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ON STOCHASTIC ORDERS

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ABSTRACT. A sharp inequality is proved between the tails of two distributions that are in increasing convex order.

Stochastic orders and inequalities are widely used in applied probability and statistics. Several types of such orders, together with their properties, and connections between different notions are presented in [5].

In the sequel let X and Y be (integrable) random variables. Let us recollect the notions of the so-called *usual stochastic order* and *increasing convex order*.

DEFINITION 1. Y is said to be smaller than X in the usual stochastic order (denoted by $Y \le_{\rm st} X$) if any of the following equivalent conditions holds.

$$P(Y > u) \le P(X > u), \quad \forall u \in (-\infty, +\infty),$$

$$P(Y \ge u) \le P(X \ge u), \quad \forall u \in (-\infty, +\infty),$$

$E[\phi(Y)] \le E[\phi(X)]$ for all increasing functions ϕ for which the expectations exist.

DEFINITION 2. Y is said to be smaller than X in the increasing convex order (denoted by $Y \le_{\rm icx} X$) if any of the following equivalent conditions holds.

$$\int_x^\infty P(Y > u)du \le \int_x^\infty P(X > u)du, \quad \forall x \in (-\infty, +\infty),$$

$E[\phi(Y)] \le E[\phi(X)]$ for all increasing convex functions ϕ for which the expectations exist.

Clearly, $Y \le_{\rm st} X \Rightarrow Y \le_{\rm icx} X$, but not conversely. The aim of the present note is to characterize the smallest (in the usual stochastic order) random variable Z with the property $Y \le_{\rm icx} X \Rightarrow Y \le_{\rm st} Z$.

The following auxiliary facts are easy and well-known.

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Lemma. *Let* $\omega = \sup\{t : P(X > t) > 0\}$ *and* $h(t) = E(X \mid X > t)$, $t < \omega$. *Then* $t < h(t) \le \omega$, h *is increasing and right continuous.*

Proof. $h(t) - t$ is sometimes called *the mean residual life* of X at time t, see [2]. From the representation

$$h(t) - t = \frac{1}{P(X > t)} \int_t^\omega P(X > u)du$$

all listed properties easily follow. For instance, let us consider the monotonicity. For $t < s < \omega$ we have

$$\begin{aligned} h(t) = t + \frac{1}{P(X > t)} \int_t^\omega P(X > u)du &\le t + (s - t) + \frac{1}{P(X > t)} \int_s^\omega P(X > u)du \\ &\le s + \frac{1}{P(X > s)} \int_s^\omega P(X > u)du = h(s). \end{aligned}$$

Now, we are in position to state and prove our results.

Theorem 1. $\max\{P(Y \ge h(t)) : Y \le_{\mathrm{icx}} X\} = P(X > t)$, $t < \omega$.

Remark. It is easy to see that $Y \le_{\mathrm{icx}} X$ implies $P(Y > \omega) = 0$.

Proof. $\phi(x) = \dfrac{(x - t)^+}{h(t) - t}$ is convex and increasing, $\phi(h(t)) = 1$, and $E[\phi(X)] = P(X > t)$. Therefore

$$P(Y \ge h(t)) \le P(\phi(Y) \ge 1) \le E[\phi(Y)] \le E[\phi(X)] = P(X > t).$$

On the other hand, let

$$Y = \begin{cases} E(X \mid X > t) = h(t), & \text{if } X > t, \\ E(X \mid X \le t), & \text{if } X \le t, \end{cases} \tag{1}$$

that is, $Y = E(X \mid \mathcal{F})$, where $\mathcal{F}$ is the σ-field generated by the event $\{X > t\}$. By the conditional Jensen inequality we have $Y \le_{\mathrm{icx}} X$, and $P(Y \ge h(t)) = P(X > t)$.

Theorem 2. $Y \le_{\mathrm{icx}} X \Rightarrow Y \le_{\mathrm{st}} h(X)$, *and this is sharp in the following sense: if for every* Y *satisfying* $Y \le_{\mathrm{icx}} X$ *it follows that* $Y \le_{\mathrm{st}} Z$, *then* $h(X) \le_{\mathrm{st}} Z$.

Proof. For $h(t - 0) \le s \le h(t)$ we have

$$P(Y \ge s) \le P(Y \ge h(t - 0)) \le P(Y \ge h(t - \varepsilon)) \le P(X > t - \varepsilon), \quad \varepsilon > 0.$$

From this it follows that $P(Y \ge s) \le P(X \ge t)$, hence

$$P(Y \ge s) \le P(X \ge t) \le P(h(X) \ge h(t)) \le P(h(X) \ge s)$$

for every $s < \omega$.

On the other hand, with Y defined in (1)

$$\begin{aligned} P(Z \ge s) \ge P(Z \ge h(t)) &\ge P(Y \ge h(t)) \\ &= P(X > t) \ge P(h(X) > h(t)) = P(h(X) > s). \end{aligned}$$

Taking limit from the right we obtain that $P(Z > s) \ge P(h(X) > s)$.

Application. Sometimes it is necessary to estimate the tail distribution of a convex combination

$$Y = \sum_{n=1}^{\infty} \lambda_n X_n, \tag{2}$$

where $X_1, X_2, \ldots$ are i.i.d. random variables, and the weights λ_n are nonnegative numbers with sum 1. This is the case when a U-statistic $\sum_{1 \le i < j \le n} K(X_i, X_j)$ with degenerate kernel K is used in statistical hypothesis testing (degenerate means $E[K(X_1, \cdot)] = 0$), see [4] for examples. Then, under some mild conditions, the limit distribution as $n \to \infty$ of

$$T_n(X_1, \ldots, X_n) = \frac{\sum_{1 \le i < j \le n} K(X_i, X_j)}{\sum_{1 \le i \le n} K(X_i, X_i)}$$

is of the form (2), where the coefficients λ_n depend on the distribution of the sample elements and the random variables X_n are of χ^2 distribution with 1 degree of freedom (i.e., the square of the standard Gaussian distribution), see [3]. In order to compute asymptotic critical values it is necessary to have good estimations for the exceedance probabilities $P(Y > t)$. Again, by Jensen's inequality we have $Y \le_{\mathrm{icx}} X$.

Let us introduce $\varphi(t) = (2\pi)^{-1/2} \exp(-t^2/2)$, and $\Phi(t) = \int_{-\infty}^{u} \varphi(u) du$, the standard Gaussian density and distribution functions, resp. Then

$$P(X > t) = 2\left(1 - \Phi(\sqrt{t})\right) \ge 2\varphi(\sqrt{t}) \frac{\sqrt{t}}{t+1}, \quad t > 0,$$

hence

$$\begin{aligned} h(t) &= \frac{1}{1 - \Phi\left(\sqrt{t}\right)} \int_{\sqrt{t}}^{\infty} u^2 \varphi(u) du \\ &= \frac{1}{1 - \Phi\left(\sqrt{t}\right)} \left(\Big[-u\varphi(u) \Big]_{\sqrt{t}}^{\infty} + \int_{\sqrt{t}}^{\infty} \varphi(u) du \right) \\ &= \frac{\sqrt{t}\varphi\left(\sqrt{t}\right)}{1 - \Phi(\sqrt{t})} + 1 \\ &\le t + 2, \end{aligned}$$

thus $P(Y > t) \le P(X > t + 2) = 2\left(1 - \Phi\left(\sqrt{t+2}\right)\right)$. However, in this very important particular case even more can be said: $P(Y > t) \le P(X > t)$ for all sufficiently large values of t, by a result of Bakirov [1].

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