequently for every polynomial P on (0,1)

$$\lim_{x\to 1-}(1-x)\lim_n \sum_{k=0}^{n} u_k P(x^k) x^k = A(u)\int_0^1 P.$$

For every Riemann integrable g on (0,1) and every $\varepsilon > 0$ there are two polynomials $p \leqslant P$ on $(0,1)$ such that $p \leqslant g \leqslant P$ on $(0,1)$ and $\int_0^1 (P-p) \leqslant \varepsilon$. Hence (2.3) follows. □

The parametric form of (1.1) with respect to the space $R(0,1)$ of all Riemann integrable functions on (0,1), i.e., (2,3), provides many opportunities by choosing g. For instance choosing $x = e^{-1/n}$ and

$$g_0(t) = \begin{cases} 0, & 0 \leqslant t < e^{-1} \\ 1/t, & e^{-1} \leqslant t < 1 \end{cases} \tag{2.4}$$

the limits (1.1) that is an iteration of a discreet and a continuous limit, becomes quite manageable and yields the following important special case of *Karamata's Hauptsatz.*

COROLLARY TO KARAMATA'S HAUPTSATZ. *Let* (2.1) *and* (2.2) *hold. Then*

$$\lim_n \sigma_n^{(1)}(u) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} u_k = A(u).$$

PROOF. In the proof of *Hauptsatz* take g_0 as defined in (2.4). □

The two major results of the classical Tauberian theory are now examples for the *Corollary to Karamata's Hauptsatz.*

EXAMPLE 2.1. For the real $u = \{u_n\}$ let (1.1) exist. If (1.5) holds, then

$$\lim_n u_n = A(u).$$

From (1.5) it follows that $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$. Thus for some $C \geqslant 0$ and all nonnegative integers $V_n^{(0)}(\Delta u) \geqslant -C$. From (1.1) we have that $A(V^{(0)}(\Delta u)) = 0$.

Therefore, applying the *Corollary to Karamata's Hauptsatz* we have

$$V_n^{(1)}(\Delta u) = \frac{1}{n+1} \sum_{k=0}^{n} V_k^{(0)}(\Delta u) = o(1), \quad n \to \infty.$$

Since $A(\sigma^{(1)}(u)) = A(u)$, the identity $\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = V_n^{(1)}(\Delta U)$, $\sigma_n^{(2)}(u) = \sigma_n^{(1)}(\sigma^{(1)}(u))$ yields $\lim_n \sigma_n^{(1)}(u) = A(u)$. Finally (1.5) recovers

$$\lim_n u_n = \lim_n \sigma_n^{(1)}(u) = A(u).$$

EXAMPLE 2.2. For $\{n\Delta u_n\}$ as in Definition of 1.3 let (1.1) exist; then $\lim_n u_n = A(u)$. From

$$n\Delta u_n \geqslant -C \tag{2.5}$$

for some $C \geqslant 0$ and all nonnegative integers n, it follows $V_n^{(0)}(\Delta u) \geqslant -C$. Then as in Example 2.1 $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$ and $\lim_n \sigma_n^{(1)}(u) = A(u)$. For the recovery of $\lim_n u_n$ as $A(u)$ we have only (2.5). However, Definition 1.3 suggests the following identities.

$$u_n = \sigma_n^{(1)}(u) + \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)\right) - \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} \Delta u_j \tag{2.6}$$

for $\lambda > 1$; and for $1 < \lambda < 2$

$$u_n = \sigma_{n-[(\lambda-1)n]-1}^{(1)}(u) - \frac{n+1}{[(\lambda-1)n]+1}\left(\sigma_{n-[(\lambda-1)n]-1}^{(1)}(1)(u) - \sigma_n^{(1)}(u)\right) + \frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^{n} \sum_{j=k+1}^{n} \Delta u_j.$$

From (2.6) we have

$$\overline{\lim_n}\, u_n \leqslant \overline{\lim_n}\, \sigma_n^{(1)}(u) \frac{\lambda}{\lambda-1} \lim_n \left(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)\right) + \overline{\lim_n} \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} (-\Delta u_j).$$

or, noticing that $-\Delta u_j \leqslant C/j$ and that the second term above vanishes,

$$\overline{\lim_n} u_n \leqslant \overline{\lim_n} \sigma_n^{(1)}(u) + \overline{\lim_n} \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} \frac{C}{j} \leqslant \lim_n \sigma_n^{(1)}(u) + C \lg \lambda.$$

Finally letting $\lambda \to 1+$ we obtain

$$\overline{\lim_n} u_n \leqslant \lim_n \sigma_n^{(1)}(u).$$

In a similar way from the identity (2.7) we have

$$\underline{\lim_n} u_n \geqslant \underline{\lim_n} \sigma_n^{(1)}(u) - C \lg \lambda$$

and

$$\underline{\lim_n} u_n \geqslant \lim_n \sigma_n^{(1)}(u).$$

For a different set of identities (2.6) and (2.7) see [**9**]. Thus

$$\lim_n u_n = \lim_n \sigma_n^{(1)}(u).$$

Example 2.2 is celebrated Hardy–Littlewood theorem. These two examples demonstrate the power of Karamata's discovery and its methodology. In the next section we shall use this methodology to expend classical and neoclassical Tauberian theory into Tauberian retrieval theory.

3. Tauberian Retrieval Theorems

In this section we will extend the classical Tauberian theory that recovers convergence of sequences out of the existence of (1.1) and certain additional conditions that control the oscillatory behavior of sequences. Instead $u = \{u_n\}$ we shall consider sequences that are related to $u = \{u_n\}$ such as $\sigma_n^{(m)}(u) = \sigma_n^{(1)}\big(\sigma^{(m-1)}(u)\big)$, $m \geqslant 1$, where $\sigma_n^{(0)}(u) = u_n$ and $\sigma_n^{(1)}(u) = \dfrac{1}{n+1} \sum_{k=0}^{n} u_k$; and

$$V_n^{(m)}(\Delta u) = \sigma_n^{(1)}\big(V^{(m-1)}(\Delta u)\big), \quad m \leqslant 1.$$

In our main theorem we shall show that assuming the existence of $A\big(V^{(1)}(\Delta u)\big)$ and left onesided boundedness of $\big\{n\Delta\big(n\Delta V_n^{(1)}(\Delta u)\big)\big\}$ with respect to some nonnegative sequence $M = \{M_n\}$, the sequence $\{u_n\}$ will be retrieved as slowly oscillating. As a corollary to this theorem we have the classical convergence recovery of $\{u_n\}$.

THEOREM 3.1. *For the real sequence $u = \{u_n\}$ let there exist a nonnegative sequence $M = \{M_n\}$ such that*

$$\left\{\sum_{k=1}^{n} \frac{M_k}{k}\right\} \tag{3.1}$$

is slowly oscillating, and

$$n\Delta\big(n\Delta V_n^{(1)}(\Delta u)\big) \geqslant -M_n \tag{3.2}$$

for all integers $n \geqslant 1$. If

$$\lim_{x\to 1-}(1-x)\lim_n \sum_{k=0}^{n} V_k^{(1)}(\Delta u)x^k = A\big(V^{(1)}(\Delta u)\big) \tag{3.3}$$

exists, then $u=\{u_n\}$ is slowly oscillating.

PROOF. The condition (3.1) implies that $\sigma_n^{(1)}(M) = \dfrac{1}{n+1}\sum_{k=0}^{n} M_k$ is bounded. Hence there is a constant $C \geqslant 0$ such that $\dfrac{1}{n+1}\sum_{k=0}^{n} k\Delta\left(k\Delta V_k^{(1)}(\Delta u)\right) \geqslant -C$ for all integers $n \geqslant 1$. Thus

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) - \{V_n^{(2)}(\Delta u) - V_n^{(3)}(\Delta u)\} \geqslant -C.$$

The existence of the limit (3.3) implies that

$$\lim_{x\to 1-}(1-x)\lim_n \sum_{k=0}^{n}\left[V_k^{(1)}(\Delta u) - V_k^{(2)}(\Delta u) - \left(V_k^{(2)}(\Delta u) - V_k^{(3)}(\Delta u)\right)\right] x^k = 0.$$

Hence by the corollary to Karamata's Hauptsatz

$$\frac{1}{n+1}\sum_{k=0}^{n} k\Delta\left(V_k^{(2)}(\Delta u) - V_k^{(3)}(\Delta u)\right) = o(1), \quad n\to\infty$$

and consequently both $\{V_n^{(3)}(\Delta u)\}$ and $\{V_n^{(2)}(\Delta u)\}$ are vanishing as $n \to \infty$. Therefore

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = n\Delta V_n^{(1)}(\Delta u) \geqslant -C.$$

Applying the identities and inequalities from Example 2.2 for the pair $\{V_n^{(1)}(\Delta u)\}$ and vanishing $\{V_n^{(2)}(\Delta u)\}$ we obtain that $V_n^{(1)}(\Delta U) = o(1)$, $n \to \infty$. Recalling that

$$n\Delta\big(n\Delta V_n^{(1)}(\Delta u)\big) = n\Delta V^{(0)}(\Delta u) - \big(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)\big) \geqslant -M_n$$

and that

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) \geqslant -C$$

we get

$$n\Delta V_n^{(0)}(\Delta u) \geqslant -M_n.$$

Applying identities and inequalities from Example 2.2 to $\{V_n^{(0)}(\Delta u)\}$ and vanishing $\{V_n^{(1)}(\Delta u)\}$ we have $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. Finally from

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=0}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0$$

it follows that $\{u_n\}$ is slowly oscillating, and for some slowly varying $\{L(n)\}$

$$u_n = O(L(n)), \quad n \to \infty. \qquad \square$$

As a corollary to above observation we obtain that there exists a finite interval I such that for every $r \in I$ there is a subsequence $\left\{\frac{u_{n(r)}}{L(n(r))}\right\}$ such that $\lim_{n(r)} \frac{u_{n(r)}}{L(n(r))} = r$. For more details of this kind of results see [**10**].

COROLLARY 3.1. *In Theorem* 3.1 *replace* (3.3) *by the existence of* $\lim_{x\to 1-}(1-x) \times \lim_n \sum_{k=0}^{n} \sigma_k^{(1)}(u)x^n = A(\sigma^{(1)}(u))$. *Then* $\lim_n u_n = A(\sigma^{(1)}(u))$.

As an example to this we have the classical Tauberian theorem. That is, replace (3.3) in Corollary 3.1 by the existence of $\lim_{x\to 1-}(1-x)\lim_n \sum_{k=0}^{n} u_k x^k = A(u)$. Then $\lim_n u_n = A(u)$.

Notice that Theorem 3.1 can be generalized by replacing (3.1) with requiring that $\{\sum_{k=0}^{n} M_k/k\}$ is moderately oscillatory. In this case we retrieve $\{u_n\}$ as moderately oscillatory sequence.

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202 Rolla Building, UMR
Rolla, MO 65409-0020
USA

Department of Mathematics
Southwest Missouri State University
Springfield, MO 65804 USA