

**ON THE CONVERGENCE OF  
FINITE-DIFFERENCE SCHEME  
FOR A NONLOCAL ELLIPTIC  
BOUNDARY VALUE PROBLEM**

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ABSTRACT. The finite-difference scheme approximating nonlocal boundary value problem for a second order elliptic equation is studied. A convergence rate estimate in discrete  $W_2^1$ -norm is obtained, assuming that the coefficients and the solution to the original problem belongs to Sobolev spaces.

**1. Introduction**

The generalization of Bitsadze–Samarski problem [1] was investigated by many authors (see e.g., [2–6]). In [5] for the Poisson equation is considered a difference scheme, which converges by the rate  $O(h^2)$  in the discrete  $W_2^2$ -norm to the exact solution from the class  $C^4(\bar{\Omega})$ .

In the present paper a nonlocal boundary value problem of Bitsadze–Samarski type is considered in a domain  $\Omega = (0, 1)^2$  for a second order elliptic equation with variable coefficients. The investigation of the corresponding difference scheme is carried out in Sobolev weight space and under assumption that the coefficients and the solution to the original problem belong to Sobolev spaces, the estimate of convergence rate

$$(1) \quad \|y - u\|_{W_2^1(\omega, r)} \leq ch^{s-1} \|u\|_{W_2^s(\Omega)}, \quad s \in (1; 3]$$

is obtained, where  $r = r(x_1) = 1 - x_1$ ,  $\omega$  is a uniform grid in  $\Omega$  with the step  $h$ .

The main idea is to introduce an auxiliary (equivalent to  $r$ ) weight function  $\rho(x_1)$ , which gives possibility to state the positive definitivity of the difference scheme operator, and validity of the first (energetic) fundamental inequality too. The inner product of the indicated type and induced by it norm were firstly in [2] to

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prove uniqueness of classic, and afterwards also difference solution [4] of nonlocal boundary value problems.

## 2. The problem and its approximation

Let  $\Omega = \{(x_1, x_2) : 0 < x_k < 1, k = 1, 2\}$  be a unit square with a boundary  $\Gamma$ ;  $\alpha_1, \alpha_2, \dots, \alpha_m$  arbitrary real numbers;  $\xi_1, \xi_2, \dots, \xi_m$  fixed points from  $(0, 1)$ ; note that  $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1, \xi_0 = 0, \xi_{m+1} = 1$ .

$$\Gamma_{(i)} = \{(\xi_i, x_2) : 0 < x_2 < 1\}, i = 1, \dots, m+1, \Gamma_* = \Gamma_{(m+1)}, \Gamma_0 = \Gamma \setminus \Gamma_*.$$

Consider the nonlocal boundary value problem

$$(2) \quad Lu \equiv \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) - au = f(x), \quad x \in \Omega,$$

$$(3) \quad u(x) = 0, \quad x \in \Gamma_0, \quad u(1, x_2) = \sum_{k=1}^m \alpha_k u(\xi_k, x_2), \quad 0 < x_2 < 1.$$

We assume that the problem (2), (3) with the right-hand side  $f \in W_2^{s-2}(\Omega)$ , is uniquely solvable in  $W_2^s(\Omega)$ ,  $1 < s \leq 3$ , and the coefficients  $a_{1j} = a_{1j}(x_{3-j})$ ,  $a_{2j} = a_{2j}(x)$  ( $j = 1, 2$ ), and  $a = a(x)$  satisfy to the following conditions:

$$(4) \quad \sum_{i,j=1}^2 a_{ij} t_i t_j \geq \nu_1 (t_1^2 + t_2^2), \quad \nu_1 = \text{const} > 0, \quad a \geq 0.$$

$$a_{1j} \in W_p^{s-1}(0; 1), \quad p > \max(1/(s-1), 2) \text{ for } s \in (1; 2], \quad p = 2 \text{ for } s \in (2; 3],$$

$$a_{2j} \in W_q^{s-1}(\Omega), \quad q > 2/(s-1) \text{ for } s \in (1; 2], \quad q = 2 \text{ for } s \in (2; 3],$$

$$a \in L_{2+\varepsilon}(\Omega), \quad 1 < s \leq 2, \quad \varepsilon > 0, \quad a \in W_2^{s-2}(\Omega), \quad 2 < s \leq 3.$$

Consider the following grid domains in  $\bar{\Omega}$ :

$$\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h : i_\alpha = 0, 1, \dots, n, h = 1/n\},$$

$$\omega_\alpha = \bar{\omega}_\alpha \cap (0, 1), \quad \omega_\alpha^+ = \bar{\omega}_\alpha \cap (0; 1], \quad \alpha = 1, 2,$$

$$\omega = \omega_1 \times \omega_2, \quad \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2, \quad \gamma_0 = \Gamma_0 \cap \bar{\omega}.$$

For grid functions and difference ratios, we use the standard notation from [7]:

$$v_{x_i} = (v^{(+1i)} - v)/h, \quad v_{\bar{x}_i} = (v - v^{(-1i)})/h,$$

where  $v^{(\pm 1i)}(x) = v(x_1 \pm h, x_2)$ ,  $v^{(\pm 12)}(x) = v(x_1, x_2 \pm h)$ . The notations  $v^{(\pm 0.5i)}$ ,  $v^{(+0.5i, -1i)}$  will have similar sense. Let

$$\xi_k = (n_k + \theta_k)h, \quad 0 \leq \theta_k < 1, \quad k = 1, 2, \dots, m,$$

$$\omega_{1,k} = \{x_1 : x_1 = ih, i = 1, 2, \dots, n_k\}.$$

where  $n_k$  are nonnegative integers  $0 \leq n_1 \leq n_2 \leq \dots \leq n_m < n$ , the equality between which will take place if in the corresponding subinterval (between the neighboring points of the grid  $\omega_1$ ) more than one point  $\xi_k$  is situated. Suppose

$$(5) \quad h/2 \leq 1 - \xi_m - \nu, \quad \nu = \text{const} > 0.$$

Let

$$G_k u = (1 - \theta_k) \int_0^{\theta_k} t u(n_k h + th, x_2) dt + \theta_k \int_{\theta_k}^1 (1 - t) u(n_k h + th, x_2) dt,$$

$k = 1, 2, \dots, n_m$ . We also need the following averaging operators:

$$\begin{aligned} S_1^- u &= \int_{-1}^0 u(x_1 + th, x_2) dt, \\ S_1^+ u &= \int_0^1 u(x_1 + th, x_2) dt, \\ T_1 u &= \int_{-1}^1 (1 - |t|) u(x_1 + th, x_2) dt. \end{aligned}$$

The operators  $S_2^-$  and  $T_2$  are defined likewise. For these operators the following relations hold:

$$\begin{aligned} T_k \frac{\partial^2 u}{\partial x_k^2} &= u_{\bar{x}_k x_k}, \quad T_k \frac{\partial u}{\partial x_k} = S_k^- u_{x_k}, \quad k = 1, 2, \\ G_k \frac{\partial^2 u}{\partial x_1^2} &= \frac{1}{h^2} ((1 - \theta_k) u(n_k h, x_2) + \theta_k u(n_k h + h, x_2) - u(\xi_k, x_2)), \quad k = 1, 2, \dots, n_m. \end{aligned}$$

By  $Y_k(x_2)$ ,  $Z_k(x_2)$ ,  $\bar{Z}_k(x_2)$ ,  $\tilde{Z}(x_2)$  we will denote the expressions

$$Y_k(x_2) = (1 - \theta_k) y(n_k h, x_2) + \theta_k y(n_k h + h, x_2) \quad \text{etc.}$$

We approximate the problem (2), (3) by finite-difference scheme

$$(6) \quad Ay \equiv \sum_{i,j=1}^2 A_{ij} y + ay = -\varphi(x), \quad x \in \omega, \quad \varphi = T_1 T_2 f,$$

$$(7) \quad y = 0, \quad x \in \gamma_0, \quad y(1, x_2) = \sum_{k=1}^m \alpha_k Y_k(x_2), \quad x_2 \in \omega_2,$$

where  $A_{ij} y = -0.5(a_{ij}^{(-0.5, 1)} y_{\bar{x}_j})_{x_i} - 0.5(a_{ij}^{(+0.5, 1)} y_{x_j})_{\bar{x}_i}$ .

### 3. The first fundamental inequality

Let  $H$  be the set of all grid functions, defined on  $\bar{\omega}$  and vanishing on  $\gamma_0$ , with the inner product and the norm:

$$(y, v)_r = \sum_{\omega} h^2 r(x_1) y(x) v(x), \quad \|y\|_r^2 = (y, y)_r, \quad r(x_1) = 1 - x_1.$$

Further, we put

$$\begin{aligned} \|y\|_r^2 &= \sum_{\omega_1^+ \times \omega_2} h^2 \bar{r} y^2, \quad \|y\|_r^2 = \sum_{\omega_1 \times \omega_2^+} h^2 r y^2, \\ |y|_{1,\omega,r}^2 &= \|y_{\bar{x}_1}\|_r^2 + \|y_{\bar{x}_2}\|_r^2, \quad \|y\|_{1,\omega,r}^2 = \|y\|^2 + |y|_{1,\omega,r}^2, \\ \|y\|^2 &= \sum_{\omega_1^+ \times \omega_2} \bar{h} h y^2, \quad \|y\|_0^2 = \sum_{\omega} h^2 y^2, \quad \|y\|^2 = \sum_{\omega_1^+ \times \omega_2} h^2 y^2, \quad \|y\|^2 = \sum_{\omega_1 \times \omega_2^+} h^2 y^2, \\ \|y\|_*^2 &= \sum_{\omega_2} h y^2, \quad \|y\|_*^2 = \sum_{\omega_2^+} h y^2, \quad \bar{r} = r + \frac{h}{2}, \quad \bar{h} = h \text{ for } x_1 \in \omega_1, \quad \bar{h} = \frac{h}{2} \text{ for } x_1 = 1. \end{aligned}$$

Define the following weight function

$$(8) \quad \rho(x_1) = \begin{cases} \rho_i(x_1), & \xi_i \leq x_1 < \xi_{i+1}, \quad i = 0, 1, 2, \dots, m-1, \\ r(x_1), & \xi_m \leq x_1 \leq 1, \end{cases}$$

where

$$\rho_i(x_1) = r(x_1) - \varkappa \sum_{k=i+1}^m \sigma_k r_k(x_1), \quad r_k(x_1) = \xi_k - x_1, \quad \varkappa = \sum_{k=1}^m |\alpha_k| \sqrt{\xi_k}, \quad \sigma_k = \frac{|\alpha_k|}{\sqrt{\xi_k}}.$$

Suppose that  $\varkappa < 1$ . Then [6]

$$(9) \quad (1 - \varkappa^2)r(x_1) \leq \rho(x_1) \leq r(x_1).$$

In future we will consider that the inner product and norms, involving  $\rho$  in index have similar to the expression with index  $r$  sense.

In order to use below the results obtained in this section, for a priori estimate of the error of method (when the nonlocal condition will not be homogeneous any more) we will get the estimates for the function  $y(x)$  in such form, in which the nonlocal condition still will not be taken into account.

LEMMA 1. *For any  $y \in H$  the following estimates are valid:*

$$(10) \quad \|Y_k\|_*^2 \leq (\xi_k / \nu) \|y_{\bar{x}_1}\|_r^2,$$

$$(11) \quad \|y\| \leq \|y_{\bar{x}_1}\|_r,$$

$$(12) \quad \|y\|_\rho \leq (1/\sqrt{8}) \|y_{\bar{x}_2}\|_r.$$

PROOF. By virtue of the Cauchy–Buniakovski inequality and via (5) the estimate (10) follows from

$$Y_k(x_2) = \sum_{i=1}^{n_k+1} h \tau_k(i) y_{\bar{x}_1}(ih, x_2), \quad \tau_k(i) = 1 \text{ for } i = 1, 2, \dots, n_k, \quad \tau_k(n_k + 1) = \theta_k$$

taking into account that  $\tau_k^2 \leq \tau_k$ ,  $\bar{r}(ih) > \nu$ ,  $i = 1, 2, \dots, n_m + 1$ . Next,

$$\begin{aligned} \|y\|^2 &= \sum_{i=1}^n \sum_{j=1}^i \sum_{\omega_2} \bar{h}(i) h^2 y(ih, x_2) y_{\bar{x}_1}(jh, x_2) \\ &\leq \left( \sum_{i=1}^n \sum_{j=1}^i \sum_{\omega_2} \bar{h}(i) h^2 y^2(ih, x_2) \right)^{1/2} \left( \sum_{i=1}^n \sum_{j=1}^i \sum_{\omega_2} \bar{h}(i) h^2 y_{\bar{x}_1}^2(jh, x_2) \right)^{1/2} \\ &\leq \|y\| \|y_{\bar{x}_1}\|_r, \end{aligned}$$

which yields (11).

The estimate (12) follows from (see e.g., [7, p.120])  $\|y\|_*^2 \leq (1/8)\|y_{\bar{x}_2}\|_*^2$ , noting that  $\rho \leq r$ .  $\square$

Denote

$$(13) \quad \Phi(a_{11}, y) = \frac{1}{2} \sum_{\omega_2} h a_{11}(x_2) \left( \varkappa \sum_{k=1}^m \sigma_k Y_k^2(x_2) - y^2(1, x_2) \right).$$

The following estimate (so called ‘‘first fundamental inequality’’ in terminology used by Ladyzhenskaya) is valid.

LEMMA 2. *If  $y(x) \in H$ , then*

$$(14) \quad (Ay, y)_\rho \geq c_1 \|y\|_{1, \omega, r}^2 + \Phi(a_{11}, y), \quad c_1 = \nu_1(1 - \varkappa^2)/2.$$

*If moreover,  $y(x)$  satisfies the nonlocal condition (7), then in the right-hand side (14) the second term may be neglected.*

PROOF. Partial summation yields

$$(15) \quad -2 \sum_{\omega_1} h \rho y_{\bar{x}_1 x_1} y = \sum_{\omega_1^+} h \rho y_{\bar{x}_1}^2 + \sum_{\omega_1^-} h \rho y_{x_1}^2 - y^2(1, x_2) - \sum_{\omega_1} h y^2 \rho_{\bar{x}_1 x_1}.$$

Represent now the weight function  $\rho$  in the following way

$$\rho(x_1) = 1 - x_1 - \sum_{k=1}^m \varkappa \sigma_k \chi(\xi_k - x_1), \quad \chi(t) = \begin{cases} t, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

It is not hard to check that  $h \chi_{\bar{x}_1 x_1}(\xi_k - ih) = (1 - \theta_k) \delta(n_k, i) + \theta_k \delta(n_k + 1, i)$ , where  $\delta(\cdot, \cdot)$  is the Kronecker delta. Consequently,

$$h \rho_{\bar{x}_1 x_1}(ih) = -\varkappa \sum_{k=1}^m \sigma_k ((1 - \theta_k) \delta(n_k, i) + \theta_k \delta(n_k + 1, i)),$$

and

$$(16) \quad \sum_{\omega_1} h y^2 \rho_{\bar{x}_1 x_1} = - \sum_{k=1}^m \varkappa \sigma_k ((1 - \theta_k) y^2(n_k h, x_2) + \theta_k y^2(n_k h + h, x_2)).$$

Taking into account that

$$\begin{aligned} (1 - \theta_k) y^2(n_k h, x_2) + \theta_k y^2(n_k h + h, x_2) &= Y_k^2(x_2) + h^2 \theta_k (1 - \theta_k) y_{x_1}^2(n_k h, x_2) \\ &\geq Y_k^2(x_2), \end{aligned}$$

from (15), (16) we get

$$(A_{11}y, y)_\rho \geq \frac{1}{2} \sum_{\omega_1^+ \times \omega_2} h^2 \rho a_{11} y_{\bar{x}_1}^2 + \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \rho a_{11} y_{x_1}^2 + \Phi(a_{11}, y).$$

Furthermore, by using partial summation we have

$$(A_{12}y, y)_\rho = \frac{1}{2} \sum_{\omega_1^+ \times \omega_2} h^2 \rho a_{12}^{(-0.51)} y_{\bar{x}_1} y_{\bar{x}_2} + \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \rho a_{12}^{(+0.51)} y_{x_1} y_{x_2},$$

$$(A_{2j}y, y)_\rho = \frac{1}{2} \sum_{\omega_1 \times \omega_2^+} h^2 \rho a_{2j}^{(-0.51)} y_{\bar{x}_j} y_{\bar{x}_2} + \frac{1}{2} \sum_{\omega_1 \times \omega_2^-} h^2 \rho a_{2j}^{(+0.51)} y_{x_j} y_{x_2}, \quad j = 1, 2.$$

Consequently,

$$(Ay, y)_\rho \geq \frac{1}{2} \sum_{\omega^+} h^2 \rho \sum_{i,j=1,2}^2 a_{ij}^{(-0.51)} y_{\bar{x}_i} y_{\bar{x}_j} + \frac{1}{2} \sum_{\omega^-} h^2 \rho \sum_{i,j=1,2}^2 a_{ij}^{(+0.51)} y_{x_i} y_{x_j} + (a, y^2)_\rho + \Phi(a_{11}, y).$$

Therefore (4), (9) implies  $(Ay, y)_\rho \geq \nu_1(1 - \varkappa^2) |y|_{1, \omega, r}^2 + \Phi(a_{11}, y)$ , and taking into account (11), we finally obtain (14).

If  $y(x)$  satisfies the nonlocal condition (7) too, then

$$y^2(1, x_2) \leq \sum_{k=1}^m \varkappa \sigma_k Y_k^2(x_2)$$

and  $\Phi(a_{11}, y) \geq 0$ . This completes the proof of the lemma.  $\square$

Due to Lemma 2 the problem (6), (7) has an unique solution.

#### 4. A priori estimate of error of difference solution

Let  $u$  be a solution of the problem (2), (3) and  $y$  – a solution of the difference scheme (6), (7). Then for the error  $z = y - u$  we obtain the problem

$$(17) \quad Az = \psi, \quad x \in \omega, \quad z = 0, \quad x \in \gamma_0, \quad z(1, x_2) = \sum_{k=1}^m \alpha_k Z_k + R, \quad x_2 \in \omega_2,$$

where

$$\psi = \sum_{i,j=1}^2 (\eta_{ij})_{x_i} + \eta, \quad R = \sum_{k=1}^m \alpha_k R_k, \quad R_k = h^2 G_k \frac{\partial^2 u}{\partial x_1^2}, \quad \eta = T_1 T_2 (au) - T_1 T_2 a u$$

$$\eta_{ij} = \frac{1}{2} a_{ij}^{(-0.51)} u_{\bar{x}_j} + \frac{1}{2} a_{ij}^{(+0.51, -1_i)} u_{x_j}^{(-1_i)} - S_i^- T_{3-i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right), \quad i, j = 1, 2.$$

LEMMA 3. *The a priori estimate*

$$(18) \quad \|z\|_{1, \omega, r} \leq c(\|a_{11} R\|_* + \|\eta_{11}\| + \|\eta_{12}\| + \|\eta_{21}\| + \|\eta_{22}\| + \|\eta\|_0).$$

is valid for the solution of the problem (17).

PROOF. Represent the solution  $z$  to the problem (17) as a sum  $z = \bar{z} + \tilde{z}$  of solutions of the following two problems:

$$(19) \quad A\bar{z} = 0, \quad x \in \omega, \quad \bar{z} = 0, \quad x \in \gamma_0, \quad \bar{z}(1, x_2) = \sum_{k=1}^m \alpha_k \bar{Z}_k + R, \quad x_2 \in \omega_2,$$

$$(20) \quad A\tilde{z} = \psi, \quad x \in \omega, \quad \tilde{z} = 0, \quad x \in \gamma_0, \quad \tilde{z}(1, x_2) = \sum_{k=1}^m \alpha_k \tilde{Z}_k, \quad x_2 \in \omega_2.$$

It follows from nonlocal condition (19) that

$$-2\Phi(a_{11}, \bar{z}) \leq \sum_{\omega_2} h a_{11}(x_2) \left( R^2 + 2R \sum_{k=1}^m \alpha_k \bar{Z}_k \right),$$

therefore (10) implies  $-2\Phi(a_{11}, \bar{z}) \leq \|a_{11}R\|_*^2 + (2\kappa/\sqrt{\nu})\|a_{11}R\|_*\|\bar{z}_{\bar{x}_1}\|_r$ , and we obtain

$$(21) \quad -2\Phi(a_{11}, \bar{z}) \leq (1 + \kappa/(\varepsilon_1\sqrt{\nu}))\|a_{11}R\|_*^2 + (\kappa\varepsilon_1/\sqrt{\nu})\|\bar{z}_{\bar{x}_1}\|_{1,\omega,r}^2, \quad \forall \varepsilon_1 > 0.$$

Further, applying Lemma 2 we conclude that  $2c_1\|\bar{z}\|_{1,\omega,r}^2 + 2\Phi(a_{11}, \bar{z}) \leq 0$ , the addition of which to (21) (with  $\varepsilon_1$  chosen properly) gives

$$(22) \quad \|\bar{z}\|_{1,\omega,r}^2 \leq c_2\|a_{11}R\|_*^2.$$

On the other hand, applying Lemma 2 to the solution of the problem (20) we come to

$$(23) \quad c_1\|\tilde{z}\|_{1,\omega,r}^2 \leq \sum_{i,j=1}^2 ((\eta_{ij})_{x_i}, \tilde{z})_\rho + (\eta, \tilde{z})_\rho$$

Since

$$\begin{aligned} \sum_{\omega_{1,k} \times \omega_2} h^2 r_k \eta_{1\beta x_1} \tilde{z} &= - \sum_{\omega_{1,k} \times \omega_2} h^2 r_k \eta_{1\beta} \tilde{z}_{\bar{x}_1} \\ &\quad + \sum_{i=1}^{n_k} \sum_{j=1}^i \sum_{\omega_2} h^3 \tau_k(i+1) \eta_{1\beta}(ih+h, x_2) \tilde{z}_{\bar{x}_1}(jh, x_2) \end{aligned}$$

using the Cauchy–Buniakovski inequality we get

$$\left| \sum_{\omega_{1,k} \times \omega_2} h^2 r_k \eta_{1\beta x_1} \tilde{z} \right| \leq 2\xi_k \|\tilde{z}_{\bar{x}_1}\|_r \|\eta_{1\beta}\|,$$

noting that  $r_k \leq \xi_k \bar{r}$ . Similarly,  $|(\eta_{1\beta x_1}, \tilde{z})_r| \leq 2\|\tilde{z}_{\bar{x}_1}\|_r \|\eta_{1\beta}\|$ . Therefore from

$$(\eta_{1\beta x_1}, \tilde{z})_\rho = (\eta_{1\beta x_1}, \tilde{z})_r - \sum_{k=1}^m \kappa \sigma_k \sum_{\omega_{1,k} \times \omega_2} h^2 r_k \eta_{1\beta x_1} \tilde{z}$$

it follows

$$(24) \quad |(\eta_{1\beta x_1}, \tilde{z})_\rho| \leq 2(1 + \kappa^2)\|\tilde{z}_{\bar{x}_1}\|_r \|\eta_{1\beta}\|, \quad \beta = 1, 2.$$

Now, it is not difficult to verify that

$$(25) \quad |(\eta_{2\beta x_2}, \tilde{z})_\rho| \leq \|\tilde{z}_{\bar{x}_2}\|_r \|\eta_{2\beta}\|, \quad \beta = 1, 2.$$

The inequality (12) immediately implies that

$$(26) \quad |(\eta, \tilde{z})_\rho| \leq (1/\sqrt{8}) \|\eta\|_0 \|\tilde{z}_{\bar{x}_2}\|_r.$$

Substituting (24)–(26) into (23) we finally obtain

$$(27) \quad \|\tilde{z}\|_{1,\omega,r}^2 \leq c \left( \sum_{\beta=1}^2 (\|\eta_{1\beta}\| + \|\eta_{2\beta}\|) + \|\eta\|_0 \right).$$

The inequality (18) follows directly from (22), (27).  $\square$

### 5. Estimate of the convergence rate

In order to estimate the convergence rate of finite-difference scheme (6), (7), it is enough to estimate the norm of error functionals in (18). For this we apply the well-known technique (see e.g., [8], [9]), which uses the generalized Bramble–Hilbert lemma [10].

We will show, that

$$(28) \quad \|a_{11}R\|_* \leq ch^{s-1} \|u\|_{W_2^s(\Omega)}, \quad s \in (1, 3].$$

Let  $e_k = (n_k h, n_k h + h) \times (x_2 - h/2, x_2 + h/2)$ ,  $\Omega_k = (n_k h, n_k h + h) \times (0, 1)$ . We now represent  $R_k$  in the form of sum

$$R_k = \left( h^2 G_k \frac{\partial^2 u}{\partial x_1^2} - h^2 G_k S_2^- \frac{\partial^2 u}{\partial x_1^2} \right) + h^2 G_k S_2^- \frac{\partial^2 u}{\partial x_1^2} = R'_k + R''_k, \quad k = 1, 2, \dots, n_m.$$

Let us remark, that  $R'_k$  is a bounded linear functional of  $u \in W_2^s(e_k)$ ,  $s > 1$ , which vanishes if  $u$  is a second-degree polynomial. Using Bramble–Hilbert lemma we obtain

$$(29) \quad |R'_k| \leq ch^{s-1} |u|_{W_2^s(e_k)}, \quad \|R'_k\|_* \leq ch^{s-1} |u|_{W_2^s(\Omega)}, \quad s \in (1, 3].$$

For  $s > 1$ ,  $R''_k$  is a bounded linear functional of  $u \in W_2^s(e_k)$ , which vanishes if  $u$  is a first-degree polynomial. Using Bramble–Hilbert lemma we obtain

$$(30) \quad \|R''_k\|_* \leq ch^{s-1} |u|_{W_2^s(\Omega)}, \quad s \in (1, 2.5].$$

In the case  $s > 2.5$  we write

$$\|R''_k\|_*^2 \leq c \sum_{\omega_2} h^3 \int_{e_k} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx \leq ch^3 \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{L_2(\Omega_k)}^2$$

and since  $\partial^2 u / \partial x_1^2 \in W_2^{s-2}(\Omega)$ ,  $s - 2 > 0.5$ , we may use an estimate for  $L_2$ -norm of a function in a sprit near the boundary in terms of  $W_2^{s-2}$ -norm in the domain  $\Omega$  (see, e.g., [8, p. 161], [11, p. 47]) :

$$\left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{L_2(\Omega_k)} \leq ch^{1/2} \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{W_2^{s-2}(\Omega)}, \quad 0.5 < s - 2 \leq 1.$$



This gives  $\|R_k''\|_* \leq ch^2|u|_{W_2^s(\Omega)}$ ,  $s \in (2.5, 3]$ , which along with (29), (30), taking into account continuity of the coefficient  $a_{11}$ , proves the inequality (28).

We will show, that

$$(31) \quad \|\eta_{1j}'\| \leq ch^{s-1}\|u\|_{W_2^s(\Omega)}, \quad \|\eta_{2j}'\| \leq ch^{s-1}\|u\|_{W_2^s(\Omega)}, \quad 1 < s \leq 3.$$

For  $1 < s \leq 2$ , we represent  $\eta_{ij}$  in the following way

$$(32) \quad \eta_{ij} = \eta_{ij}' + 0.5a_{ij}^{(-0.5i)}\eta_{ij}''\left(\frac{\partial u}{\partial x_j}\right) + 0.5a_{ij}^{(+0.5i, -1i)}\eta_{ij}'''\left(\frac{\partial u}{\partial x_j}\right),$$

where

$$\begin{aligned} \eta_{ij}' &= 0.5\left(a_{ij}^{(-0.5i)} + a_{ij}^{(+0.5i, -1i)}\right)S_i^-T_{3-i}\frac{\partial u}{\partial x_j} - S_i^-T_{3-i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right), \\ \eta_{ij}''(v) &= S_j^-v - S_i^-T_{3-i}v, \quad \eta_{ij}'''(v) = (S_j^+v)^{(-1i)} - S_i^-T_{3-i}v. \end{aligned}$$

Let  $e = (x_1 - h, x_1 + h) \times (x_2 - h, x_2)$ . Since  $\eta_{2j}'$  is a linear functional with respect to  $a_{2j}$ , is bounded in  $W_q^{s-1}(e)$ ,  $s > 1$  and vanishes if  $a_{2j}$  is constant, we have

$$|\eta_{2j}'| \leq ch^{s-3-2/q}|a_{2j}|_{W_q^{s-1}(e)} \int_e \left| \frac{\partial u}{\partial x_j} \right| dx.$$

It is easy to see that

$$\int_e \left| \frac{\partial u}{\partial x_j} \right| dx \leq \left( \int_e \left| \frac{\partial u}{\partial x_j} \right|^{2q/(q-2)} dx \right)^{(q-2)/(2q)} h^{1+2/q},$$

therefore

$$|\eta_{2j}'| \leq ch^{s-2}|a_{2j}|_{W_q^{s-1}(e)}|u|_{W_{2q/(q-2)}^1(e)}$$

and

$$(33) \quad \|\eta_{2j}'\| \leq ch^{s-1}\|a_{2j}\|_{W_q^{s-1}(\Omega)}\|u\|_{W_{2q/(q-2)}^1(\Omega)}.$$

Analogously,

$$(34) \quad \|\eta_{1j}'\| \leq ch^{s-1}\|a_{1j}\|_{W_p^{s-1}(0;1)}\|u\|_{W_{2p/(p-2)}^1(\Omega)}.$$

Since  $W_{2q/(q-2)}^1, W_{2p/(p-2)}^1 \subset W_2^{s-2}$ , using (33), (34) and similar estimates for  $\eta_{ij}''$ ,  $\eta_{ij}'''$ , from (32) we come (31) for  $1 < s \leq 2$ .

In the case  $2 < s \leq 3$  we write

$$(35) \quad \eta_{i1} = l_1\left(a_{i1}\frac{\partial u}{\partial x_1}\right) + 0.5a_{i1}l_2\left(\frac{\partial u}{\partial x_1}\right) + 0.5a_{i1}^{(+0.5i, -1i)}l_3\left(\frac{\partial u}{\partial x_1}\right),$$

$$(36) \quad \eta_{i2} = l_4\left(a_{i2}\frac{\partial u}{\partial x_2}\right) + a_{i2}^{(-0.5i)}l_5\left(\frac{\partial u}{\partial x_2}\right) + 0.5u_{\bar{x}_2}l_6(a_{i2}),$$

where

$$l_1(v) = 0.5v^{(-0.5i)} + 0.5v^{(+0.5i, -1i)} - S_i^-T_{3-i}v, \quad l_2(v) = S_1^-v - v^{(-0.5i)},$$

$$l_3(v) = S_1^+v - v^{(+0.5i)}, \quad l_4(v) = v^{(-0.5i)} - S_i^-T_{3-i}v,$$

$$l_5(v) = 0.5S_2^-v + 0.5S_2^+v^{(-1i)} - v^{(-0.5i)}, \quad l_6(v) = v^{(-0.5i)} - 2v^{(-0.5i)} + v^{(+0.5i, -1i)}.$$

For  $s > 2$ ,  $l_j(v)$  ( $j = 1, 2, 3, 4, 5, 6$ ) are a bounded linear functionals of  $v \in W_2^{s-2}$  and vanish if  $v$  is a first-degree polynomial. Hence from (35), (36) we get (31) for  $2 < s \leq 3$ .

For  $\eta$  the estimate  $\|\eta\|_0 \leq ch^{s-1} \|u\|_{W_2^s(\Omega)}$ ,  $1 < s \leq 3$  is true.

Finally on the basis of obtained estimates, together with Lemma 3 the following convergence theorem is proved.

**THEOREM 1.** *The finite-difference scheme (6), (7) converges and the convergence rate estimate (1) holds.*

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