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ESTIMATING OF PARAMETERS: NUAR(1) PROCESS

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ABSTRACT. We applied the method of conditional least squares for estimating parameters of NUAR(1). This process can be represented as the random coefficient autoregressive time series of the form

$$X_n = U_n X_{n-1} + V_n,$$

where $\{(U_n, V_n)\}$ is the sequence of independent identically distributed random vectors such that supply the elements of the sequence $\{X_n\}$ with $\mathcal{U}(0,1)$ marginal distribution. Defined estimates were the functions of the estimates of moments $E(U_n)$ and $E(U_n V_n)$ and they are strong consistent and asymptotically normally distributed.

1. Introduction

The new uniform autoregressive time series of order one (NUAR(1)) was defined in [**7**] by Ristić and Popović. This process is a generalization of the uniformly distributed autoregressive process of order one with positive correlations of order one which was defined by Chernik [**3**].

NUAR(1) process is a stationary process with uniform $\mathcal{U}(0,1)$ marginal distribution defined as

$$X_n = \begin{cases} \alpha X_{n-1} & \text{w.p. } \alpha \\ \beta X_{n-1} + \varepsilon_n & \text{w.p. } 1-\alpha, \end{cases}$$

where α and β are parameters with the values from the interval $(0,1)$ such that $k \equiv (1-\alpha)/\beta \in \{2,3,\dots\}$. $\{\varepsilon_n\}$ is the sequence of independent identically distributed random variables with the following distribution

$$P\{\varepsilon_n = \alpha + j\beta\} = \beta/(1-\alpha), \quad j = 0,1,\dots,k-1.$$

The sequences $\{X_n\}$ and $\{\varepsilon_n\}$ are semiindependent.

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Ristić and Popović proved (theorem 4.1 [**7**]) that NUAR(1) process can be represented as the following random coefficient autoregression

$$X_n = U_n X_{n-1} + V_n, \tag{1.1}$$

where $\{(U_n, V_n)\}$ is the sequence of independent identically distributed random vectors distributed as

$U_n \backslash V_n$	0	α	$\alpha+\beta$	$\dots$	$\alpha+(k-2)\beta$	$\alpha+(k-1)\beta$
α	α	0	0	$\dots$	0	0
β	0	β	β	$\dots$	β	β

and $\{X_n\}$ and $\{(U_n, V_n)\}$ are semiindependent.

The authors proved also (theorem 4.2 [**7**]) that if the conditions of the theorem 4.1 [**7**] were satisfied, then difference equation (1.1) had nonanticipative unique σ_n-measurable, stationary, strictly stationary and ergodic solution:

$$X_n = \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} U_{n-j} \right) V_{n-i} + V_n,$$

where σ_n is the σ-field generated by the random vectors $\{(U_m, V_m),\ m \le n\}$.

In this very moment we apply the method of conditional least squares according to the procedure introduced by Popović [**6**]. Meanwhile, instead of the second moments of U_n and V_n separately, now we use mixed second moment $E(U_nV_n)$.

Section 2 of this paper is given up to the explicit representation of the estimates of parameters of NUAR(1) by the method of conditional least squares. In Section 3 we prove that the estimate of the moment $E(U_n)$ is strong consistent and satisfies the asymptotic normality. The vector of estimates of moments $E(U_n)$ and $E(U_nV_n)$ is considered in Section 4. We prove that this estimate is strong consistent and asymptotically bivariate normally distributed.

2. The application of the method of conditional least squares

The purpose of the application of the procedure of conditional least squares is to estimate parameters α and β of NUAR(1). We use only one realization, X_1, X_2, $\dots$, X_N, according to the ergodicity of the process $\{X_n\}$. Mark that the difference equation (1.1) can be represented as

$$Y_n = uY_{n-1} + \theta_n,$$

where $Y_n = X_n - 1/2$, $u = E(U_n)$ and $\theta_n = (U_n - u)Y_{n-1} + (U_n + 2V_n - 1)/2$. The solution of this equation has the same properties as the solution of the equation (1.1).

The method of conditional least squares minimizes the function

$$S(u) = \sum_{n=2}^{N} \{Y_n - E(Y_n|Y_{n-1})\}^2 = \sum_{n=2}^{N} \{Y_n - uY_{n-1}\}^2.$$

After the minimizing procedure the estimate for u follows:

$$\hat{u} = \sum_{n=2}^{N} Y_{n-1}Y_n \Big/ \sum_{n=2}^{N} Y_{n-1}^2.$$

Properties of this statistics will be considered in Section 3.

To solve the problem of estimating two parameters we need one more equation, or better to say one more moment of the process. The most convenient for the model which we discuss in this paper is the mixed second moment: $w = E(U_nV_n) = \beta(1-u)/2$.

Let $R_n = (Y_n - E(Y_n|Y_{n-1}))^2$. Simple computation will prove that

$$E(R_n|Y_{n-1}) = u_2 P_{n-1} + 2wY_{n-1} + Q_{n-1},$$

where $P_{n-1} = Y_{n-1}^2 + Y_{n-1} - 1/12$ and $Q_{n-1} = 1/12 - uY_{n-1} - u^2Y_{n-1}^2$. Now we minimize the function

$$S_1(u_2, w) = \sum_{n=2}^{N} \{R_n - E(R_n|Y_{n-1})\}^2$$

according to u_2 and w. Let $\mathbf{R} = (R_2, R_3, \dots, R_N)'$, $\mathbf{P} = (P_1, P_2, \dots, P_{N-1})'$, $\mathbf{Q} = (Q_1, Q_2, \dots, Q_{N-1})'$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{N-1})'$. The solution (u_2, w) of the system of the equations $\dfrac{\partial S_1}{\partial u_2} = 0$, $\dfrac{\partial S_1}{\partial w} = 0$ will supply us with the statistics

$$\hat{w} = \Delta^{-1}\left(\mathbf{P}'\mathbf{P}(\hat{\mathbf{R}} - \hat{\mathbf{Q}})'\mathbf{Y} - \mathbf{P}'\mathbf{Y}(\hat{\mathbf{R}} - \hat{\mathbf{Q}})'\mathbf{P}\right), \tag{2.1}$$

where $\hat{\mathbf{R}} = ((Y_2 - \hat{u}Y_1)^2, \dots, (Y_N - \hat{u}Y_{N-1})^2)'$, $\hat{\mathbf{Q}} = (1/12 - \hat{u}Y_1 - \hat{u}^2Y_1^2, \dots, 1/12 - \hat{u}Y_{N-1} - \hat{u}^2Y_{N-1}^2)'$, and $\Delta = 2(\mathbf{P}'\mathbf{P}\mathbf{Y}'\mathbf{Y} - \mathbf{P}'\mathbf{Y}\mathbf{P}'\mathbf{Y})$. Finally, from $\hat{u}$ and $\hat{w}$ we shall the estimates for α i β. The estimate for β is

$$\hat{\beta} = \frac{2\hat{w}}{1-\hat{u}},$$

and for α will be necessary to solve the quadratic equation $\hat{\alpha}^2 + (1-\hat{\alpha})\hat{\beta} = \hat{u}$.

3. Asymptotic behavior of $\hat{u}$

Properties of the statistics $\hat{u}$ are discussed in this section.

THEOREM 3.1. *The estimate $\hat{u}$ is strong consistent estimate of the parameter u and $\sqrt{N-1}(\hat{u} - u)$ is asymptotically normally distributed with zero mean and variance*

$$\sigma^2 = \frac{1}{5}(1-\alpha)(5 + 5\alpha + 5\alpha^2 + 9\alpha^3 - 18\alpha^2\beta - 5\beta^2 + 9\alpha\beta^2).$$

PROOF. The sequences of random variables $\{Y_{n-1}\theta_n\}$ and $\{Y_{n-1}^2\}$ are strictly stationary and ergodic. It is obvious that

$$\hat{u} - u = \sum_{n=2}^{N} Y_{n-1}\theta_n \Big/ \sum_{n=2}^{N} Y_{n-1}^2.$$

According to this and ergodic theorem it follows that $\hat{u}$ is strong consistent estimate for u. According to the same reasons

$$\frac{1}{N-1}\sum_{n=2}^{N} Y_{n-1}^2 \xrightarrow{\mathrm{P}} \frac{1}{12}, \tag{3.1}$$

and according to the central limit theorem for martingals [**1**] it follows that

$$\frac{1}{\sqrt{N-1}}\sum_{n=2}^{N} Y_{n-1}\theta_n \xrightarrow{\mathrm{d}} \mathcal{N}(0,\sigma^2/144). \tag{3.2}$$

At last, from (3.1) and (3.2), it follows that $\sqrt{N-1}(\hat{u}-u)$ converges in distribution to the above specified normal distribution. □

4. Asymptotic behavior of the vector $\hat{\mathbf{D}} = (\hat{u}, \hat{w})'$

Finally, we prove that the vector $\hat{\mathbf{D}}$ is strong consistent estimate of the vector $\mathbf{D} = (u, w)'$ and that $\sqrt{N-1}(\hat{\mathbf{D}} - \mathbf{D})$ is asymptotically bivariate normally distributed.

If we suppose that the true value of the parameter u is known, we can set it in (2.1) instead of $\hat{u}$ to get $\tilde{w}$. Now we shall set $\tilde{\mathbf{D}} = (\hat{u}, \tilde{w})'$ and discuss firstly properties of

$$\tilde{\mathbf{D}} - \hat{\mathbf{D}} = \begin{pmatrix} 0 \\ \tilde{w} - \hat{w} \end{pmatrix}$$

It is obvious that the asymptotic behavior of the vector $\tilde{\mathbf{D}} - \hat{\mathbf{D}}$ will follow the asymptotic behavior of its component $\tilde{w} - \hat{w}$. This behavior is displayed in the next lemma.

LEMMA 4.1. *$\tilde{w} - \hat{w}$ converges almost sure to 0 and $\sqrt{N-1}(\tilde{w} - \hat{w})$ converges in probability to 0.*

PROOF. Simple calculation will prove that

$$\tilde{w} - \hat{w} = \Delta^{-1}\left[\mathbf{P}'\mathbf{P}\left(\mathbf{R} - \hat{\mathbf{R}} - \mathbf{Q} + \hat{\mathbf{Q}}\right)'\mathbf{Y} - \mathbf{P}'\mathbf{Y}\left(\mathbf{R} - \hat{\mathbf{R}} - \mathbf{Q} + \hat{\mathbf{Q}}\right)'\mathbf{P}\right].$$

Let us consider the elements of the vectors $\mathbf{R} - \hat{\mathbf{R}}$ i $\mathbf{Q} - \hat{\mathbf{Q}}$:

$$\begin{aligned} R_n - \hat{R}_n &= (\hat{u}-u)^2 Y_{n-1}^2 - 2(\hat{u}-u)Y_{n-1}\theta_n, \\ Q_{n-1} - \hat{Q}_{n-1} &= (\hat{u}-u)^2 Y_{n-1}^2 + (\hat{u}-u)(Y_{n-1} + 2uY_{n-1}^2). \end{aligned}$$

Then we shall have that

$$\begin{aligned} &\frac{1}{N-1}\left(\mathbf{R} - \hat{\mathbf{R}} - \mathbf{Q} + \hat{\mathbf{Q}}\right)'\mathbf{Y} \\ &\qquad = \frac{1}{N-1}\left(-2(\hat{u}-u)\sum_{n=2}^{N} Y_{n-1}^2\theta_n - (\hat{u}-u)\sum_{n=2}^{N}(Y_{n-1}^2 + 2uY_{n-1}^3)\right). \end{aligned}$$

According to the ergodic theorem, each of the random variables $\frac{1}{N-1}\sum\limits_{n=2}^{N} Y_{n-1}^2\theta$ and $\frac{1}{N-1}\sum\limits_{n=2}^{N} Y_{n-1}^3$ converges to zero almost sure, and that $\frac{1}{N-1}\sum\limits_{n=2}^{N} Y_{n-1}^2$ converges almost sure to $1/12$. Because of the same reasons, it follows that $\frac{1}{N-1}\mathbf{P}'\mathbf{P}$ converges almost sure to $4/45$. Following also the arguments of the theorem 3.1, finally we shall have that

$$\frac{1}{(N-1)^2}\mathbf{P}'\mathbf{P}\left(\mathbf{R}-\hat{\mathbf{R}}-\mathbf{Q}+\hat{\mathbf{Q}}\right)'\mathbf{Y}$$

converges almost sure to 0.

In the similar way it can be concluded that

$$\frac{1}{(N-1)^2}\mathbf{P}'\mathbf{Y}\left(\mathbf{R}-\hat{\mathbf{R}}-\mathbf{Q}+\hat{\mathbf{Q}}\right)'\mathbf{P}$$

and $\frac{1}{N-1}\Delta$ converge almost sure to 0 and $1/1080$, respectively. It will imply the same convergence of $\tilde{w}-\hat{w}$ to 0. So, it follows that $\sqrt{N-1}(\tilde{w}-\hat{w})$ converges in probability to 0. □

Next we shall prove that there exists a vector $\mathbf{D}^*$ such that:

$$\mathbf{D}^*-\tilde{\mathbf{D}} \overset{a.s.}{\to} \mathbf{0}, \tag{4.1}$$

$$\mathbf{D}^*-\mathbf{D} \overset{a.s.}{\to} \mathbf{0}, \tag{4.2}$$

$$\sqrt{N-1}(\mathbf{D}^*-\mathbf{D}) \overset{\mathrm{d}}{\to} \mathcal{N}(\mathbf{0},\mathbf{H}), \tag{4.3}$$

where $\mathbf{H}$ is a covariance matrix.

LEMMA 4.2. *If we define vector* $\mathbf{D}^*=(u^*,w^*)'$ *such that*

$$\begin{aligned} u^* &= \frac{12}{N-1}\sum_{n=2}^{N} Y_{n-1}\theta_n + u,\\ w^* &= \frac{1080}{N-1}\mathbf{P}'\mathbf{A}+w, \end{aligned}$$

where $\mathbf{A}=\mathbf{P}(\mathbf{R}-u_2\mathbf{P}-2w\mathbf{Y}-\mathbf{Q})'\mathbf{Y}-\mathbf{Y}(\mathbf{R}-u_2\mathbf{P}-2w\mathbf{Y}-\mathbf{Q})'\mathbf{P}$, *it will be the truth that* $\mathbf{D}^*$ *satisfies* $(4.1)-(4.3)$, *where the covariance matrix* $\mathbf{H}$ *will be*

$$\begin{pmatrix} 144E(Y_1^2\theta_2^2) & 12960E(Y_1\theta_2P_1A_1)\\ 12960E(Y_1\theta_2P_1A_1) & 1080^2E(P_1^2A_1^2)\end{pmatrix}. \tag{4.4}$$

PROOF. The convergence of (4.1) and (4.2) is to be proved in the similar way as it was done in the previous lemma. So, we shall prove (4.3) only.

According to the central limit theorem for martingals [**1**] it follows that for any vector $\lambda=(\lambda_1,\lambda_2)'$

$$\sqrt{N-1}\lambda'(\mathbf{D}^*-\mathbf{D})=\frac{1}{\sqrt{N-1}}\sum_{n=2}^{N}(12\lambda_1Y_{n-1}\theta_n+1080\lambda_2P_{n-1}A_{n-1})$$

has normal limit distribution with zero mean and variance $\lambda'\mathbf{H}\lambda$, where $\mathbf{H}$ is defined by (4.4). As it is the truth for any λ, (4.3) will be satisfied according to the Cramer-Wold's device (Brockwell and Davis [**2**], proposition 6.3.1). $\square$

These results imply the theorem:

THEOREM 4.1. *The vector* $\hat{\mathbf{D}}$ *is a strong consistent estimate for* $\mathbf{D}$ *and the convergence*

$$\sqrt{N-1}(\hat{\mathbf{D}}-\mathbf{D}) \stackrel{\mathrm{d}}{\to} \mathcal{N}(\mathbf{0},\mathbf{H})$$

is satisfied, where $\mathbf{H}$ *is the covariance matrix defined by* (4.4).

PROOF. The proof follows from two previous lemmas and the following fact

$$\hat{\mathbf{D}}-\mathbf{D} = (\hat{\mathbf{D}}-\tilde{\mathbf{D}}) + (\tilde{\mathbf{D}}-\mathbf{D}^*) + (\mathbf{D}^*-\mathbf{D}).$$

$\square$

References

[1] P. Billingsley, *The Lindeberg-Levy theorem for martingales*, Proc. Amer. Math. Soc. **12** (1961), 788–792.

[2] P. Brockwell, R. Davis, *Time Series: Theory and Methods*, Springer-Verlag, New York, 1987.

[3] M.R. Chernick, *A limit theorem for the maximum of autoregressive processes with uniform marginal distributions*, Ann. Probab. **9** (1981), 145–149.

[4] M.R. Chernick, R.A. Davis, *Extremes in autoregressive processes with uniform marginal distributions*, Statistics and Probability Letters **1** (1982), 85–88.

[5] A.J. Lawrance, *Uniformly distributed first-order autoregressive time series models and multiplicative congruential random number generators*, J. Appl. Prob. **29** (1992), 896–903.

[6] B.Č. Popović, *Estimation of parameters of RCA with exponential marginals*, Publ. Inst. Math. (Beograd) (N.S.) **54(68)** (1993), 135–143.

[7] M.M. Ristić, B.Č. Popović, *A new uniform* $AR(1)$ *time series model* ($NUAR(1)$), Publ. Inst. Math. (Beograd) (N.S.) **68(82)** (2000), 145–152.

[8] M.M. Ristić, M. M., B.Č. Popović, *Parameter estimation for uniform autoregressive processes*, Novi Sad J. Math. **30** (2000), 89–95.

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