PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 71(85) (2002), 27–39

ON SUMMATORY FUNCTIONS OF ADDITIVE FUNCTIONS AND REGULAR VARIATION

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ABSTRACT. An overview of results and problems concerning the asymptotic behaviour for summatory functions of a certain class of additive functions is given. The class of functions in question involves regular variation. Some new Abelian and Tauberian results for additive functions of the form $F(n) = \sum_{p^\alpha \| n} \alpha h(p)$ are obtained.

1. Introduction

The purpose of this paper is to provide an overview of results and problems concerning the asymptotic behaviour of summatory functions for a certain class of additive functions. An arithmetic function $g(n)$ is *additive* if $g(mn) = g(m) + g(n)$ whenever $(m,n) = 1$. Many additive functions can be represented in the form

$$f(n) = \sum_{p|n} h(p) \quad \text{or} \quad F(n) = \sum_{p^\alpha \| n} \alpha h(p), \quad h(p) = p^\rho L(p), \tag{1.1}$$

where $\rho \in \mathbb{R}$ and $p^\alpha \| n$ means (p denotes primes) that p^α divides n, but $p^{\alpha+1}$ does not. The function $h(x)$ appearing in (1.1) is a *regularly varying* function. It is a positive, continuous function for $x \geqslant x_0\,(> 0)$, for which there exists $\rho \in \mathbb{R}$ (called the index of h) such that

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = c^\rho, \quad \text{for all } \ c > 0. \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 11N37; Secondary 26A12.
Key words and phrases. Regularly and slowly varying functions; arithmetic sums; additive functions; Abelian and Tauberian theorems.

We shall denote the set of all regularly varying functions by $\Re$. We shall also denote by $\mathcal{L}$ the set of slowly *slowly varying* (or *slowly oscillating*) functions, namely those functions in $\Re$ for which the index $\rho = 0$. It is easy to show that if $h \in \Re$, then there exists $L \in \mathcal{L}$ such that $h(x) = x^\rho L(x)$, with ρ being the index of h.

Slowly varying functions arise naturally in many branches of number theory, especially in the theory of arithmetic functions. Namely if $a(n)$ is an arithmetic function, then the summatory function $A(x) := \sum_{n \leqslant x} a(n)$ is usually written in the form

$$A(x) = M(x) + E(x), \tag{1.3}$$

where $M(x)$ is the *main term* and $E(x)$ the *error term* in the asymptotic formula (1.3), meaning

$$\lim_{x\to\infty} \frac{E(x)}{M(x)} = 0. \tag{1.4}$$

The main term $M(x)$ is usually represented in terms of elementary functions such as powers, exponentials, logarithms etc. The error term function $E(x)$ is usually not smooth, but is bounded in terms of smooth functions, and one often looks for estimates of the form

$$E(x) = O(\Delta(x)), \tag{1.5}$$

where $\Delta(x)$ is a regularly varying function.

For a comprehensive account of regularly varying functions the reader is referred to the monographs of Bingham et al. [1] and E. Seneta [21]. By a fundamental result of J. Karamata [20], who founded the theory of regular variation, the limit in (1.2) is uniform for $0 < a \leqslant c \leqslant b < \infty$ and any $0 < a < b$. This is known as the *uniform convergence theorem*. It is used to show that any slowly varying function $L(x)$ is necessarily of the form

$$L(x) = A(x) \exp\left(\int_{x_0}^{x} \eta(t)\frac{\mathrm{d}t}{t}\right), \quad \lim_{x\to\infty} A(x) = A > 0, \ \lim_{x\to\infty} \eta(x) = 0. \tag{1.6}$$

Usually the function $\Delta(x)$ in (1.5) is assumed to be of the form $\Delta(x) = x^\rho L(x)$, where $L(x)$ is of the form (1.6) with $A(x) \equiv A\,(> 0)$. This is convenient, since we are interested in the asymptotic behaviour of $\Delta(x)$ and $\lim_{x\to\infty} A(x) = A$ has to hold in (1.6).

Special cases of (1.1) (with $\rho = 0, L(x) \equiv 1$) are the well-known functions

$$\omega(n) = \sum_{p|n} 1, \qquad \Omega(n) = \sum_{p^\alpha || n} \alpha, \tag{1.7}$$

which represent the number of distinct prime factors of n and the total number of prime factors of n, respectively. Another pair of additive functions, which in distinction with the "small additive" functions of (1.7) are called "large additive functions", is (see e.g., [10], [12] and [19])

$$\beta(n) = \sum_{p|n} p, \qquad B(n) = \sum_{p^\alpha \| n} \alpha p. \tag{1.8}$$

This pair is the special case $\rho = 1$, $L(x) \equiv 1$ of (1.1).

Given an additive arithmetic function $g = f$ or $g = F$, defined by (1.1), one can ask for properties of the summatory function

$$G(x) \; = \; \sum_{n \leqslant x} g(n)$$

from the properties of $g(n)$. Such types of results are commonly called *Abelian theorems.* Conversely, if from the properties of $G(x)$ we can deduce properties of $g(n)$ itself, then these kinds of results are called *Tauberian theorems.* Abelian and Tauberian theorems appear in various contexts in many branches of analysis and, in general, it is Tauberian theorems which are more difficult to prove than their corresponding Abelian counterparts. Tauberian theorems commonly necessitate an additional so-called *Tauberian condition*, which usually involves monotonicity of some kind or non-negativity.

2. Abelian theorems for sums of additive functions

If $f(n)$ is given by (1.1) with $L \in \mathcal{L}$ and $\rho > 0$, then we have

$$\sum_{n \leqslant x} f(n) = \left(\frac{\zeta(\rho+1)}{\rho+1} + o(1)\right) \frac{x^{\rho+1} L(x)}{\log x} \quad (x \to \infty), \tag{2.1}$$

where $\zeta(s)$ is the Riemann zeta-function (see [15]). This Abelian theorem was proved in [6], with the remark that the very generality of $L(x)$ hinders a result sharper than (2.1), namely to have an O–term in place of $o(1)$. The asymptotic formula (2.1) cannot hold for $\rho = 0$, since $\zeta(s)$ has a pole at $s = 1$. In this case the sum in question equals ($[y]$ is the greatest integer not exceeding y)

$$\int_2^x \frac{L(t)}{\log t} \left[\frac{x}{t}\right] \mathrm{d}t \tag{2.2}$$

plus an error term, which depends on the prime number theorem (see [15, Chapter 12]). Bingham–Inoue [3] considered the case $\rho = 0$ in (2.1) and obtained that

$$\sum_{n \leqslant x} \sum_{p|n} L(p) \; \sim \; x g(x) \qquad (x \to \infty), \tag{2.3}$$

where

$$g(x) \; := \; \int_1^{x/2} \frac{[u]}{u^2} \tilde{\ell}\left(\frac{x}{u}\right) \mathrm{d}u, \qquad \tilde{\ell}(x) \; := \; \frac{L(x)}{\log x}. \tag{2.4}$$

Moreover, $g(x) \in \Pi_{\tilde{\ell}}$ with $\tilde{\ell}$–index 1, where the *de Haan class* Π_ℓ, for given $\ell \in \mathcal{L}$, consists of measurable functions g satisfying

$$\lim_{x\to\infty} \frac{g(\lambda x) - g(x)}{\ell(x)} = c\log\lambda \qquad (\forall\lambda > 0,\, x \to \infty), \tag{2.5}$$

where c is called the *ℓ-index* of Π_ℓ. We shall give now a direct proof of the fact that $g(x) \in \Pi_{\tilde{\ell}}$ with $\tilde{\ell}$–index 1, which establishes then the Abelian result, since the sum in (2.1) is seen to be asymptotic to (2.2), similarly as was done in [6], and thus by a change of variable it follows that (2.3) holds. We have

$$\begin{aligned} g(\lambda x) - g(x) = \int_1^{\lambda A} \frac{[u]}{u^2}\tilde{\ell}\left(\frac{\lambda x}{u}\right) \mathrm{d}u - \int_1^A \frac{[u]}{u^2}\tilde{\ell}\left(\frac{x}{u}\right) \mathrm{d}u \\ + \int_A^{x/2} \left(\frac{\{u\}}{u^2} - \frac{\{\lambda u\}}{\lambda u^2}\right)\tilde{\ell}\left(\frac{x}{u}\right) \mathrm{d}u, \end{aligned}$$

where $\{u\} = u - [u]$ is the fractional part of u and A is a large positive constant. By the uniform convergence theorem we obtain

$$\begin{aligned} &\int_1^{\lambda A} \frac{[u]}{u^2}\tilde{\ell}\left(\frac{\lambda x}{u}\right) \mathrm{d}u - \int_1^A \frac{[u]}{u^2}\tilde{\ell}\left(\frac{x}{u}\right) \mathrm{d}u = (1+o(1))\tilde{\ell}(x)\int_A^{\lambda A} \frac{[u]}{u^2}\mathrm{d}u \\ &= \left(1 + o(1) + O\left(\frac{1}{A}\right)\right)\tilde{\ell}(x)\int_A^{lA} \frac{\mathrm{d}u}{u} = \left(\log\lambda + o(1) + O\left(\frac{1}{A}\right)\right)\tilde{\ell}(x). \end{aligned}$$

We also have

$$\begin{aligned} &\int_A^{x/2} \left(\frac{\{u\}}{u^2} - \frac{\{\lambda u\}}{\lambda u^2}\right)\tilde{\ell}\left(\frac{x}{u}\right) \mathrm{d}u \\ &\qquad = (1+o(1))\tilde{\ell}(x)\left(\int_A^\infty \frac{\{u\}}{u^2}\mathrm{d}u - \int_A^\infty \frac{\{\lambda u\}}{\lambda u^2}\mathrm{d}u\right) \ll \frac{\tilde{\ell}(x)}{A}, \end{aligned}$$

hence by letting $A \to \infty$ it follows that $g \in \Pi_{\tilde{\ell}}$ with $\tilde{\ell}$–index 1. Here we used the fact that, for $L \in \mathcal{L}$, we have

$$\int_{x_0}^{Bx} L\left(\frac{x}{u}\right) h(u)\,\mathrm{d}u \sim L(x)\int_{x_0}^\infty h(u)\,\mathrm{d}u \qquad (x \to \infty) \tag{2.6}$$

if $B > 0$ is a constant, and h is an integrable function satisfying $h(u) \ll u^{-c}$ for some $c > 1$. One obtains (2.6) by writing

$$\int_{x_0}^{Bx} = \int_{x_0}^{A} + \int_A^{x/A} + \int_{x/A}^{Bx} = I_1 + I_2 + I_3,$$

say, where again A is a large constant. By the uniform convergence theorem

$$I_1 = (1+o(1))L(x)\int_{x_0}^A h(u)\,\mathrm{d}u = \left(\int_{x_0}^\infty h(u)\,\mathrm{d}u + o(1) + O\left(A^{1-c}\right)\right)L(x).$$

Next, since $x^{-\varepsilon}L(x)$ (see (1.6)) is asymptotic to a non-increasing function for $x \geqslant x_1(\varepsilon)$ and any given $\varepsilon > 0$, we have

$$\begin{aligned} I_2 &= \int_A^{x/A} \left(\frac{x}{u}\right)^{-\varepsilon} L\left(\frac{x}{u}\right)\left(\frac{x}{u}\right)^{\varepsilon} h(u)\,\mathrm{d}u \\ &\ll \left(\frac{x}{A}\right)^{-\varepsilon} L\left(\frac{x}{A}\right)\int_A^{\infty}\left(\frac{x}{u}\right)^{\varepsilon}|h(u)|\,\mathrm{d}u \ll L\left(\frac{x}{A}\right)A^{1-c} = (A^{1-c}+o(1))L(x). \end{aligned}$$

Finally

$$I_3 \ll \int_{x/A}^{\infty} u^{-c}\,\mathrm{d}u \ll \left(\frac{x}{A}\right)^{1-c},$$

and (2.6) follows on taking A sufficiently large, since $c > 1$.

The above analysis shows that the case $\rho = 0$ is more delicate than the case $\rho > 0$. Namely the function $g(x)$ given by (2.3)–(2.4) is not so simple to evaluate asymptotically. To illustrate our point, consider the additive function

$$G_\alpha(n) := \sum_{p|n} \log^\alpha p \qquad (\alpha \in \mathbb{R}),$$

which is of the form (1.1) with $\rho = 0$ and $L(x) = \log^\alpha x$ ($G_0(n) \equiv \omega(n)$). Then

$$\sum_{n\leqslant x} G_\alpha(n) = \sum_{pm\leqslant x} \log^\alpha p = \sum_{p\leqslant x}\left[\frac{x}{p}\right]\log^\alpha p. \tag{2.7}$$

We use the prime number theorem in its strongest known form, namely

$$\pi(x) := \sum_{p\leqslant x} 1 = \int_2^x \frac{\mathrm{d}t}{\log t} + R(x)$$

with

$$R(x) \ll x\mathrm{e}^{-c\varepsilon(x)}, \quad \varepsilon(x) := \log^{3/5} x(\log\log x)^{-1/5}, \quad c > 0.$$

Then we have

$$\begin{aligned} \sum_{p\leqslant x}\left[\frac{x}{p}\right]\log^\alpha p &= \int_{2-0}^{x}\left[\frac{x}{t}\right](\log t)^\alpha\,\mathrm{d}\pi(t) \\ &= \int_2^x \left[\frac{x}{t}\right](\log t)^{\alpha-1}\,\mathrm{d}t + \int_{2-0}^x \left[\frac{x}{t}\right](\log t)^\alpha\,\mathrm{d}R(t). \end{aligned} \tag{2.8}$$

One can show, similarly as was done in [6], that

$$\int_{2-0}^x \left[\frac{x}{t}\right](\log t)^\alpha\,\mathrm{d}R(t) \ll x\mathrm{e}^{-c\varepsilon(x)/2}.$$

We have

$$\int_2^x \left[\frac{x}{t}\right](\log t)^{\alpha-1}\,\mathrm{d}t = x\int_2^x t^{-1}(\log t)^{\alpha-1}\,\mathrm{d}t + O(x(\log x)^{\alpha-1}) \tag{2.9}$$

and

$$\int_2^x t^{-1}(\log t)^{\alpha-1}\,\mathrm{d}t = \begin{cases} \frac{1}{\alpha}\log^\alpha x + O(1), & (\alpha > 0), \\ \log\log x + O(1), & (\alpha = 0), \\ \int_2^\infty (\log t)^{\alpha-1}\frac{\mathrm{d}t}{t} + O((\log x)^\alpha), & (\alpha < 0). \end{cases} \tag{2.10}$$

The asymptotic formula for the summatory function of $G_\alpha(n)$ follows then from (2.7)–(2.10). Its shape changes according to the cases $\alpha < 0$, $\alpha = 0$ or $\alpha > 0$, respectively.

It was pointed out in [6], without details of proof, that (2.1) remains valid if $f(n)$ is replaced by $F(n)$ (both given by (1.1)). To see this, note first that for any additive function $g(n)$ one has

$$\begin{aligned} \sum_{n\leqslant x} g(n) &= \sum_{n\leqslant x}\sum_{p^\nu\|n} g(p^\nu) = \sum_{p^\nu m\leqslant x,(p,m)=1} g(p^\nu) \\ &= \sum_{p^\nu m\leqslant x} g(p^\nu) - \sum_{p^{\nu+1}m\leqslant x} g(p^\nu) = \sum_{p^\nu\leqslant x}(g(p^\nu)-g(p^{\nu-1}))\left[\frac{x}{p^\nu}\right] \\ &= \sum_{p\leqslant x} g(p)\left[\frac{x}{p}\right] + \sum_{p^\nu\leqslant x,\nu\geqslant 2}(g(p^\nu)-g(p^{\nu-1}))\left[\frac{x}{p^\nu}\right]. \end{aligned} \tag{2.11}$$

Setting $g(n) = F(n) - f(n)$ in (2.11) and noting that

$$g(p) = 0, \quad g(p^\nu) = (\nu-1)p^\rho L(p)\ (\nu\geqslant 2),$$

it follows that

$$\begin{aligned} \sum_{n\leqslant x}(F(n)-f(n)) &= \sum_{p^\nu\leqslant x,\nu\geqslant 2} p^\rho L(p)\left[\frac{x}{p^\nu}\right] \\ &\leqslant x\sum_{p\leqslant\sqrt{x}} p^\rho L(p)\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right) \ll x\sum_{p\leqslant\sqrt{x}} p^{\rho-2}L(p). \end{aligned} \tag{2.12}$$

Since $x^{-\varepsilon} \ll L(x) \ll x^\varepsilon$ for any given $\varepsilon > 0$, we have

$$\sum_{p\leqslant\sqrt{x}} p^{\rho-2}L(p) \ll \sum_{p\leqslant\sqrt{x}} p^{\rho-2+\varepsilon} \ll_\varepsilon \begin{cases} 1, & (\rho < 1), \\ x^{(\rho-1)/2+\varepsilon}, & (\rho\geqslant 1). \end{cases} \tag{2.13}$$

Hence for sufficiently small ε we obtain, in view of (2.1),

$$\begin{aligned} \sum_{n\leqslant x} F(n) &= \sum_{n\leqslant x} f(n) + O_\varepsilon\left(\max(x,\, x^{(\rho+1)/2+\varepsilon})\right) \\ &= \left(\frac{\zeta(\rho+1)}{\rho+1} + o(1)\right)\frac{x^{1+\rho}L(x)}{\log x} \quad (\rho > 0,\ x\to\infty), \end{aligned}$$

which proves our assertion. In the case $\rho = 0$ the asymptotic formula (2.3)–(2.4) remains valid if and only if

$$\lim_{x\to\infty} g(x) = +\infty. \tag{2.14}$$

To see this note that if (2.14) holds, then (2.12) gives (in case $\rho = 0$) that

$$\sum_{n\leqslant x}(F(n) - f(n)) \ll x,$$

hence

$$\sum_{n\leqslant x} F(n) = (1 + o(1))xg(x) \qquad (x \to \infty). \tag{2.15}$$

On the other hand, if $g(x)$ is bounded, then (2.15) cannot hold. Namely for $\rho = 0$ we obtain from (2.12)

$$\sum_{n\leqslant x} F(n) = \sum_{n\leqslant x} f(n) + \sum_{p^\nu\leqslant x,\nu\geqslant 2} L(p)\Big[\frac{x}{p^\nu}\Big].$$

Then the last sum above equals

$$x\sum_{p^\nu\leqslant x,\nu\geqslant 2} L(p)p^{-\nu} + O_\varepsilon(x^{1/2+\varepsilon}) = Cx + O_\varepsilon(x^{1/2+\varepsilon})$$

with

$$C := \sum_p \frac{L(p)}{p(p-1)} \;(< +\infty).$$

This means that

$$\sum_{n\leqslant x} F(n) = x(g(x) + C + o(1)) \qquad (x \to \infty),$$

which proves our claim.

3. Tauberian theorems for sums of additive functions

Tauberian theorems for arithmetic sums, including the Tauberian counterpart of (2.1) (both the case $\rho > 0$ and the limiting case $\rho = 0$), have been recently obtained by Bingham–Inoue [3]. In previous works (see e.g., [2]) they developed powerful Tauberian theorems for systems of kernels. In this way they succeeded in simplifying and extending Tauberian results of De Koninck and the author [8]. In the case $\rho > 0$ they proved that

$$\sum_{n\leqslant x}\sum_{p|n} h(p) = \left(\frac{\zeta(\rho+1)}{\rho+1} + o(1)\right)\frac{x^{\rho+1}L(x)}{\log x} \quad (x\to\infty) \tag{3.1}$$

with $L \in \mathcal{L}$, h positive and non-decreasing implies that

$$h(x) \sim x^\rho L(x) \qquad (x \to \infty). \tag{3.2}$$

This result they call Tauberian, and more generally they proved that if

$$\sum_{n \leqslant x} \sum_{p|n} h(p) = (C + o(1))h(x)\frac{x}{\log x} \quad (x \to \infty) \tag{3.3}$$

holds for some positive constant C, and $h : [2, \infty) \to (0, \infty)$ is continuous and non-decreasing, then h is regularly varying with index ρ, where ρ is the unique positive solution of the equation $C = \zeta(\rho + 1)/(\rho + 1)$. This type of result they called *Mercerian.* In the limiting case $\rho = 0$ they prove the following Tauberian theorem: If $L \in \mathcal{L}$,

$$\int_{x_0}^{\infty} L(x) \mathrm{e}^{-\sqrt{\log x}} \frac{\mathrm{d}x}{x} < \infty, \tag{3.4}$$

$$\log x = O(L(x)), \tag{3.5}$$

then if $\tilde{L}(x) = L(x)/\log x$ and

$$\frac{1}{x} \sum_{n \leqslant x} \sum_{p|n} h(p) \in \Pi_{\tilde{L}} \qquad (\text{with}\, \tilde{L} - \text{index } 1) \tag{3.6}$$

with $h : [2, \infty) \to (0, \infty)$ continuous and non-decreasing, we have

$$h(x) \sim L(x) \qquad (x \to \infty). \tag{3.7}$$

The Tauberian condition (3.4) comes from using the prime theorem with the classical error term (see Section 2) $R(x) \ll x\mathrm{e}^{-\sqrt{\log x}}$. It could be replaced with a slightly sharper condition coming from the strongest known error term, namely

$$\int_{x_0}^{\infty} L(x) \mathrm{e}^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} \frac{\mathrm{d}x}{x} < \infty,$$

with suitable $c > 0$. However, as remarked by Bingham-Inoue [3], the Tauberian condition (3.4) is much less restrictive than (3.5). It is an open problem to relax (3.5), or dispense with it altogether. The method of proof of the above results consists of using Tauberian (Mercerian) theorems for the *Mellin convolution*

$$(f \star k)(x) := \int_0^{\infty} k\left(\frac{x}{t}\right) f(t) \frac{\mathrm{d}t}{t},$$

on showing that the relevant sums are asymptotic to an appropriate Mellin convolution.

One can also ask for the Tauberian analogue of (3.1) if we suppose that

$$\sum_{n\leqslant x}\sum_{p^\nu||n}\nu h(p)=\Bigl(\frac{\zeta(\rho+1)}{\rho+1}+o(1)\Bigr)\frac{x^{\rho+1}L(x)}{\log x}\quad(x\to\infty)\tag{3.8}$$

with $\rho>0$, h positive and non-decreasing. Note that

$$\begin{aligned}\sum_{n\leqslant 2x}\sum_{p^\nu||n}\nu h(p)&\geqslant\sum_{n\leqslant 2x}\sum_{p|n}h(p)=\sum_{p\leqslant 2x}h(p)\Bigl[\frac{2x}{p}\Bigr]\\&\geqslant\sum_{x<p\leqslant 2x}h(p)\geqslant h(x)(\pi(2x)-\pi(x))\gg\frac{xh(x)}{\log x},\end{aligned}\tag{3.9}$$

and consequently (3.8) yields

$$h(x)\ \ll\ x^\rho L(x).\tag{3.10}$$

Therefore by using (2.12), (2.13) and (3.10) we find that

$$\begin{aligned}\sum_{n\leqslant x}\sum_{p^\nu||n}\nu h(p)&=\sum_{n\leqslant x}\sum_{p|n}h(p)+\sum_{p^\nu\leqslant x,\nu\geqslant 2}h(p)\Bigl[\frac{x}{p^\nu}\Bigr]\\&=\sum_{n\leqslant x}\sum_{p|n}h(p)+O\Bigl(\sum_{p^\nu\leqslant x,\nu\geqslant 2}p^\rho L(p)\Bigl[\frac{x}{p^\nu}\Bigr]\Bigr)\\&=\sum_{n\leqslant x}\sum_{p|n}h(p)+O\Bigl(x\sum_{p\leqslant\sqrt{x}}p^{\rho-2}L(p)\Bigr)\\&=\sum_{n\leqslant x}\sum_{p|n}h(p)+O_\varepsilon\Bigl(\max(x,\,x^{(\rho+1)/2+\varepsilon})\Bigr).\end{aligned}\tag{3.11}$$

Since $\rho>0$ it follows that (3.8) yields (3.1), hence by the Tauberian theorem of Bingham–Inoue we have (3.2). A similar discussion shows that the corresponding Mercerian analogue of (3.3) also holds for the sum in (3.8), as well as does the Tauberian anlogue of (3.6). For the latter note that, by the analogue of (3.9) when $\rho=0$, we obtain that

$$h(x)\ll g(x)\log x,\quad g(x)\in\Pi_{\tilde L}\qquad(\text{with}\,\tilde L-\text{index }1),$$

and consequently $h(x)\ll_\varepsilon x^\varepsilon$. Supposing $\lambda>1$ (the case $\lambda<1$ is analogous), we have

$$\begin{aligned}&\frac{1}{\lambda x}\sum_{p^\nu\leqslant\lambda x,\nu\geqslant 2}h(p)\Bigl[\frac{\lambda x}{p^\nu}\Bigr]-\frac{1}{x}\sum_{p^\nu\leqslant x,\nu\geqslant 2}h(p)\Bigl[\frac{x}{p^\nu}\Bigr]\\&=\sum_{x<p^\nu\leqslant\lambda x,\nu\geqslant 2}\frac{h(p)}{p^\nu}+O\left(\frac{1}{x}\sum_{p^\nu\leqslant\lambda x,\nu\geqslant 2}\frac{|h(p)|}{p^\nu}\right)=O_\varepsilon(x^{\varepsilon-1/2}).\end{aligned}$$

In view of (3.6) and (3.11) this means that

$$\sum_{n\leqslant x}\sum_{p^\nu||n}\nu h(p)\in\Pi_{\tilde L}\qquad\Big(\text{with }\tilde L-\text{index }1,\ \tilde L(x)=\frac{L(x)}{\log x}\Big),$$

implies (3.6), and therefore (3.7) follows if (3.4)–(3.5) holds and $h:[2,\infty)\to(0,\infty)$ is continuous and non-decreasing.

4. Sums of reciprocals

We shall conclude our exposition by considering sums of reciprocals of certain arithmetic functions. It is a classical result of prime number theory that

$$\sum_{p\leqslant x}\frac{1}{p}=\log\log x+C+O\left(\frac{1}{\log x}\right).$$

Sums of reciprocals of large additive functions $\beta(n)$, $B(n)$ (see (1.8)) and $P(n)$ are much more difficult to handle, where $P(n)$ is the largest prime factor of $n\,(\geqslant 2)$ and $P(1)=1$. They were investigated by De Koninck, Erdős, Pomerance, Xuan and the author (see [4]–[7], [9]–[14], [16]–[19], [22], [23]). It was proved by Erdős, Pomerance and the author [13] that

$$\sum_{n\leqslant x}\frac{1}{P(n)}=x\delta(x)\left(1+O\left(\sqrt{\frac{\log\log x}{\log x}}\right)\right)\tag{4.1}$$

with

$$\delta(x)\ :=\ \int_2^x\rho\left(\frac{\log x}{\log t}\right)\frac{\mathrm{d}t}{t^2},\tag{4.2}$$

where the Dickman–de Bruijn function $\rho(u)$ is the continuous solution to the differential delay equation

$$u\rho'(u)=-\rho(u-1),\ \rho(u)=1\text{ for }0\leqslant u\leqslant 1,\ \rho(u)=0\text{ for }u<0\,.$$

It is known (see [21]) that ($\log_k x=\log(\log_{k-1}x)$)

$$\rho(u)=\exp\left\{-u\left(\log u+\log_2 u-1+\frac{\log_2 u-1}{\log u}+O\left(\left(\frac{\log_2 u}{\log u}\right)^2\right)\right)\right\}.$$

It was shown by Pomerance and the author [19] that one has

$$\delta(x)=\exp\left\{-(2\log x\log_2 x)^{1/2}\left(1+g_0(x)+O\left(\frac{\log_3^3 x}{\log_2^3 x}\right)\right)\right\},$$

where

$$g_r(x) = \frac{\log_3 x + \log(1+r) - 2 - \log 2}{2\log_2 x}\left(1 + \frac{2}{\log_2 x}\right) - \frac{(\log_3 x + \log(1+r) - \log 2)^2}{8\log_2^2 x},$$

and the expression for $\delta(x)$ was sharpened by the author [17]. Already in 1977 Paul Erdős told the author that the function $\delta(x)$ is slowly varying, but it is only in 1986 that this fact was established in [13], by the use of (4.2) and properties of the function $\rho(u)$. A corollary of (4.1) and the fact that $\delta(x)$ is slowly varying is the asymptotic formula

$$\sum_{n\leqslant x}\frac{1}{P(n)} \sim \sum_{x<n\leqslant 2x}\frac{1}{P(n)} \qquad (x\to\infty),$$

which is by no means obvious. In fact we have (see [13, p. 291]) that

$$\delta\left(\frac{x}{t}\right) = \left(1 + O\left(\frac{(\log\log x)^{3/2}}{(\log x)^{1/2}}\right)\right)\delta(x) \qquad (1\leqslant t\leqslant \log^{12} x).$$

One may ask: precisely for what $t\geqslant 1$ does one have $\delta(x/t)\sim\delta(x)$ as $x\to\infty$?

The asymptotic formula (4.1) remains valid if $P(n)$ is replaced by $\beta(n)$ or $B(n)$, and the asymptotic formula for the summatory function of $\frac{B(n)}{\beta(n)} - 1$ is of the same shape as the right-hand side of (4.1). Furthermore we have

$$\begin{aligned}(4.3)\qquad &\sum_{2\leqslant n\leqslant x}\left(\frac{1}{\beta(n)} - \frac{1}{B(n)}\right)\\ &= x\exp\left\{-2(\log x\log_2 x)^{1/2}\left(1 + g_1(x) + O\left(\frac{\log_3^3 x}{\log_2^3 x}\right)\right)\right\}.\end{aligned}$$

One can also show that the sum on the left-hand side of (4.3) is asymptotic to a regularly varying function with index 1.

Finally let us mention that, by using results of the joint paper with Erdős and Pomerance [13], the author [16] sharpened some of the asymptotic formulas proved in earlier works and obtained (see (1.7)), for example,

$$\sum_{n\leqslant x}\frac{\Omega(n)-\omega(n)}{P(n)} = \left\{\sum_p \frac{1}{p^2-p} + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right\}\sum_{n\leqslant x}\frac{1}{P(n)},$$

$$\sum_{n\leqslant x}\frac{\omega(n)}{P(n)} = \left\{\left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\right\}\sum_{n\leqslant x}\frac{1}{P(n)},$$

and this remains valid if $\omega(n)$ is replaced by $\Omega(n)$. We also have

$$\sum_{n\leqslant x}\frac{\mu^2(n)}{P(n)}=\left\{\frac{6}{\pi^2}+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right\}\sum_{n\leqslant x}\frac{1}{P(n)}, \tag{4.4}$$

where μ is the Möbius function. On the left hand-side of (4.4) we have summation over squarefree numbers, whose density is precisely $6/\pi^2$, so that this result shows that the sum of reciprocals of $P(n)$ over squarefree numbers $\leqslant x$ asymptotically equals the corresponding density times the sum of $1/P(n)$ over all $n \leqslant x$.

References

[1] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[2] N.H. Bingham and A. Inoue, *Tauberian and Mercerian theorems for systems of kernels*, J. Math. Anal. Appl. **252** (2000), 177–197

[3] N.H. Bingham and A. Inoue, *Abelian, Tauberian and Mercerian theorems for arithmetic sums*, J. Math. Anal. Appl. **250** (2000), 465–493

[4] J.-M. De Koninck and A. Ivić, *Topics in Arithmetical Functions*, Mathematics Studies **43**, North-Holland, Amsterdam, 1980.

[5] J.-M. De Koninck and A. Ivić, *The distribution of the average prime divisor of an integer*, Archiv Math. **43** (1984), 37–43.

[6] J.-M. De Koninck and A. Ivić, *On the average prime factor of an integer and some related problems*, Ricerche di Matematica **39** (1990), 131–140.

[7] J.-M. De Koninck and A. Ivić, *Random sums related to prime divisors of an integer*, Publ. Inst. Math. (Beograd) **48(62)** (1990), 7–14.

[8] J.-M. De Koninck and A. Ivić, *Arithmetic characterization of regularly varying functions*, Ricerche di Matematica **44** (1995), 41–64.

[9] J.-M. De Koninck, P. Erdős and A. Ivić, *Reciprocals of certain large additive functions*, Canad. Math. Bull. **24** (1981), 225–231.

[10] P. Erdős and A. Ivić, *Estimates for sums involving the largest prime factor of an integer and certain related additive functions*, Studia Sci. Math. Hungar. **15** (1980), 183–199.

[11] P. Erdős and A. Ivić, *On sums involving reciprocals of certain arithmetical functions*, Publ. Inst. Math. (Beograd) **32(46)** (1982), 49–56.

[12] P. Erdős and A. Ivić, *The distribution of small and large additive functions II*, Proceedings of the Amalfi Conference on Analytic Number Theory (Amalfi, Sep. 1989), Università di Salerno, Salerno 1992, 83–93.

[13] P. Erdős, A. Ivić and C. Pomerance, *On sums involving reciprocals of the largest prime factor of an integer*, Glasnik Mat. **21(41)** (1986), 283–300.

[14] A. Ivić, *Sum of reciprocals of the largest prime factor of an integer*, Archiv Math. **36** (1981), 57–61.

[15] A. Ivić, *The Riemann Zeta-function*, John Wiley & Sons, New York, 1985.

[16] A. Ivić, *On some estimates involving the number of prime divisors of an integer*, Acta Arithmetica **49** (1987), 21–32.

[17] A. Ivić, *On sums involving reciprocals of the largest prime factor of an integer II*, Acta Arithmetica **71** (1995), 241–245.

[18] A. Ivić, *On large additive functions over primes of positive density*, Mathematica Balkanica **10** (1996), 103–120.

[19] A. Ivić and C. Pomerance, *Estimates for certain sums involving the largest prime factor of an integer*, Proc. Budapest Conf. in Number Theory July 1981, Coll. Math. Soc. J. Bolyai **34**, North-Holland, Amsterdam 1984, 769–789.

[20] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.
[21] E. Seneta, *Regularly Varying Functions*, LNM **508**, Springer-Verlag, Berlin–Heidelberg–New York, 1976.
[21] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, Société Math. de France, Paris, 1995.
[22] T.Z. Xuan, *On sums involving reciprocals of certain large additive functions*, Publ. Inst. Math. (Beograd) **45(59)**, 41–55 and II, ibid. **46(60)** (1989), 25–32.
[23] T.Z. Xuan, *On a result of Erdős and Ivić*, Archiv Math. **62** (1994), 143–154.

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