

## ABOUT AN OLD PROBLEM OF KARAMATA

Slavko Simić

ABSTRACT. We give a unified solution of problems posed by Karamata–Blanuša–Mitrinović more than 50 years ago on the determination of algebraic functions with prescribed asymptotic and monotonicity properties.

### 1. Introduction

One of the oldest unsolved problems published in the American Mathematical Monthly is Problem 5626 [6], proposed in 1968 by D. S. Mitrinović:

PROBLEM A. *Determine those algebraic functions  $A_n(x)$ ,  $n = 2, 3, \dots$  which have the following properties:*

1.  $\frac{\log x}{x-1} \leq A_n(x)$  ( $x > 0$ );
2.  $A_n(x) \sim x^{-1/n}$  ( $x \rightarrow 0^+$ );
3.  $x A_n(x) \sim x^{1/n}$  ( $x \rightarrow +\infty$ );
4.  $A_n(x) - \frac{\log x}{x-1} \sim a_n(x-1)^{2n-2}$  ( $x \rightarrow 1$ ),

where  $a_n$  is independent of  $x$ .

PROBLEM B. *Find also algebraic functions  $A_n(x)$ ,  $n = 2, 3, \dots$  such that, for  $x > 0$ ,*

$$\frac{\log x}{x-1} \leq A_m(x) \text{ and } A_m(x) \geq A_n(x) \text{ for } 2 \leq m \leq n.$$

Mitrinović has also put this problem in his well-known book “Analytic Inequalities” [7] with a comment that “so far no solution has been published to the above problems”.

In fact, this problem is much older and was originally posed in 1949 [3] by the distinguished Serbian mathematician Jovan Karamata (1902–1967) in connection with some approximation problems of Ramanujan [4]. In [3] he indicated that

$$A_2(x) = \frac{1}{\sqrt{x}}; \quad A_3(x) = \frac{1 + x^{1/3}}{x + x^{1/3}}.$$

D. Blanuša in [2] gives the form of  $A_4(x)$ :

$$A_4(x) = \frac{7 + 16t + 7t^2}{7t - t^2 + 18t^3 - t^4 + 7t^5} \quad \text{for } t = \sqrt[3]{x}.$$

He also noticed that, for each  $x > 0$ ,

$$A_2(x) \geq A_3(x) \geq A_4(x).$$

Therefore Problem 5626 stays unsolved for more than 50 years and there is a logical question: what is the reason for this?

Our investigation, which includes a lot of digging among old papers, shows the following: while it is quite obvious that Mitrinović just rewrote problems posed by Karamata and Blanuša into the form of 5626, his demand to determine (all of) those algebraic functions  $A_n(x)$  satisfying the conditions of Problems A and B, leads to difficulties. They seem to be so big that it makes 5626 practically unsolvable.

To illustrate this point of view, we give just an example.

Let  $A_k(x)$  be a solution of Problem A and let  $B_m(x)$  be some algebraic function defined and positive for  $x \geq 0$ . Then

$$(i) \quad A_n(x) := A_k(x) + b_n B_m(x), \quad n = 2, 3, \dots,$$

is a solution of Problem B for any positive decreasing sequence  $(b_n)$ .

$$(ii) \quad A_k(x) \cdot \frac{B_m(x) + x(x-1)^{m-1}}{B_m(x)},$$

is also a solution of Problem A for any  $m \geq 2k - 1$  and  $B_m(x)/x^m \rightarrow \infty$  ( $x \rightarrow \infty$ ).

The presence of an almost arbitrary algebraic function  $B_m(x)$  in (i) and (ii), shows that it is hard to give a definite general form of the solutions of Problems A and B.

Nevertheless, some symmetry among solutions should also persist since  $A_k(x)$  and  $A_k^*(x) := x^{-1}A_k(x^{-1})$  are both solutions of cited problems.

Anyway, the situation becomes entirely different if we take an insight into the original problems posed by Karamata and Blanuša in [2] and [3]. Instead of asking for “all possible solutions”, they looked for “the simplest algebraic function  $A_n(x)$ ” satisfying the conditions of Problems A and B. Although the word “simplest” is a little bit ambiguous, we can conclude from the given forms of  $A_2(x)$ ,  $A_3(x)$  and  $A_4(x)$  what the authors meant by it. The algebraic function  $A_n(x)$  is “simple” if it

is a *rational function* of  $\sqrt[n]{x}$  with integer coefficients and with the further symmetry property that  $A_n(x) = A_n^*(x)$ .

If there is a couple of solutions of this form, we can always call “better” the one which gives better approximation to  $\frac{\log x}{x-1}$  in the neighborhood of the point  $x = 1$  i.e., for which  $a_n$  is smaller.

In this way, we can consider solutions of 5626 “in the sense of Karamata”.

Our task in this article is, beside the former discussion, to give an explicit form of a “simple solution”  $A_n(x)$  for each  $n \geq 2$ . It happens that our solutions coincide with Karamata–Blanuša versions for  $n = 2$  and are “better” for  $n = 3$  and  $n = 4$ .

### 2. Results

**PROPOSITION.** *The algebraic functions  $A_n(x)$ ,  $x > 0$ ,  $n = 2, 3, \dots$  which represent a common solution of Problems A and B in the sense of Karamata, are defined by the following expression*

$$A_n(x) := \frac{1}{x-1} \left( (x^{1/2n} + x^{-1/2n})^2 \left( \frac{x^{1/n} - 1}{x^{1/n} + 1} \right)^{2n-1} + 2n \sum_{k=1}^{n-1} \frac{1}{2k-1} \left( \frac{x^{1/n} - 1}{x^{1/n} + 1} \right)^{2k-1} \right).$$

*Proof.* We shall prove first that  $A_n(x)$  satisfy the conditions 1, 2, 3 and 4 of Problem A.

Since for fixed  $n$ ,

$$\frac{x}{x-1} \sim 1, \quad \frac{x^{1/n} - 1}{x^{1/n} + 1} \sim 1 \quad (x \rightarrow +\infty); \quad \frac{x^{1/n} - 1}{x^{1/n} + 1} \sim -1 \quad (x \rightarrow +0),$$

it is evident that  $A_n(x)$  satisfy the conditions 2 and 3.

To prove the validity of the assertion 1, put  $x = e^t$ ,  $t \in R$  and rewrite the inequality  $\frac{\log x}{x-1} \leq A_n(x)$  in the form

$$(A.1) \quad \frac{t}{2 \sinh(t/2)} \leq \frac{n}{\sinh(t/2)} \left( \frac{2 X_n^{2n-1}}{n(1 - X_n^2)} + \sum_{k=1}^{n-1} \frac{X_n^{2k-1}}{2k-1} \right),$$

where, for the sake of simplicity, we put  $X_n = X_n(t) := \tanh(t/2n)$ .

Now, for  $0 \leq w \leq X_n < 1$ ;  $n = 2, 3, \dots$ , integrating the identity

$$(A.2) \quad \frac{1}{1-w^2} = \frac{w^{2n-2}}{1-w^2} + \sum_{k=1}^{n-1} w^{2k-2}$$

we get

$$(A.3) \quad \frac{t}{2n} = \frac{1}{2} \log \frac{1+X_n}{1-X_n} = \int_0^{X_n} \frac{dw}{1-w^2} = \sum_{k=1}^{n-1} \frac{1}{2k-1} X_n^{2k-1} + \int_0^{X_n} \frac{w^{2n-2}}{1-w^2} dw$$

Since

$$\int_0^{X_n} \frac{w^{2n-2}}{1-w^2} dw \leq \frac{1}{1-X_n^2} \int_0^{X_n} w^{2n-2} dw \leq \frac{2X_n^{2n-1}}{n(1-X_n^2)},$$

putting this in (A.3) and dividing by  $\frac{\sinh(t/2)}{n}$ ,  $t > 0$ , we obtain (A.1) for  $t > 0$ .

Since both functions on the left and right-hand side of (A.1) are even, we see that this inequality is also valid for the negative values of  $t$ .

Therefore (A.1) is proved.

In the sequel we need this simple lemma.

LEMMA 1.

$$\tanh^m y \sim y^m; \quad \frac{1}{2} \log \frac{1+y}{1-y} - \sum_{k=1}^n \frac{y^{2k-1}}{2k-1} \sim \frac{y^{2n+1}}{2n+1} \quad (y \rightarrow 0).$$

Now, we shall prove the assertion 4. Applying Lemma 1 and A.3, we have

$$\begin{aligned} A_n(e^t) - \frac{t}{e^t - 1} &= \frac{2n}{e^t - 1} \left( \frac{2X_n^{2n-1}}{n(1-X_n^2)} - \left( \frac{1}{2} \log \frac{1+X_n}{1-X_n} - \sum_{k=1}^{n-1} \frac{1}{2k-1} X_n^{2k-1} \right) \right) \\ &\sim \frac{2n}{t} \left( \frac{2}{n} X_n^{2n-1} - \frac{1}{2n-1} X_n^{2n-1} \right) \sim \frac{1}{t} \left( 4 - \frac{2n}{2n-1} \right) \left( \frac{t}{2n} \right)^{2n-1} \\ &\sim \frac{2(3n-2)}{(2n-1)(2n)^{2n-1}} t^{2n-2} \quad (t \rightarrow 0). \end{aligned}$$

Hence we obtain a very precise approximation

$$A_n(x) - \frac{\log x}{x-1} \sim \frac{3n-2}{(2n-1)4^{n-1}n^{2n-1}} (x-1)^{2n-2} \quad (x \rightarrow 1),$$

and the proof that  $A_n(x)$  are solutions of Problem A is finished.

We shall also prove that the algebraic functions  $A_n(x)$  are monotone decreasing in  $n$  i.e., that they satisfy the conditions of Problem B.

For this purpose, note that we can restrict ourselves to the case  $t > 0$ ; hence  $(X_n)$  is monotone decreasing in  $n$ .

Therefore, for fixed  $t > 0$ , using A.3 and partial integration, we have

$$\begin{aligned} A_n(e^t) &= \frac{1}{e^t - 1} \left( 4 \frac{X_n^{2n-1}}{1-X_n^2} + 2n \sum_{k=1}^{n-1} \frac{1}{2k-1} X_n^{2k-1} \right) \\ &= \frac{1}{e^t - 1} \left( t + 4 \frac{X_n^{2n-1}}{1-X_n^2} - 2n \int_0^{X_n} \frac{w^{2n-2}}{1-w^2} dw \right) \\ &= \frac{1}{e^t - 1} \left( t + \int_0^{X_n} 4w^{n/2} d \left( \frac{w^{3n/2-1}}{1-w^2} \right) \right). \end{aligned}$$

For  $0 \leq w \leq X_n < 1$  an easy computation shows that  $W_n(w) := 4w^{n/2} \frac{d}{dw} \left( \frac{w^{3n/2-1}}{1-w^2} \right)$  is positive and decreasing in  $n$ .

Hence

$$\begin{aligned} A_n(e^t) &= \frac{1}{e^t - 1} \left( t + \int_0^{X_n} W_n(w) dw \right) \geq \frac{1}{e^t - 1} \left( t + \int_0^{X_n} W_{n+1}(w) dw \right) \\ &\geq \frac{1}{e^t - 1} \left( t + \int_0^{X_{n+1}} W_{n+1}(w) dw \right) = A_{n+1}(e^t), \end{aligned}$$

and the proof of Proposition is completed.

Since this year is the hundredth anniversary of Karamata's birthday, we should say a few words about this extraordinary man. He became famous around 1930 by his brilliant (and short) proof of Hardy–Littlewood's Tauberian Theorem for Power Series [5].

His proof is cited in many books on Real or Complex Analysis as an example of an ingenious thinking (see [8, pp. 234–236]). This spirit of beauty and clearness is present in each of Karamata's 130 papers.

Karamata is also the founder of well-known Theory of Regular Variation. This theory has large applications in Probability, Real and Complex Analysis, Number Theory, Theory of Distributions (see [1], [9]) etc.

In honor of his anniversary, the Serbian Academy of Sciences is preparing the Collected Papers of Jovan Karamata.

*Remark 1.* This article is originally written for “The American Mathematical Monthly” but a very shortened version is published in [11].

*Remark 2.* Another solution of Karamata's problem is in [10]. There we treat separately Problems A and B. Among other results, we construct two monotone sequences of algebraic functions which are well-approximating the target function from both sides. This paper completes the investigations started in [10].

## References

- [1] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1987.
- [2] D. Blanuša, *Problem 13*, Bull. Soc. Math. Phys. Serbie **1**(3-4) (1949), 156–157.
- [3] J. Karamata, *Problem 1*, Bull. Soc. Math. Phys. Serbie **1**(1) (1949), 77–78.
- [4] J. Karamata, *Sur quelques problèmes posés par Ramanujan*, J. Indian Math. Soc. **3-4** (1960), 343–365.
- [5] J. Karamata, *Über die Hardy–Littlewoodschen Umkehrungen des Abelschen Stetigkeitssatzes*, Math. Zeitschrift **32** (1930), 319–320.
- [6] D.S. Mitrinović, *Problem 5626*, Amer. Math. Monthly **75** (1968), 911–912.
- [7] D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
- [8] E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939.
- [9] E. Seneta, *Functions of Regular Variation*, Springer-Verlag, New York, 1976.

- [10] S. Simić, *A solution of an old problem of Karamata*, Publ. Inst. Math. Nouv. Sér. **70(84)** (2001), 19–25
- [11] S. Simić, *Special functions from long ago*, Amer. Math. Monthly **109** (2002),

Matematički institut  
Kneza Mihaila 35  
11001 Beograd, p.p. 367  
Yugoslavia  
`ssimic@mi.sanu.ac.yu`