

REMAINDER TERM IN CHAKALOV–POPOVICIU QUADRATURES OF RADAU AND LOBATTO TYPE AND INFLUENCE FUNCTION

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ABSTRACT. Let f be a given real function defined on $[a, b]$, $-\infty < a < b \leq \infty$. We develop estimates of the remainder term in the quadrature formulas with multiple nodes (Q) below, where $\sigma = \sigma_n = (s_1, s_2, \dots, s_n)$ is a given sequence of nonnegative integers, $p, q \in \mathbb{N}_0$, and $w(t)$ is a given weight function on (a, b) . Let $N = 2(\sum_{\nu=1}^n s_\nu + n) + p + q$, and denote by $AC^k[a, b]$, $B^k[a, b]$, $C^k[a, b]$ the classes of functions whose the k -th derivative is absolutely continuous, bounded or continuous on $[a, b]$, respectively. An influence function is introduced, its relevant properties are investigated, and in classes of functions $AC^{N-1}[a, b]$, $B^N[a, b]$, $C^N[a, b]$ the error estimates are given. A numerical example is included.

1. Introduction

Let $d\varphi(t)$ be a given nonnegative measure on the real line \mathbb{R} , with compact or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k d\varphi(t)$ ($k = 0, 1, \dots$) exist and are finite, and $\mu_0 > 0$.

A quadrature formula of the form

$$(1.1) \quad \int_{\mathbb{R}} f(t) d\varphi(t) = \sum_{\nu=1}^n \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f),$$

where $A_{i,\nu} = A_{i,\nu}^{(n,s)}$, $\tau_\nu = \tau_\nu^{(n,s)}$ ($i = 0, 1, \dots, 2s; \nu = 1, \dots, n$), which is exact for all algebraic polynomials of degree at most $2(s+1)n-1$, was considered firstly by P. Turán (see [24]), in the case when $d\varphi(t) = dt$ on $[-1, 1]$. The case with a weight

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function $d\varphi(t) = w(t) dt$ on $[a, b]$ has been investigated by Italian mathematicians Ossicini, Ghizzetti, Guerra, Rosati, and also by Chakalov, Stroud, Stancu, Ionescu, Pavel, etc. (see the survey paper [11] for references).

The nodes τ_ν in (1.1) must be zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the integral

$$\Phi \equiv \Phi(a_0, a_1, \dots, a_{n-1}) = \int_{\mathbb{R}} \pi_n(t)^{2s+2} d\varphi(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$. In order to minimize Φ we must have

$$(1.2) \quad \int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k d\varphi(t) = 0, \quad k = 0, 1, \dots, n-1.$$

Such polynomials $\pi_n(t)$, which satisfy this new type of orthogonality so called “*power orthogonality*” are known as s -orthogonal (or s -self associated) polynomials with respect to the measure $d\varphi(t)$.

For $s = 0$ we have the standard case of orthogonal polynomials.

Take now a sequence of nonnegative integers $\sigma = (s_1, s_2, \dots)$. Consider a generalization of Gauss-Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities

$$(1.3) \quad \int_{\mathbb{R}} f(t) d\varphi(t) = \sum_{\nu=1}^n \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f),$$

where $A_{i,\nu} = A_{i,\nu}^{(n,\sigma)}$, $\tau_\nu = \tau_\nu^{(n,\sigma)}$ ($i = 0, 1, \dots, 2s_\nu$; $\nu = 1, \dots, n$). Such formulas were derived independently by Chakalov [2, 3] and Popoviciu [18]. A deep theoretical progress in this subject was made by Stancu (see [23] and references in it).

In this case, it is important to assume that the nodes $\tau_\nu (= \tau_\nu^{(n,\sigma)})$ are ordered, say

$$(1.4) \quad \tau_1 < \tau_2 < \dots < \tau_n, \quad \tau_\nu \in \text{supp}(d\lambda),$$

with odd multiplicities $2s_1 + 1, 2s_2 + 1, \dots, 2s_n + 1$, respectively, in order to have uniqueness of Chakalov-Popoviciu quadrature formula (1.3) (cf. Karlin and Pinkus [8]). Then this quadrature formula has the maximum degree of exactness

$$d_{\max} = 2 \sum_{\nu=1}^n s_\nu + 2n - 1$$

if and only if

$$(1.5) \quad \int_{\mathbb{R}} \prod_{\nu=1}^n (t - \tau_\nu)^{2s_\nu+1} t^k d\varphi(t) = 0, \quad k = 0, 1, \dots, n-1.$$

The last *orthogonality conditions* correspond to (1.2). The existence of such quadrature rules was proved by Chakalov [2], Popoviciu [18], Morelli and Verna [15], and existence and uniqueness subject to (1.4) by Ghizzetti and Ossicini [6].

The conditions (1.5) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n \in \mathbb{N}_0}$,

$$\pi_{n,\sigma}(t) = \prod_{\nu=1}^n (t - \tau_{\nu}^{(n,\sigma)}), \quad \tau_1^{(n,\sigma)} < \tau_2^{(n,\sigma)} < \dots < \tau_n^{(n,\sigma)}, \quad \tau_{\nu}^{(n,\sigma)} \in \text{supp}(d\lambda),$$

such that

$$\int_{\mathbb{R}} \pi_{k,\sigma}(t) \prod_{\nu=1}^n (t - \tau_{\nu}^{(n,\sigma)})^{2s_{\nu}+1} d\varphi(t) = 0, \quad k = 0, 1, \dots, n-1.$$

These polynomials are called σ -orthogonal polynomials and they correspond to the sequence $\sigma = (s_1, s_2, \dots)$. If we have $\sigma = (s, s, \dots)$, the above polynomials reduce to the s -orthogonal polynomials.

In this paper we consider the generalized Chakalov-Popoviciu quadrature formulae

$$(Q) \quad \int_a^b w(t)f(t) dt = \sum_{i=0}^{p-1} A_{i,0} f^{(i)}(a) + \sum_{\nu=1}^n \sum_{i=0}^{2s_{\nu}} A_{i,\nu} f^{(i)}(\tau_{\nu}) \\ + \sum_{i=0}^{q-1} A_{i,n+1} f^{(i)}(b) + R(f),$$

with arbitrary, $\tau_{\nu} (\nu = 1, \dots, n)$, and fixed, a and (or) b , multiple nodes. A such quadrature formula has maximum degree of exactness $N-1$ if and only if

$$\int_a^b w(t)(1+t)^p(1-t)^q \prod_{\nu=1}^n (t - \tau_{\nu})^{2s_{\nu}+1} t^k dt = 0, \quad k = 0, 1, \dots, n-1.$$

Recent proofs of the existence and the uniqueness of such quadrature rules have been obtained in [19], [22]. Proofs of convergence of such formulas can be found in [17], [20].

In Section 2 an influence function is considered, its relevant properties are investigated, and in the classes of functions $AC^{N-1}[a, b]$, $B^N[a, b]$, $C^N[a, b]$ the error estimates are given. In order to illustrate the possibility of use these error estimates we give a numerical example.

2. Error estimates for quadrature formulae of Radau and Lobatto type connected to σ -orthogonal polynomials

In [16] (see also [5]) Ossicini, for the Gauss-Turán quadratures (formula (Q) with $p = q = 0$, $s_1 = \dots = s_n = s$, or (1.1) with $d\varphi(t) = w(t) dt$), and the Gauss-Turán quadratures of Lobatto type (formula (Q) with $s_1 = \dots = s_n = s$

and $p = q = 2s + 1$), constructed an influence function, investigated its relevant properties, and in some classes of functions gave error estimates. Recently, we have generalized those results to the formula (Q) with $p = q = 0$ (or (1.3) with $d\varphi(t) = w(t) dt$) (see [14]). In this section we will consider the quadrature formula (Q), the general case.

For all undefined notions and notations we refer to [5].

Radau formula. Let

$$(2.1) \quad \int_a^b w(t)f(t) dt = \sum_{i=0}^{p-1} A_{i,0}^R f^{(i)}(a) + \sum_{\nu=1}^n \sum_{i=0}^{2s_\nu} A_{i,\nu}^R f^{(i)}(\tau_\nu) + R^R(f),$$

$-\infty < a < \infty$, $p \in \mathbb{N}$, with $R^R(f) = 0$ for $f \in \mathcal{P}_{2(\sum_{\nu=1}^n s_\nu + n) + p - 1}$, be the generalized Chakalov-Popoviciu quadrature formula of Radau type. With \mathcal{P}_k we denoted the set of all polynomials of degree at most k , $k \in \mathbb{N}_0$. Denote $N = 2(\sum_{\nu=1}^n s_\nu + n) + p$. Concerning the assumptions on $w(t), f(t)$ for the validity of (2.1) we have the following theorem:

THEOREM 2.1. *Formula (2.1) is valid under the following hypotheses:*

$$\begin{aligned} w(t) &\in L[a, b], \quad f(t) \in AC^{N-1}[a, b], \quad \text{if } b \text{ is finite,} \\ t^N w(t) &\in L[a, \infty), \quad f(t) \in AC_{loc}^{N-1}[a, \infty), \\ f^{(N)}(t) &\int_t^\infty \xi^{N-1} w(\xi) d\xi \in L[0, \infty). \end{aligned}$$

The proof is the same as the one of Theorem 4.13.I in [5, pp. 132–133] and will be omitted.

Consider, for example, the case:

$$(2.2) \quad p - 1 < 2s_1 < 2s_2 < \cdots < 2s_n,$$

i. e., $N - p - 1 > N - 2s_1 - 2 > \cdots > N - 2s_n - 2$. Let p be odd, without loss of generality. Then, we have that N is odd. Assuming already computed the nodes τ_ν and the coefficients $A_{i,\nu}^R$ for the remainder in (2.1) we have (see [5]):

$$(2.3) \quad R^R(f) \equiv R(f) = \int_a^b \Phi(t) f^{(N)}(t) dt,$$

where the influence-function $\Phi(t)$ is expressed by

$$(2.4) \quad \Phi(t) = \varphi_{\nu+1}(t) \quad \text{for } \tau_\nu < t < \tau_{\nu+1}, \quad \nu = 0, 1, \dots, n; \quad \tau_0 = a, \tau_{n+1} = b,$$

and the functions $\varphi_\nu(t)$, integrals of the differential equation $\varphi^{(N)}(t) = -w(t)$ (since N is odd), are given by the formulae

$$(2.5) \quad \begin{aligned} \varphi_\nu(t) = - \int_a^t w(\xi) \frac{(t-\xi)^{N-1}}{(N-1)!} d\xi &+ \sum_{j=1}^{\nu-1} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-1}}{(N-i-1)!} \\ &+ \sum_{i=0}^{p-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-1}}{(N-i-1)!}, \end{aligned}$$

where $\nu = 1, \dots, n+1$, and $A_{h,j} = A_{h,j}^R$.

For $\varphi_{n+1}(t)$ we have

$$(2.6) \quad \varphi_{n+1}(t) = \int_t^b w(\xi) \frac{(t-\xi)^{N-1}}{(N-1)!} d\xi.$$

From (2.4), (2.5) it follows, differentiating k times (with $0 \leq k \leq N-1$):

$$(2.7) \quad \Phi^{(k)}(t) = \varphi_{\nu}^{(k)}(t) \quad \text{for } t \in (\tau_{\nu-1}, \tau_{\nu}), \quad \nu = 1, \dots, n+1,$$

where for $l = \nu, \nu-1, \dots, 1$,

$$\begin{aligned} \varphi_{\nu}^{(k)}(t) = & - \int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)!} d\xi + \sum_{j=1}^{l-1} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!} \\ & + \sum_{j=l}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!} + \sum_{i=0}^{p-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)!}, \end{aligned}$$

with: (i) $0 \leq k \leq N - 2s_{\nu-1} - 2$, for $l = \nu$, (ii) $N - 2s_l - 1 \leq k \leq N - 2s_{l-1} - 2$, for $l = \nu - 1, \nu - 2, \dots, 1$, where we put $s_0 = (p-1)/2$, and

$$(2.8) \quad \begin{aligned} \varphi_{\nu}^{(k)}(t) = & - \int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)!} d\xi + \sum_{j=1}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!} \\ & + \sum_{i=0}^{N-k-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)!}, \end{aligned}$$

for $N-p \leq k \leq N-1$. (We used the convention $\sum_i^j \cdot \equiv 0$ for $j < i$.)

For the derivatives of $\varphi_{n+1}(t)$ ($t \in (\tau_n, b)$) we can use the following formulas:

$$(2.9) \quad \varphi_{n+1}^{(k)}(t) = \int_t^b w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)!} d\xi, \quad t \in (\tau_n, b).$$

Now, we can conclude that

$$(2.10) \quad \begin{aligned} \Phi^{(k)}(a) &= 0, \quad k = 0, 1, \dots, N-p-1, \\ \Phi^{(k)}(b) &= 0, \quad k = 0, 1, \dots, N-1, \end{aligned}$$

and that the functions $\Phi(t), \Phi'(t), \dots, \Phi^{(n-2n-2)}(t)$ are continuous in $[a, b]$, since $N - 2s_n - 2 = \min_{1 \leq \nu \leq n} (N - 2s_{\nu} - 2)$, while $\Phi^{(N-2s_n-1)}(t), \dots, \Phi^{(N-1)}(t)$ have discontinuities of the first kind at the points $\tau_1, \tau_2, \dots, \tau_n$. From (2.9) we conclude

$$(2.11) \quad (-1)^k \Phi^{(k)}(t) > 0 \quad \text{for } t \in (\tau_n, b), \quad k = 0, 1, \dots, N-1,$$

and, particularly, $\Phi(t) > 0$ on (τ_n, b) .

Let the weight function $w(t)$ be not identically zero in any interval contained in $[a, b]$. We will prove that the influence function $\Phi(t)$ is positive inside $[a, b]$. We give the proof for the case (2.2). (The general case can be considered in analogous way, see the Lobatto case.) If we identify the function $\Phi(t)$ as a monospline, then the property $\Phi(t) > 0$ on (a, b) is a corollary from the Micchelli estimate [9] of the number of zeros for monosplines with multiplicities (cf. Braess [p. 241, 1]). Our proof is direct and use only Rolle theorem.

We show that $\Phi^{(N-2s_l-2)}(t)$ ($l = 1, 2, \dots, n$) has at most $2s_l + 2$ zeros in each interval $[\tau_{\nu-1}, \tau_\nu]$, $\nu = 1, 2, \dots, n$. In fact, should it have $2s_l + 3$ of them, for the Rolle theorem, $\Phi^{(N-2s_l-1)}(t)$ would have at least $2s_l + 2$ zeros inside $[\tau_{\nu-1}, \tau_\nu]$, $\Phi^{(N-2s_l)}(t)$ would have at least $2s_l + 1$ zeros and so on, until we may conclude that $\Phi^{(N-1)}(t)$ would have at least two zeros inside $[\tau_{\nu-1}, \tau_\nu]$. But this is absurd since from (2.8) there follows that, for $t \in (\tau_{\nu-1}, \tau_\nu)$, we have

$$\Phi^{(N-1)}(t) = \varphi_\nu^{(N-1)}(t) = - \int_a^t w(\xi) d\xi + \sum_{j=0}^{\nu-1} A_{0,j}$$

and this function is decreasing (for the hypothesis on $w(t)$).

$\Phi^{(N-p)}(t)$ is continuous in $[a, \tau_1]$ and, we can prove as for $\Phi^{(N-2s_1-2)}(t)$, it has at most p zeros in $[a, \tau_1]$. $\Phi^{(N-2s_\nu-2)}(t)$ ($\nu = 1, 2, \dots, n$) is continuous in $[a, \tau_1]$ and let it have α zeros in $[a, \tau_1]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-p)}(t)$ has at least α zeros in (a, τ_1) . Since $\alpha \leq p$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ ($\nu = 1, 2, \dots, n$) has at most p zeros in $[a, \tau_1]$.

$\Phi^{(N-2s_1-2)}(t)$ is continuous in $[a, \tau_2]$ and has at most $p + (2s_1 + 2)$ zeros in $[a, \tau_2]$. $\Phi^{(N-2s_\nu-2)}(t)$ ($\nu = 2, \dots, n$) is continuous in $[a, \tau_2]$ and let it have α_1 zeros in $[a, \tau_2]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-2s_\nu-1)}(t)$ has at least α_1 zeros in (a, τ_2) , etc., $\Phi^{(N-2s_1-2)}(t)$ has at least α_1 zeros in (a, τ_2) . Since $\alpha_1 \leq p + (2s_1 + 2)$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ ($\nu = 2, \dots, n$) has at most $p + (2s_1 + 2)$ zeros in $[a, \tau_2]$.

$\Phi^{(N-2s_2-2)}(t)$ is continuous in $[a, \tau_3]$ and has at most $p + (2s_1 + 2) + (2s_2 + 2)$ zeros in $[a, \tau_3]$. $\Phi^{(N-2s_\nu-2)}(t)$ ($\nu = 3, \dots, n$) is continuous in $[a, \tau_3]$ and let it have α_2 zeros in $[a, \tau_3]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-2s_\nu-1)}(t)$ has at least α_2 zeros in (a, τ_3) , etc., $\Phi^{(N-2s_2-2)}(t)$ has at least α_2 zeros in (a, τ_3) . Since $\alpha_2 \leq p + (2s_1 + 2) + (2s_2 + 2)$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ ($\nu = 3, \dots, n$) has at most $p + (2s_1 + 2) + (2s_2 + 2)$ zeros in $[a, \tau_3]$.

In analogous way, we conclude that $\Phi^{(N-2s_n-2)}(t)$ is continuous in $[a, \tau_n]$ and has at most

$$p + \sum_{\nu=1}^{n-1} (2s_\nu + 2) = N - 2s_n - 2$$

zeros in $[a, \tau_n]$, and also in (a, b) , because of (2.11).

We may then show that $\Phi(t)$ does not vanish inside $[a, b]$ and therefore it is positive, because it is such on (x_n, b) . In fact, if $\Phi(t)$ should vanish at one point in

(a, b) , using (2.2) and (2.10) and applying Rolle theorem, we find that $\Phi'(t)$ would vanish at least two times, etc., $\Phi^{(N-2s_n-2)}(t)$ would vanish at least $N-2s_n-1$ times, in contraposition with the preceding deduction, because $N-2s_n-1 \leq N-2s_n-2$ gives $1 \leq 0$.

So, we proved the theorem:

THEOREM 2.2. *Under the hypothesis that the weight function $w(t)$ is not identically zero in any interval contained in $[a, b]$, the influence function $\Phi(t)$ defined by (2.4) (together with (2.5) and (2.6)) belongs to the class $C^{N-2s_k-2}[a, b]$, where $N-2s_k-2 = \min_{1 \leq \nu \leq n}(N-2s_\nu-2)$, and it is positive inside $[a, b]$.*

Now, we can estimate the remainder in the formulas of the type (2.1), by using (2.3).

1° If $f(t) \in AC^{N-1}[a, b]$ and $a, b \in R$ we have

$$|R(f)| \leq \max_{a \leq t \leq b} \Phi(t) V_{N-1} = \Phi(x_0) V_{N-1},$$

where V_{N-1} denotes the total variation of the function $f^{(N-1)}(t)$ absolutely continuous on the interval $[a, b]$. Because $\Phi'(t)$ vanish in exact one point of the interval (a, b) it holds $(\exists x_0 \in (a, b)) \max_{a \leq t \leq b} \Phi(t) = \Phi(x_0)$.

2° If $f^{(N)}(t)$ is bounded in $[a, b]$, i. e., $M_N = \sup_{a \leq t \leq b} |f^{(N)}(t)|$, $b < \infty$, then we have

$$|R(f)| \leq M_N \int_a^b \Phi(t) dt.$$

3° If $f \in C^N[a, b]$, $b < \infty$, because $\Phi(t) > 0$ on (a, b) we may apply the mean value theorem and write

$$R(f) = f^{(N)}(\xi) \int_a^b \Phi(t) dt, \quad \xi \in (a, b).$$

Lobatto formula. Let

$$(2.12) \quad \int_a^b w(t) f(t) dt = \sum_{i=0}^{p-1} A_{i,0}^L f^{(i)}(a) + \sum_{\nu=1}^n \sum_{i=0}^{2s_\nu} A_{i,\nu}^L f^{(i)}(\tau_\nu) + \sum_{i=0}^{q-1} A_{i,n+1}^L f^{(i)}(b) + R^L(f),$$

$-\infty < a < b < \infty$, $p, q \in \mathbb{N}$, with $R^L(f) = 0$ for $f \in \mathcal{P}_{2(\sum_{\nu=1}^n s_\nu + n) + p + q - 1}$, be the generalized Chakalov-Popoviciu quadrature formula of Lobatto type. Denote $N = 2(\sum_{\nu=1}^n s_\nu + n) + p + q$. Concerning the assumptions on $w(t), f(t)$ for the validity of (2.12) we have the following theorem:

THEOREM 2.3. *Formula (2.12) is valid under the following hypothesis:*

$$w(t) \in L[a, b], \quad f(t) \in AC^{N-1}[a, b].$$

The proof is the same as one of the theorem 4.13.I in [5, pp. 132–133] and will be omitted.

Let, for simplicity,

$$(2.13) \quad p-1 < 2s_1 < 2s_2 < \dots < 2s_n < q-1,$$

i. e.,

$$N-p-1 > N-2s_1-2 > \dots > N-2s_n-2 > N-q-1,$$

and, let $p+q$ be even, without loss of generality. Then, N is even.

Assuming already computed the nodes τ_ν and the coefficients $A_{i,\nu}^L$ for the remainder in (2.12) we have (see [5]):

$$(2.14) \quad R^L(f) \equiv R(f) = \int_a^b \Phi(t) f^{(N)}(t) dt,$$

where the influence-function $\Phi(t)$ is expressed by

$$(2.15) \quad \Phi(t) = \varphi_{\nu+1}(t) \quad \text{for } \tau_\nu < t < \tau_{\nu+1}, \quad \nu = 0, 1, \dots, n; \quad \tau_0 = a, \tau_{n+1} = b,$$

and the functions $\varphi_\nu(t)$, integrals of the differential equation $\varphi^{(N)}(t) = w(t)$ (since N is even), are given by the formulae

$$(2.16) \quad \begin{aligned} \varphi_\nu(t) = \int_a^t w(\xi) \frac{(t-\xi)^{N-1}}{(N-1)!} d\xi - \sum_{j=1}^{\nu-1} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-1}}{(N-i-1)!} \\ - \sum_{i=0}^{p-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-1}}{(N-i-1)!}, \end{aligned}$$

where $\nu = 1, \dots, n+1$, and $A_{h,j} = A_{h,j}^L$.

For $\varphi_{n+1}(t)$ we have

$$(2.17) \quad \varphi_{n+1}(t) = - \int_t^b w(\xi) \frac{(t-\xi)^{N-1}}{(N-1)!} d\xi + \sum_{i=0}^{q-1} (-1)^i A_{i,n+1} \frac{(t-b)^{N-i-1}}{(N-i-1)!}.$$

From (2.15), (2.16) it follows, differentiating k times (with $0 \leq k \leq N-1$):

$$\Phi^{(k)}(t) = \varphi_\nu^{(k)}(t) \quad \text{for } t \in (\tau_{\nu-1}, \tau_\nu), \quad \nu = 1, \dots, n+1,$$

where for $l = \nu, \nu - 1, \dots, 1$,

$$\begin{aligned} \varphi_\nu^{(k)}(t) &= \int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)!} d\xi - \sum_{j=1}^{l-1} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!} \\ &\quad - \sum_{j=l}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!} - \sum_{i=0}^{p-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)!}, \end{aligned}$$

with

(i) $0 \leq k \leq N - 2s_{\nu-1} - 2$, for $l = \nu$,

(ii) $N - 2s_l - 1 \leq k \leq N - 2s_{l-1} - 2$, for $l = \nu - 1, \nu - 2, \dots, 1$, where we put $s_0 = (p-1)/2$,

and

$$\begin{aligned} \varphi_\nu^{(k)}(t) &= \int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)!} d\xi - \sum_{j=1}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!} \\ &\quad - \sum_{i=0}^{N-k-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)!}, \end{aligned}$$

for $N - p \leq k \leq N - 1$.

For the derivatives of $\varphi_{n+1}(t)$ ($t \in (\tau_n, b)$) we can use the following formulas:

$$\varphi_{n+1}^{(k)}(t) = - \int_t^b w(\xi) \frac{(t-\xi)^{N-k-1}}{(n-k-1)!} d\xi + \sum_{h=0}^I (-1)^h A_{h,n+1} \frac{(t-b)^{N-h-k-1}}{(N-h-k-1)!},$$

with $I = q - 1$, for $0 \leq k \leq N - q - 1$; $I = N - k - 1$, for $N - q \leq k \leq N - 1$.

Now, we can conclude that

$$(2.18) \quad \begin{aligned} \Phi^{(k)}(a) &= 0, \quad k = 0, 1, \dots, N - p - 1, \\ \Phi^{(k)}(b) &= 0, \quad k = 0, 1, \dots, N - q - 1, \end{aligned}$$

and that the functions $\Phi(t), \Phi'(t), \dots, \Phi^{(N-2s_n-2)}(t)$ are continuous in $[a, b]$, since $N - 2s_n - 2 = \min_{1 \leq \nu \leq n} (N - 2s_\nu - 2)$, while $\Phi^{(n-2s_n-1)}(t), \dots, \Phi^{(n-1)}(t)$ have discontinuities of first kind at the points $\tau_1, \tau_2, \dots, \tau_n$.

The same conclusions can be derived for an arbitrary case $\sigma = (s_1, s_2, \dots, s_n)$, $p, q \in \mathbb{N}$, $s_\nu \in \mathbb{N}_0$ ($\nu = 1, 2, \dots, n$). So, we have just proved that the influence function $\Phi(t)$ defined by (2.15) (together with (2.16) and (2.17)) belongs to the class $C^{N-2s_k-2}[a, b]$, where $N - 2s_k - 2 = \min_{1 \leq \nu \leq n} (N - 2s_\nu - 2)$.

If we put $f(t) = (t-a)^p(b-t)^q \prod_{\nu=1}^n (t-\tau_\nu)^{2s_\nu+2}$ in (2.14), then

$$(-1)^q N! \int_a^b \Phi(t) dt = \int_a^b w(t) (t-a)^p (b-t)^q \prod_{\nu=1}^n (t-\tau_\nu)^{2s_\nu+2} dt.$$

So, we obtain that

$$\int_a^b \Phi(t) dt \begin{cases} < 0, & \text{if } q \text{ is odd,} \\ > 0, & \text{if } q \text{ is even.} \end{cases}$$

Therefore, if $\Phi(t)$ does not vanish in (a, b) it holds a sign on this interval.

Let the weight function $w(t)$ be not identically zero in any interval contained in $[a, b]$. We will prove that the influence function $\Phi(t)$ holds a sign in (a, b) .

We will give a proof for a sufficient general case. Then, proceed in analogous way, a proof for any other case can be performed. The consideration will be given in detail.

Let $n = 9$ and

$$q - 1 > 2s_1 > 2s_5 > 2s_4 > 2s_8 > p - 1 > 2s_9 > 2s_6 > 2s_3 > 2s_7 > 2s_2,$$

i. e.,

$$N - q - 1 < N - 2s_1 - 2 < \dots < N - 2s_2 - 2.$$

A point τ_ν , $\nu = 2, 3, \dots, n - 1$, we will call a *point of partition* of $[a, b]$ if for it holds $s_{\nu-1} \leq s_\nu > s_{\nu+1}$, i. e., $N - 2s_{\nu-1} - 2 \geq N - 2s_\nu - 2 < N - 2s_{\nu+1} - 2$. τ_1 is the *point of partition* of $[a, b]$ if for it holds $p - 1 \leq s_1 > s_2$. τ_n is the *point of partition* of $[a, b]$ if for it holds $s_{n-1} \leq s_n > q - 1$. Denote by I the *index set* whose elements are the indices of the points of partition of $[a, b]$. It is clearly that $I \subset \{1, 2, \dots, n\}$. Therefore, in our case, the points of partition of $[a, b]$ are τ_1, τ_5, τ_8 , and $I = \{1, 5, 8\}$. $[a, b]$ by the points of partition we divide into the *intervals of partition*, in our case $[a, \tau_1], [\tau_1, \tau_5], [\tau_5, \tau_8], [\tau_8, b]$, on which we consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, $\nu \in I$. It is clearly that $[a, b]$ can be represented as the union of the intervals of partition.

For $\nu \in I$, order in the decreasing sequence the values $N - 2s_\nu - 2$, and consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_8-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, $\Phi^{(N-2s_1-2)}(t)$, respectively.

a) Firstly, consider $\Phi^{(N-2s_8-2)}(t)$, which is continuous in $[\tau_5, b] = [\tau_5, \tau_8] \cup [\tau_8, b]$.

a.1) Firstly, consider $[\tau_5, \tau_8]$. For $\nu \in \{5, 6, 7, 8\}$ (the indices of the nodes belong to $[\tau_5, \tau_8]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last be one which correspond to the point of partition τ_8 , and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_7-2)}(t)$, $\Phi^{(N-2s_6-2)}(t)$, $\Phi^{(N-2s_8-2)}(t)$, respectively.

$\Phi^{(N-2s_7-2)}(t)$ is continuous in $[\tau_6, \tau_8] = [\tau_6, \tau_7] \cup [\tau_7, \tau_8]$ and has at most $(2s_7 + 2) + (2s_7 + 2)$ zeros in it.

$\Phi^{(N-2s_6-2)}(t)$ is continuous in $[\tau_5, \tau_8] = [\tau_5, \tau_6] \cup [\tau_6, \tau_8]$. Let $\Phi^{(N-2s_6-2)}(t)$ have β_6 zeros in $[\tau_6, \tau_8]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_6-1)}(t)$ has at least $\beta_6 - 1$ zeros in (τ_6, τ_8) , etc., $\Phi^{(N-2s_7-2)}(t)$ has at least $\beta_6 - (2s_6 - 2s_7)$ zeros in (τ_6, τ_8) . Therefore, we have $\beta_6 - (2s_6 - 2s_7) \leq (2s_7 + 2) + (2s_7 + 2)$, i. e., $\beta_6 \leq (2s_6 + 2) + (2s_7 + 2)$.

Therefore, $\Phi^{(N-2s_6-2)}(t)$ has at most $(2s_6 + 2) + (2s_6 + 2) + (2s_7 + 2)$ zeros in $[\tau_5, \tau_8]$.

Let $\Phi^{(N-2s_8-2)}(t)$ have α_8 zeros in $[\tau_5, \tau_8]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_8-1)}(t)$ has at least $\alpha_8 - 1$ zeros in (τ_5, τ_8) , etc., $\Phi^{(N-2s_6-2)}(t)$ has at least $\alpha_8 - (2s_8 - 2s_6)$ zeros in (τ_5, τ_8) . Therefore, we have $\alpha_8 - (2s_8 - 2s_6) \leq (2s_6 + 2) + (2s_6 + 2) + (2s_7 + 2)$, i. e., $\alpha_8 \leq (2s_6 + 2) + (2s_7 + 2) + (2s_8 + 2)$.

a.2) Consider $[\tau_8, b]$. For $\nu \in \{8, 9\}$ (the indices of the nodes belong to $[\tau_8, b]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last be one which correspond to the point of partition τ_8 , and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_9-2)}(t)$, $\Phi^{(N-2s_8-2)}(t)$, respectively.

$\Phi^{(N-2s_9-2)}(t)$ is continuous in $[\tau_8, b] = [\tau_8, \tau_9] \cup [\tau_9, b]$ and has at most $(2s_9 + 2) + (2s_9 + 2)$ zeros in it.

Let $\Phi^{(N-2s_8-2)}(t)$ have β_8 zeros in $[\tau_8, b]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_8-1)}(t)$ has at least $\beta_8 - 1$ zeros in (τ_8, b) , etc., $\Phi^{(N-2s_9-2)}(t)$ has at least $\beta_8 - (2s_8 - 2s_9)$ zeros in (τ_8, b) . Therefore, we have $\beta_8 - (2s_8 - 2s_9) \leq (2s_9 + 2) + (2s_9 + 2)$, i. e., $\beta_8 \leq (2s_8 + 2) + (2s_9 + 2)$.

Therefore, using a.1), a.2), we conclude that $\Phi^{(N-2s_8-2)}(t)$ has at most $(2s_8 + 2) + \sum_{\nu=6}^9 (2s_\nu + 2)$ zeros in $[\tau_5, b]$.

b) Now, consider $\Phi^{(N-2s_5-2)}(t)$, which is continuous in $[\tau_1, b] = [\tau_1, \tau_5] \cup [\tau_5, b]$.

b.1) Consider $[\tau_1, \tau_5]$. For $\nu \in \{1, 2, 3, 4, 5\}$ (the indices of the nodes belong to $[\tau_1, \tau_5]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last be one which correspond to the point of partition τ_5 , and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_2-2)}(t)$, $\Phi^{(N-2s_3-2)}(t)$, $\Phi^{(N-2s_4-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, respectively.

$\Phi^{(N-2s_2-2)}(t)$ is continuous in $[\tau_1, \tau_3] = [\tau_1, \tau_2] \cup [\tau_2, \tau_3]$ and has at most $(2s_2 + 2) + (2s_2 + 2)$ zeros in it.

$\Phi^{(N-2s_3-2)}(t)$ is continuous in $[\tau_1, \tau_4] = [\tau_1, \tau_3] \cup [\tau_3, \tau_4]$.

Let $\Phi^{(N-2s_3-2)}(t)$ have α_3 zeros in $[\tau_1, \tau_3]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_3-1)}(t)$ has at least $\alpha_3 - 1$ zeros in (τ_1, τ_3) , etc., $\Phi^{(N-2s_2-2)}(t)$ has at least $\alpha_3 - (2s_3 - 2s_2)$ zeros in (τ_1, τ_3) . Therefore, we have $\alpha_3 - (2s_3 - 2s_2) \leq (2s_2 + 2) + (2s_2 + 2)$, i. e., $\alpha_3 \leq (2s_2 + 2) + (2s_3 + 2)$.

Therefore, $\Phi^{(N-2s_3-2)}(t)$ has at most $(2s_2 + 2) + (2s_3 + 2) + (2s_3 + 2)$ zeros in $[\tau_1, \tau_4]$.

$\Phi^{(N-2s_4-2)}(t)$ is continuous in $[\tau_1, \tau_5] = [\tau_1, \tau_4] \cup [\tau_4, \tau_5]$.

Let $\Phi^{(N-2s_4-2)}(t)$ have α_4 zeros in $[\tau_1, \tau_4]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_4-1)}(t)$ has at least $\alpha_4 - 1$ zeros in (τ_1, τ_4) , etc., $\Phi^{(N-2s_3-2)}(t)$ has at least $\alpha_4 - (2s_4 - 2s_3)$ zeros in (τ_1, τ_4) . Therefore, we have $\alpha_4 - (2s_4 - 2s_3) \leq (2s_2 + 2) + (2s_3 + 2) + (2s_3 + 2)$, i. e., $\alpha_4 \leq (2s_2 + 2) + (2s_3 + 2) + (2s_4 + 2)$.

Therefore, $\Phi^{(N-2s_4-2)}(t)$ has at most $\sum_{\nu=2}^4 (2s_\nu + 2) + (2s_4 + 2)$ zeros in $[\tau_1, \tau_5]$.

Finally, let $\Phi^{(N-2s_5-2)}(t)$ have α_5 zeros in $[\tau_1, \tau_5]$. As above, we conclude that $\alpha_5 \leq \sum_{\nu=2}^5 (2s_\nu + 2)$.

b.2) Let $\Phi^{(N-2s_5-2)}(t)$ have β_5 zeros in $[\tau_5, b]$. In analogous way as above, by

using the conclusions from a), we conclude that $\beta_5 \leq \sum_{\nu=5}^9 (2s_\nu + 2)$.

Therefore, $\Phi^{(N-2s_5-2)}(t)$ has at most $\sum_{\nu=2}^9 (2s_\nu + 2) + (2s_5 + 2)$ zeros in $[\tau_1, b]$.

c) Finally, consider $\Phi^{(N-2s_1-2)}(t)$, which is continuous in $[a, b] = [a, \tau_1] \cup [\tau_1, b]$.

c.1) Consider $[a, \tau_1]$. $\Phi^{(N-p)}(t)$ has at most p zeros in $[a, \tau_1]$. Let $\Phi^{(N-2s_1-2)}(t)$ have α_1 zeros in $[a, \tau_1]$. Then, using (2.18) (for the point a) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_1-1)}(t)$ has at least α_1 zeros in (a, τ_1) , etc., $\Phi^{(N-p)}(t)$ has at least α_1 zeros in (a, τ_1) . Therefore, $\alpha_1 \leq p$.

c.2) By using the conclusions for $\Phi^{(N-2s_5-2)}(t)$, from b), in analogous way as above, we conclude that $\Phi^{(N-2s_1-2)}(t)$ has at most $\sum_{\nu=1}^9 (2s_\nu + 2)$ zeros in $[\tau_1, b]$.

Therefore, on the basis of c.1), c.2), we conclude that $\Phi^{(N-2s_1-2)}(t)$ has at most $p + \sum_{\nu=1}^9 (2s_\nu + 2) = N - q$ zeros in $[a, b]$.

We may then show that $\Phi(t)$ does not vanish inside $[a, b]$ and therefore holds a sign in it. In fact, if $\Phi(t)$ should vanish at one point in (a, b) , using (2.18) and applying Rolle theorem, we find that $\Phi'(t)$ would vanish at least two times, $\Phi^{(N-q-1)}(t)$ would vanish at least $N - q$ times, $\Phi^{(N-q)}(t)$ would vanish at least $N - q + 1$ times, etc., $\Phi^{(N-2s_1-2)}(t)$ would vanish at least $N - q + 1$ times, in contraposition with the preceding deduction, because $N - q + 1 \leq N - q$ gives $1 \leq 0$.

On the basis of the upper considerations we have just proved the following statement:

THEOREM 2.4. *Under the hypothesis that the weight function $w(t)$ is not identically zero in any interval contained in $[a, b]$, the influence function $\Phi(t)$ defined by (2.15) (together with (2.16) and (2.17)) belongs to the class $C^{n-2s_k-2}[a, b]$, where $N - 2s_k - 2 = \min_{1 \leq \nu \leq n} (N - 2s_\nu - 2)$, and one holds a sign inside $[a, b]$.*

Now, we can estimate the remainder in the formulas of the type (2.12), by using (2.14).

1^o If $f(t) \in AC^{N-1}[a, b]$ we have

$$|R(f)| \leq \max_{a \leq t \leq b} |\Phi(t)| V_{N-1} = |\Phi(x_0)| V_{N-1},$$

where V_{N-1} denotes the total variation of the function $f^{(N-1)}(t)$ absolutely continuous on the interval $[a, b]$. Because $\Phi'(t)$ vanish in exact one point of the interval (a, b) it holds $(\exists x_0 \in (a, b)) \max_{a \leq t \leq b} \Phi(t) = \Phi(x_0)$.

2^o If $f^{(N)}(t)$ is bounded in $[a, b]$, i. e., $M_N = \sup_{a \leq t \leq b} |f^{(N)}(t)|$, then we have

$$|R(f)| \leq M_N \int_a^b |\Phi(t)| dt.$$

3^o If $f \in C^N[a, b]$, because $\Phi(t)$ holds a sign on (a, b) we may apply the mean value theorem and write

$$R(f) = f^{(N)}(\xi) \int_a^b \Phi(t) dt, \quad \xi \in (a, b).$$

Example. An iterative process for computing the coefficients of s -orthogonal polynomials in a special case, when the interval $[a, b]$ is symmetric with respect to the origin and the weight function w is an even function, was proposed by Vincenti [25]. He applied his process to the Legendre case. When n and s increase, the process becomes numerically unstable.

In [10] (see also [4]) a stable procedure for numerical construction of s -orthogonal polynomials with respect to $d\varphi(t)$ on \mathbb{R} is given.

Recently, a simple and numerically stable procedure for construction of σ -orthogonal polynomials is proposed by Milovanović and Spalević [14].

A stable numerical procedure for calculating the coefficients $A_{i,\nu}$ in (1.1) was given by Gautschi and Milovanović [4]. Some alternative methods were proposed by Stroud and Stancu [23], Golub and Kautsky [7], and Milovanović and Spalević [12] (see also [21]). A generalization of methods, for the weights, from [4, 12] to the general case when $s_\nu \in \mathbb{N}_0$, $\nu = 1, \dots, n$, was derived recently by Milovanović and Spalević [13].

Finally, a method for calculating the nodes and the coefficients in the generalized Chakalov-Popoviciu quadrature formulae of Radau and Lobatto type, by using the results from [13, 14], has been proposed in [22]. We use that method, for calculating in this example, in order to tabulate the corresponding influence function. Consider the Legendre case with $w(t) = 1$ on $[-1, 1]$. Let $p = q = 1$ and $\sigma = (1, 0, 1)$ in (2.12). Therefore, we have a symmetric task. The results show that the nodes of the corresponding quadrature are symmetrically distributed with respect to the origin, namely,

$$\tau_1 = -\tau_3 = -0.66772435790692, \quad \tau_2 = 0.$$

The coefficients of the corresponding Chakalov-Popoviciu quadrature formula of Lobatto type are:

$$\begin{aligned} A_{0,1} = A_{0,3} &= 6.68946557387391(-01), & A_{1,1} = -A_{1,3} &= 2.90757109134605(-02), \\ A_{2,1} = A_{2,3} &= 8.27917955975223(-03), & A_{0,2} &= 5.47406124470793(-01), \\ A_{0,0} = A_{0,4} &= 5.7350380377213(-02). \end{aligned}$$

The results show that the influence function is even (see Table), and

$$\max_{-1 \leq t \leq 1} |\Phi(t)| = -\Phi(0) = 7.06(-12).$$

TABLE

t	∓ 1	$\pm \tau_1$	∓ 0.5	∓ 0.4	∓ 0.3	∓ 0.2	∓ 0.1
$\Phi(t)$	0	-4.05(-15)	-1.94(-13)	-8.23(-13)	-2.23(-12)	-4.31(-12)	-6.25(-12)

Therefore, these results can be use in estimations of the remainder given above for the corresponding Chakalov-Popoviciu quadrature formula of Lobatto type.

All computations were done using FORTRAN in double precision arithmetics. The numbers in parentheses denoted the decimal exponents.

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