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A CLASSIFICATION OF 2-TYPE CURVES IN THE MINKOWSKI SPACE E_1^n

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Abstract. We complete a classification of 2-type curves in Minkowski spaces. Namely, we give a classification of 2-type spacelike and timelike curves, lying fully in the Minkowski spaces E_1^4 and E_1^5.

1. Introduction

A submanifolds of finite type were defined by Chen in [1]. Recall that a submanifold M is said to be of *finite type* (finite Chen type) if its position vector field x can be written as a finite sum of the eigenfunctions $\mathrm{x}_0, \mathrm{x}_1, \dots, \mathrm{x}_k$ of the Laplace operator Δ of M. More explicitly,

$$\mathrm{x} = \mathrm{x}_0 + \sum_{i=1}^{k} \mathrm{x}_\mathrm{i}, \quad \Delta \mathrm{x}_i = \lambda_i \mathrm{x}_i, \tag{1.1}$$

where x_0 is a constant vector, x_i are non-constant vectors and $\lambda_1 < \dots < \lambda_k$ are eigenvalues of Δ.

The simplest submanifolds of finite type are the curves of finite type. A curve α is said to be of (finite) k-type for some natural number k if its Laplace operator Δ has exactly k eigenvalues $\{\lambda_1, \dots, \lambda_k\}$ in decomposition (1.1) which are all different. In particular, if one of the eigenvalues $\{\lambda_1, \dots, \lambda_k\}$ in decomposition (1.1) is equal to zero, α is said to be of (finite) null k-type. Moreover, in the case of null k-type submanifolds, decomposition (1.1) is not unique.

Finite type curves in the Euclidean space E^n were studied in [1], [2] and [3]. A full classification of 1-type, 2-type and 3-type curves in the space E^n is given respectively in [4], [6] and [8]. On the other hand, finite type curves in Minkowski

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spaces were studied in [5] and [7]. It is interesting that hyperbolas and straight lines are the only 1-type curves in Minkowski spaces. A classification of non-planar 3-type curves in the Minkowski 3-space is given in [10]. It is proved in [1] that arbitrary curve of k-type is contained in at most $2k$-dimensional subspace of the space E^n. This implies that the dimension n of the space E^n is not greater than $2k$. Similarly, arbitrary curve of k-type is contained in at most $2k$-dimensional subspace of the Minkowski space E_1^n, so the dimension n of the space E_1^n is not greater than $2k+1$. Spacelike and timelike curves of 2-type, lying fully in the Minkowski space E_1^3 are classified in [11]. In this paper, we give a classification of all 2-type spacelike and timelike curves, lying fully in the Minkowski spaces E_1^4 and E_1^5. In this way, classification of such curves in Minkowski spaces is completed.

2. Preliminaries

In the sequel, we introduce some basic definitions and notions. Let E_1^n denote the n-dimensional Minkowski space, i.e., the Euclidean space E^n with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + \cdots + dx_n^2,$$

where $(x_1, \ldots, x_n)$ is a rectangular coordinate system of E_1^n.

Recall that arbitrary subspace W of the Minkowski space E_1^n is said to be:

(1) spacelike if $g|_W$ is positive definite;
(2) timelike if $g|_W$ is nondegenerate of index 1;
(3) lightlike (isotropic) if $g|_W$ is degenerate.

The type into which W falls is called its causal character. In the same manner, a vector a in the space E_1^n is said to be:

(1) spacelike if $g(a,a) > 0$ or $a = 0$;
(2) timelike if $g(a,a) < 0$;
(3) null (lightlike) if $g(a,a) = 0$ and $a \neq 0$.

Similarly, a curve $\alpha = \alpha(s)$ in E_1^n is said to be spacelike, timelike or null (lightlike), if respectively all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike).

The norm of a vector a is given by $\|a\| = \sqrt{|g(a,a)|}$ and two vectors a, b are said to be orthogonal if $g(a,b) = 0$.

Next, the curve $\alpha(s)$ is said to be of unit speed, if for its velocity holds $v = \|\alpha'(s)\| = 1$, i.e., if $g(\alpha'(s), \alpha'(s)) = \pm 1$. The Laplace operator Δ of the curve $\alpha(s)$ is defined by $\Delta = \pm d^2/ds^2$. Its eigenfunctions are the functions s, $\cos(ps)$, $\sin(ps)$, $\cosh(ps)$ and $\sinh(ps)$. Following the definition of Chen, the curve $\alpha(s)$ is of finite type in the space E_1^n if and only if it can be written in the form

$$\alpha(s) = a_0 + b_0 s + \sum_{i=1}^{m} (a_i \cos(p_i s) + b_i \sin(p_i s)) + \sum_{j=1}^{t} (c_j \cosh(q_j s) + d_j \sinh(q_j s)),$$

where $a_0, b_0, a_i, b_i, c_j, d_j \in R^n$, $p_i, q_j \in N$, $0 < p_1 < \dots < p_m$, $0 < q_1 < \dots < q_t$.

Further, recall that an isometry of the space E_1^n is a diffeomorphism $I : E_1^n \to E_1^n$ that preserves metric. More explicitly, $g(I(a), I(b)) = g(a, b)$ for each a, $b \in E_1^n$. We mention here that the spacelike rotation (in the spacelike 2-plane $\{x_3, x_4\}$) and the timelike rotation (in the timelike 2-plane $\{x_1, x_2\}$) in the space E_1^4 may be expressed respectively by matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & 0 & -\sin(\varphi) & \cos(\varphi) \end{bmatrix}, \qquad \begin{bmatrix} \cosh(\varphi) & \sinh(\varphi) & 0 & 0 \\ \sinh(\varphi) & \cosh(\varphi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Such rotations in the space E_1^5 may be expressed in a similar way ([9]).

3. A classification of 2-type curves in the Minkowski space $\mathbf{E}_1^4$

It is known that the space E_1^3 is contained in the space E_1^4 as its timelike subspace, i.e., as its timelike hyperplane. Therefore, if α is a 2-type curve in E_1^3, it is also a 2-type curve in E_1^4. All 2-type spacelike and timelike curves, lying fully in E_1^3 are classified in [11]. In the following two theorems, we give a classification of null 2-type and 2-type spacelike and timelike curves, lying fully in E_1^4 and not lying in its timelike hyperplane.

THEOREM 3.1. *Let $\alpha(s)$ be a unit speed spacelike or timelike curve, with one eigenvalue of its Laplace operator Δ in decomposition (1.1) equal to zero, lying fully in E_1^4 and not lying in its timelike hyperplane. Then up to isometries of E_1^4, α is a null 2-type curve if and only if α is a part of one of the following spacelike circular helices:*

(1) $\alpha(s) = (0, ms, n\cos(ps), n\sin(ps))$, $\quad m^2 + p^2n^2 = 1$, $\quad p \in N, m, n \in R_0$;

(2) $\alpha(s) = (ms, ms, n\cos(ps), n\sin(ps))$, $\quad p^2n^2 = 1$, $\quad p \in N, m, n \in R_0$;

Proof. Suppose that $\alpha(s)$ satisfies the assumptions of the theorem and that it is a null 2-type curve. Then $\alpha(s)$ can be written in one of the following forms

$$(a) \quad \alpha(s) = a + bs + c\cos(ps) + d\sin(ps);$$
$$(b) \quad \alpha(s) = a + bs + c\cosh(ps) + d\sinh(ps);$$

where $p \in N$, $a, b, c, d \in R^4$. Denote by R_0 the set of all real numbers different from zero. Next, assume that $a = (0, 0, 0, 0)$ up to a translation and let $b = (b_1, b_2, b_3, b_4)$, $c = (c_1, c_2, c_3, c_4)$, $d = (d_1, d_2, d_3, d_4)$. In the sequel, we consider cases (a) and (b).

Case (a). Since $g(\alpha', \alpha') = \pm 1$ and using the linear independence of the functions $\sin(x)$ and $\cos(x)$, we obtain the system of equations

$$g(b, b) + \tfrac{p^2}{2}(g(c, c) + g(d, d)) = \pm 1, \tag{1}$$
$$g(b, c) = g(b, d) = g(c, d) = 0, \tag{2}$$
$$g(c, c) - g(d, d) = 0. \tag{3}$$

With respect to the causal character of the vectors c and d, we distinguish three subcases: $(a.1)$ $g(c,c) = g(d,d) > 0$; $(a.2)$ $g(c,c) = g(d,d) = 0$; $(a.3)$ $g(c,c) = g(d,d) < 0$.

$(a.1)$ We may assume that $c = (0,0,c_3,0)$, $d = (0,0,0,c_3)$, $c_3 \neq 0$. The equation (2) implies that $b = (b_1, b_2, 0, 0)$. If b is spacelike, let $b_1 = \rho\sinh(\varphi)$, $b_2 = \rho\cosh(\varphi)$, $\rho \in R_0$, $\varphi \in R$. Then α has the form

$$\begin{aligned}\alpha(s) &= (\rho s\sinh(\varphi), \rho s\cosh(\varphi), c_3\cos(ps), c_3\sin(ps))\\ &= (0, \rho s, c_3\cos(ps), c_3\sin(ps))\begin{bmatrix}\cosh(\varphi) & \sinh(\varphi) & 0 & 0\\ \sinh(\varphi) & \cosh(\varphi) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

Up to isometries of E_1^4, α is a spacelike circular helix lying fully in a spacelike hyperplane of E_1^4, where the equation (1) gives $\rho^2 + p^2c_3^2 = 1$. Next, if b is timelike, let $b_1 = \rho\cosh(\varphi)$, $b_2 = \rho\sinh(\varphi)$, $\rho \in R_0$, $\varphi \in R$. Then α has the form

$$\begin{aligned}\alpha(s) &= (\rho s\cosh(\varphi), \rho s\sinh(\varphi), c_3\cos(ps), c_3\sin(ps))\\ &= (\rho s, 0, c_3\cos(ps), c_3\sin(ps))\begin{bmatrix}\cosh(\varphi) & \sinh(\varphi) & 0 & 0\\ \sinh(\varphi) & \cosh(\varphi) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

Up to isometries of E_1^4, α lies fully in a timelike hyperplane of E_1^4, which is a contradiction to the assumption of the theorem. Finally, if b is null, let $b = (b_1, b_1, 0, 0)$, $b_1 \neq 0$. Then α has the form $\alpha(s) = (b_1 s, b_1 s, c_3\cos(ps), c_3\sin(ps))$, where the equation (1) gives $p^2c_3^2 = 1$. Therefore, α is a spacelike circular helix with a null axis lying fully in a lightlike hyperplane of E_1^4.

$(a.2)$ In this subcase, assume that $c = (c_1, 0, 0, c_1)$, $d = (d_1, 0, 0, d_1)$, c_1 and d_1 are not both equal to zero. The equation (2) implies that $b = (b_1, b_2, b_3, b_1)$. Let $b_1 = \rho\cos(\varphi)$, $b_2 = \rho\sin(\varphi)$, $\rho \in R_0$, $\varphi \in R$. Then α has the form

$$\begin{aligned}\alpha(s) &= (b_1 s + c_1\cos(ps) + d_1\sin(ps), \rho s\cos(\varphi), \rho s\sin(\varphi),\\ &\qquad b_1 s + c_1\cos(ps) + d_1\sin(ps))\\ &= (b_1 s + c_1\cos(ps) + d_1\sin(ps), \rho s, 0,\\ &\qquad b_1 s + c_1\cos(ps) + d_1\sin(ps))\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\varphi) & \sin(\varphi) & 0\\ 0 & -\sin(\varphi) & \cos(\varphi) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

Consequently, α lies fully in a 2-dimensional lightlike subspace of E_1^4 and hence lies in a timelike hyperplane of E_1^4, which is a contradiction.

$(a.3)$ It follows that c and d are mutually orthogonal timelike vectors in E_1^4, which is not possible.

Case (b). Since $g(\alpha',\alpha') = \pm 1$ and using the linear independence of the functions $\sinh(x)$ and $\cosh(x)$, we obtain the system of equations

$$g(b,b) + \frac{p^2}{2}(g(d,d) - g(c,c)) = \pm 1, \tag{1}$$

$$g(b,c) = g(b,d) = g(c,d) = 0, \tag{2}$$

$$g(c,c) + g(d,d) = 0. \tag{3}$$

With respect to the causal character of the vectors c and d, we distinguish three subcases: $(b.1)$ $g(c,c) = -g(d,d) > 0$; $(b.2)$ $g(c,c) = -g(d,d) < 0$; $(b.3)$ $g(c,c) = -g(d,d) = 0$.

$(b.1)$ We may take $c = (0,0,0,c_4)$, $d = (c_4,0,0,0)$, $c_4 \neq 0$. The equation (2) gives $b = (0,b_2,b_3,0)$. Let $b_2 = \rho\cos(\varphi)$, $b_3 = \rho\sin(\varphi)$, $\rho \in R_0$, $\varphi \in R$. Then α has the form

$$\begin{aligned}\alpha(s) &= (c_4\sinh(ps), \rho s\cos(\varphi), \rho s\sin(\varphi), c_4\cosh(ps))\\ &= (c_4\sinh(ps), \rho s, 0, c_4\cosh(ps))\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\varphi) & \sin(\varphi) & 0\\ 0 & -\sin(\varphi) & \cos(\varphi) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

This means that α lies fully in a timelike hyperplane of E^4_1, which is a contradiction.

$(b.2)$ In a similar way we get a contradiction.

$(b.3)$ We may assume that $c = (c_1,0,0,c_1)$, $d = (d_1,0,0,d_1)$, c_1 and d_1 are not both equal to zero. The equation (2) gives $b = (b_1,b_2,b_3,b_1)$. Let $b_2 = \rho\cos(\varphi)$, $b_3 = \rho\sin(\varphi)$, $\rho \in R_0$, $\varphi \in R$. It follows that α has the form

$$\begin{aligned}\alpha(s) = (&b_1 s + c_1\cosh(ps) + d_1\sinh(ps), \rho s\cos(\varphi), \rho s\sin(\varphi),\\ &b_1 s + c_1\cosh(ps) + d_1\sinh(ps))\\ = (&b_1 s + c_1\cosh(ps) + d_1\sinh(ps), \rho s, 0,\\ &b_1 s + c_1\cosh(ps) + d_1\sinh(ps))\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\varphi) & \sin(\varphi) & 0\\ 0 & -\sin(\varphi) & \cos(\varphi) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

Therefore, α lies fully in a 2-dimensional lightlike subspace and hence lies in a timelike hyperplane of E^4_1, which is a contradiction. □

Remark 3.1. All curves of null 2-type lie fully in a hyperplane of E^4_1.

THEOREM 3.2. *Let $\alpha(s)$ be a unit speed spacelike or timelike curve with both eigenvalues of its Laplace operator Δ in decomposition (1.1) different from zero, lying fully in E^4_1 and not lying in its timelike hyperplane. Then up to isometries of E^4_1, α is a 2-type curve if and only if α is a part of one of the following curves:*

(1) $\alpha(s) = (m\cosh(ts), n\cos(ps), -n\sin(ps), m\sinh(ts))$;
$p^2n^2 + t^2m^2 = 1$, $\quad p,t \in N$, $\quad m,n \in R_0$;

(2) $\alpha(s) = (m\sinh(ts), n\cos(ps), -n\sin(ps), m\cosh(ts))$;
$p^2n^2 - t^2m^2 = 1, \quad p, t \in N, \quad m, n \in R_0$;

(3) $\alpha(s) = (k\cos(ps) + l\sin(ps) + m\cosh(ts) + n\sinh(ts), r\cos(ps), -r\sin(ps), k\cos(ps) + l\sin(ps) + m\cosh(ts) + n\sinh(ts))$;
$p^2r^2 = 1, \quad m^2 + n^2 \neq 0, \quad p, t \in N, \quad m, n, k, l \in R$;

(4) $\alpha(s) = (m\cos(ps) + n\sin(ps), m\cos(ps) + n\sin(ps), k\sin(ts), k\cos(ts))$;
$t^2k^2 = 1, \quad m^2 + n^2 \neq 0, \quad p, t \in N, \quad t \neq 3p, \quad m, n \in R$;

(5) $\alpha(s) = (k\cos(ps) + l\sin(ps) + m\cos(ts) + n\sin(ts), k\cos(ps) + l\sin(ps) + m\cos(ts) + n\sin(ts), r\cos(ps), r\sin(ps))$;
$p^2r^2 = 1, \quad m^2 + n^2 \neq 0, \quad p, t \in N, \quad m, n, k, l \in R$;

(6) $\alpha(s) = (r\cos(ps), m\cos(p(\omega - s)), \frac{m^2}{12n}\cos(p(2\omega + s)) - \frac{r^2}{12n}\cos(ps) + n\cos(3ps), \frac{m^2}{12n}\sin(p(2\omega + s)) - \frac{r^2}{12n}\sin(ps) + n\sin(3ps))$;
$p^2(\frac{m^2 - r^2}{2} + (\frac{1}{12n})^2(m^4 + r^4 - 2m^2r^2\cos(2p\omega)) + 9n^2) = \pm 1, \quad p \in N$,
$r, m, n, \omega \in R_0$;

(7) $\alpha(s) = (r\sin(ps), r\sin(ps), m\cos(3ps), m\sin(3ps)), \quad (3pm)^2 = 1$,
$p \in N, \quad m, r \in R_0$;

(8) $\alpha(s) = (0, r\sin(ps), m\cos(ps) + k\cos(3ps), m\sin(ps) + k\sin(3ps))$;
$r^2 + 12mk = 0, \quad p^2(m - 3k)^2 = 1, \quad p \in N, \quad r, m, k \in R_0$;

(9) $\alpha(s) = (r\sin(ps), 0, r\cos(ps) + m\cos(3ps), r\sin(ps) + m\sin(3ps))$;
$r = 12m, \quad (9pm)^2 = 1, \quad p \in N, \quad m, r \in R_0$;

(10) $\alpha(s) = (-\frac{r^2}{12n}\cosh(ps) - \frac{m^2}{12n}\cosh(p(2\omega - s)) + n\cosh(3ps), r\cosh(ps), m\cosh(p(\omega + s)), -\frac{r^2}{12n}\sinh(ps) + \frac{m^2}{12n}\sinh(p(2\omega - s)) + n\sinh(3ps))$;
$p^2((\frac{1}{12n})^2(r^4 + m^4 + 2m^2r^2\cosh(2p\omega)) - \frac{r^2 + m^2}{2} + 9n^2) = \pm 1, \quad p \in N$,
$r, m, n, \omega \in R_0$;

(11) $\alpha(s) = (-\frac{r^2}{12n}\cosh(ps) - \frac{m^2}{12n}\cosh(p(2\omega - s)) + n\cosh(3ps), r\sinh(ps), m\cosh(p(\omega + s)), -\frac{r^2}{12n}\sinh(ps) + \frac{m^2}{12n}\sinh(p(2\omega - s)) + n\sinh(3ps))$;
$p^2((\frac{1}{12n})^2(r^4 + m^4 + 2m^2r^2\cosh(2p\omega)) + \frac{r^2 - m^2}{2} + 9n^2) = \pm 1, \quad p \in N$,
$r, m, n, \omega \in R_0$;

(12) $\alpha(s) = (-\frac{r^2}{6n}e^{-p(s+2\theta)} - \frac{m^2}{12n}\cosh(ps) + n\cosh(3ps), re^{p(s-\theta)}, m\cosh(ps), \frac{r^2}{6n}e^{-p(s+2\theta)} - \frac{m^2}{12n}\sinh(ps) + n\sinh(3ps))$;
$p^2((\frac{m}{12n})^2(m^2 + 4r^2e^{-2p\theta}) - \frac{m^2}{2} + 9n^2) = \pm 1, \ p \in N, \ r, m, n \in R_0, \ \theta \in R$;

(13) $\alpha(s) = (-\frac{r^2}{12n}\cosh(ps) - \frac{m^2}{12n}\cosh(p(2\omega - s)) + n\cosh(3ps), r\sinh(ps), m\sinh(p(\omega + s)), -\frac{r^2}{12n}\sinh(ps) + \frac{m^2}{12n}\sinh(p(2\omega - s)) + n\sinh(3ps))$;
$p^2((\frac{1}{12n})^2(r^4 + m^4 + 2r^2m^2\cosh(2p\omega)) + \frac{r^2 + m^2}{2} + 9n^2) = 1, \quad p \in N$,
$r, m, n, \omega \in R_0$;

(14) $\alpha(s) = (-\frac{r^2}{6n}e^{-p(s+2\theta)} - \frac{m^2}{12n}\cosh(ps) + n\cosh(3ps), re^{p(s-\theta)}, m\sinh(ps), \frac{r^2}{6n}e^{-p(s+2\theta)} - \frac{m^2}{12n}\sinh(ps) + n\sinh(3ps))$;
$p^2((\frac{m}{12n})^2(m^2 + 4r^2e^{-2p\theta}) + \frac{m^2}{2} + 9n^2) = 1, \ p \in N, \ r, m, n \in R_0, \ \theta \in R$;

(15) $\alpha(s) = (\frac{r^2}{12n}\sinh(ps) + \frac{m^2}{12n}\sinh(p(s - 2\omega)) + n\sinh(3ps), r\cosh(ps), m\cosh(p(\omega + s)), \frac{r^2}{12n}\cosh(ps) + \frac{m^2}{12n}\cosh(p(s - 2\omega)) + n\cosh(3ps))$;
$p^2((\frac{1}{12n})^2(-r^4 - m^4 - 2r^2m^2\cosh(2p\omega)) - \frac{r^2 + m^2}{2} - 9n^2) = -1, \quad p \in N$,

$r, m, n, \omega \in R_0$;

(16) $\alpha(s) = (\frac{r^2}{12n}\sinh(ps) + \frac{m^2}{12n}\sinh(p(s-2\omega)) + n\sinh(3ps), r\sinh(ps),$
$m\cosh(p(\omega+s)), \frac{r^2}{12n}\cosh(ps) + \frac{m^2}{12n}\cosh(p(s-2\omega)) + n\cosh(3ps));$
$p^2((\frac{1}{12n})^2(-r^4 - m^4 - 2r^2m^2\cosh(2p\omega)) + \frac{r^2-m^2}{2} - 9n^2) = \pm 1, \quad p \in N,$
$r, m, n, \omega \in R_0$;

(17) $\alpha(s) = (-\frac{r^2}{6n}e^{-p(s+2\theta)} + \frac{m^2}{12n}\sinh(ps) + n\sinh(3ps), re^{p(s-\theta)},$
$m\cosh(ps), \frac{r^2}{6n}e^{-p(s+2\theta)} + \frac{m^2}{12n}\cosh(ps) + n\cosh(3ps));$
$-p^2((\frac{m}{12n})^2(m^2 + 4r^2e^{-2p\theta}) + \frac{m^2}{2} + 9n^2) = -1$, $p \in N$, $r, m, n \in R_0$, $\theta \in R$;

(18) $\alpha(s) = (\frac{r^2}{12n}\sinh(ps) + \frac{m^2}{12n}\sinh(p(s-2\omega)) + n\sinh(3ps), r\sinh(ps),$
$m\sinh(p(\omega+s)), \frac{r^2}{12n}\cosh(ps) + \frac{m^2}{12n}\cosh(p(s-2\omega)) + n\cosh(3ps));$
$p^2((\frac{1}{12n})^2(-r^4 - m^4 - 2r^2m^2\cosh(2p\omega)) + \frac{r^2+m^2}{2} - 9n^2) = \pm 1, \quad p \in N,$
$r, m, n, \omega \in R_0$;

(19) $\alpha(s) = (-\frac{r^2}{6n}e^{-p(s+2\theta)} + \frac{m^2}{12n}\sinh(ps) + n\sinh(3ps), re^{p(s-\theta)},$
$m\sinh(ps), \frac{r^2}{6n}e^{-p(s+2\theta)} + \frac{m^2}{12n}\cosh(ps) + n\cosh(3ps));$
$p^2(-(\frac{m}{12n})^2(m^2 + 4r^2e^{-2p\theta}) + \frac{m^2}{2} - 9n^2) = \pm 1$, $p \in N$, $r, m, n \in R_0$, $\theta \in R$.

Proof. Suppose that $\alpha(s)$ satisfies the assumptions of the theorem and that it is a 2-type curve. Then $\alpha(s)$ can be written in one of the following forms

(a) $\alpha(s) = a + b\cos(ps) + c\sin(ps) + d\cosh(ts) + e\sinh(ts)$;

(b) $\alpha(s) = a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts)$;

(c) $\alpha(s) = a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts)$;

where $p, t \in N$, $a, b, c, d, e \in R^4$. Denote by R_0 the set of all real numbers different from zero. Next, assume that $0 < p < t$ and that $a = (0,0,0,0)$ up to a translation. Denote vectors b, c, d, e by $b = (b_1, b_2, b_3, b_4)$, $c = (c_1, c_2, c_3, c_4)$ and so on. In the sequel, we consider cases (a), (b) and (c).

Case(a). Since $g(\alpha', \alpha') = \pm 1$ and using the linear independence of the functions $\sin(x)$ and $\cos(x)$ as well as $\sinh(x)$ and $\cosh(x)$, we get the system of equations:

$$\frac{p^2}{2}(g(b,b) + g(c,c)) + \frac{t^2}{2}(g(e,e) - g(d,d)) = \pm 1, \tag{1}$$

$$g(c,c) - g(b,b) = 0, \tag{2}$$

$$g(e,e) + g(d,d) = 0, \tag{3}$$

$$g(b,d) = g(b,e) = g(c,d) = g(c,e) = g(b,c) = g(d,e) = 0. \tag{4}$$

With respect to the causal character of vectors d and e, we distinguish three subcases: $(a.1)$ $g(e,e) = -g(d,d) > 0$; $(a.2)$ $g(e,e) = -g(d,d) < 0$; $(a.3)$ $g(e,e) = -g(d,d) = 0$.

$(a.1)$ In this subcase, assume that $d = (e_4, 0, 0, 0)$, $e = (0, 0, 0, e_4)$, $e_4 \neq 0$. The equation (4) implies $b = (0, b_2, b_3, 0)$, $c = (0, c_2, c_3, 0)$. Next, let $b_2 = \rho\cos(p\varphi)$, $c_2 = \rho\sin(p\varphi)$, $\rho \in R_0$, $\varphi \in R$. Then the equations (2) and (4) imply $b_3 = \rho\sin(p\varphi)$,

$c_3 = -\rho\cos(p\varphi)$. Hence α has the form

$$\begin{aligned}\alpha(s) &= (e_4\cosh(ts), \rho\cos(p(\varphi - s)), \rho\sin(p(\varphi - s)), e_4\sinh(ts)) \\ &= (e_4\cosh(ts), \rho\cos(ps), -\rho\sin(ps), \\ &\qquad e_4\sinh(ts))\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(p\varphi) & \sin(p\varphi) & 0\\ 0 & -\sin(p\varphi) & \cos(p\varphi) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\end{aligned}$$

where the equation (1) becomes $p^2\rho^2 + t^2e_4^2 = 1$. Up to isometries of E_1^4, α lies fully in E_1^4, which gives form (1).

(a.2) We may assume that $e = (e_1, 0, 0, 0)$, $d = (0, 0, 0, e_1)$, $e_1 \neq 0$. This subcase is analogous to the subcase (a.1), so we get that (up to isometries of E_1^4) α has the form $\alpha(s) = (e_1\sinh(ts), \rho\cos(ps), -\rho\sin(ps), e_1\cosh(ts))$, where the equation (1) gives $p^2\rho^2 - t^2e_1^2 = \pm 1$. Therefore, α is a spacelike or a timelike curve lying fully in E_1^4, which gives form (2). The causal character of α depends on the choice of the constants.

(a.3) We may take that $d = (d_1, 0, 0, d_1)$, $e = (e_1, 0, 0, e_1)$, d_1 and e_1 are not both equal to zero. The equation (4) gives $b = (b_1, b_2, b_3, b_1)$, $c = (c_1, c_2, c_3, c_1)$. Let $b_2 = \rho\cos(p\varphi)$, $c_2 = \rho\sin(p\varphi)$, $\rho \in R_0$, $\varphi \in R$. Then equations (2) and (4) imply $b_3 = \rho\sin(p\varphi)$, $c_3 = -\rho\cos(p\varphi)$. Hence α has the form

$$\begin{aligned}\alpha(s) = (&b_1\cos(ps) + c_1\sin(ps) + d_1\cosh(ts) + e_1\sinh(ts), \rho\cos(p(\varphi - s)), \\ &\rho\sin(p(\varphi - s)), b_1\cos(ps) + c_1\sin(ps) + d_1\cosh(ts) + e_1\sinh(ts)) \\ = (&b_1\cos(ps) + c_1\sin(ps) + d_1\cosh(ts) + e_1\sinh(ts), \rho\cos(ps), -\rho\sin(ps), \\ &b_1\cos(ps) + c_1\sin(ps) + d_1\cosh(ts) \\ &+ e_1\sinh(ts))\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(p\varphi) & \sin(p\varphi) & 0\\ 0 & -\sin(p\varphi) & \cos(p\varphi) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\end{aligned}$$

where the equation (1) gives $p^2\rho^2 = 1$. Up to isometries of E_1^4, α is a spacelike curve lying fully in a lightlike hyperplane of E_1^4, which gives form (3).

Case(b). After using the equation $g(\alpha'(s), \alpha'(s)) = \pm 1$, we get that the arguments of the functions $\sin(x)$ and $\cos(x)$ are the numbers $\{2p, 2t, p+t, t-p\}$. Since $0 < p < t$, we distinguish two subcases: (b.1) $t - p \neq 2p$; (b.2) $t - p = 2p$.

(b.1) $t - p \neq 2p$. The corresponding system of equations has the form

$$\begin{aligned}&(1) \qquad \tfrac{p^2}{2}(g(b, b) + g(c, c)) + \tfrac{t^2}{2}(g(d, d) + g(e, e)) = \pm 1,\\ &(2) \qquad g(c, c) - g(b, b) = 0,\\ &(3) \qquad g(e, e) - g(d, d) = 0,\\ &(4) \qquad g(b, c) = g(b, d) = g(b, e) = g(c, d) = g(c, e) = g(d, e) = 0.\end{aligned}$$

With respect to the causal character of vectors d and e, we distinguish three subcases: $(b.1.1)$ $g(e,e) = g(d,d) > 0$; $(b.1.2)$ $g(e,e) = g(d,d) = 0$; $(b.1.3)$ $g(e,e) = g(d,d) > 0$.

$(b.1.1)$ Take that $e = (0,0,e_3,0)$, $d = (0,0,0,e_3)$, $e_3 \neq 0$. Then the equation (4) implies $b = (b_1, b_2, 0, 0,)$, $c = (c_1, c_2, 0, 0)$. If $g(b,b) = g(c,c) > 0$, then there exist four mutually orthogonal spacelike vectors b, c, d, e in E_1^4, which is impossible. If $g(b,b) = g(c,c) < 0$, then there exist two mutually orthogonal timelike vectors b and c in E_1^4, which is impossible again. Finally, if $g(b,b) = g(c,c) = 0$, take $b = (b_1, b_1, 0, 0)$, $c = (c_1, c_1, 0, 0)$, b_1 and c_1 are not both equal to zero. We obtain that α has the equation $\alpha(s) = (b_1\cos(ps) + c_1\sin(ps), b_1\cos(ps) + c_1\sin(ps), e_3\sin(ts), e_3\cos(ts))$, where the equation (1) becomes $t^2e_3^2 = 1$. Thus α is a spacelike curve, lying fully in a lightlike hyperplane of E_1^4, which gives form (4).

$(b.1.2)$ Assume that $d = (d_1, d_1, 0, 0)$, $e = (e_1, e_1, 0, 0)$, d_1 and e_1 are not both equal to zero. Then (4) implies that $b = (b_1, b_1, b_3, b_4)$, $c = (c_1, c_1, c_3, c_4)$. Let $b_3 = \rho\cos(p\varphi)$, $b_4 = \rho\sin(p\varphi)$, $\rho \in R_0$, $\varphi \in R$. The equations (2) and (4) give $c_3 = -\rho\sin(p\varphi)$, $c_4 = \rho\cos(p\varphi)$. Thus α has the form

$$\begin{aligned}
\alpha(s) = (&b_1\cos(ps) + c_1\sin(ps) + d_1\cos(ts) + e_1\sin(ts),\\
&b_1\cos(ps) + c_1\sin(ps) + d_1\cos(ts) + e_1\sin(ts),\\
&\rho\cos(p(\varphi+s)), \rho\sin(p(\varphi+s)))\\
= (&b_1\cos(ps) + c_1\sin(ps) + d_1\cos(ts) + e_1\sin(ts),\\
&b_1\cos(ps) + c_1\sin(ps) + d_1\cos(ts) + e_1\sin(ts),\\
&\rho\cos(ps), \rho\sin(ps)) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(p\varphi) & \sin(p\varphi) \\ 0 & 0 & -\sin(p\varphi) & \cos(p\varphi) \end{bmatrix},
\end{aligned}$$

where the equation (1) becomes $p^2\rho^2 = 1$. Up to isometries of E_1^4, α is a spacelike curve lying fully in a lightlike hyperplane of E_1^4, which gives form (5).

$(b.1.3)$ It follows that d and e are mutually orthogonal timelike vectors in E_1^4, which is not possible.

$(b.2)$ Since $g(\alpha', \alpha') = \pm 1$, we get the system of the equations:

$$\frac{p^2}{2}(g(b,b) + g(c,c)) + \frac{t^2}{2}(g(d,d) + g(e,e)) = \pm 1, \tag{1}$$
$$\frac{p^2}{2}(g(c,c) - g(b,b)) + pt(g(b,d) + g(c,e)) = 0, \tag{2}$$
$$g(e,e) - g(d,d) = 0, \tag{3}$$
$$-p^2 g(b,c) + pt(g(b,e) - g(c,d)) = 0, \tag{4}$$
$$g(d,e) = 0, \tag{5}$$
$$g(c,e) - g(b,d) = 0, \tag{6}$$
$$g(b,e) + g(c,d) = 0. \tag{7}$$

Again we distinguish three subcases: $(b.2.1)$ $g(e,e) = g(d,d) > 0$; $(b.2.2)$ $g(e,e) = g(d,d) = 0$; $(b.2.3)$ $g(e,e) = g(d,d) < 0$.

$(b.2.1)$ We may take $d = (0,0,d_3,0)$, $e = (0,0,0,d_3)$, $d_3 \neq 0$. The equations (6) and (7) give $b = (b_1,b_2,b_3,b_4)$, $c = (c_1,c_2,-b_4,b_3)$. If b, d, e (c, d, e) are linearly dependent vectors, then $b_1 = b_2 = 0$ ($c_1 = c_2 = 0$), so α lies fully in a 3-dimensional subspace of the space E_1^4. Moreover, if $b_1 \neq 0$ and $c_2 \neq 0$ ($b_2 \neq 0$ and $c_1 \neq 0$), then α lies fully in the space E_1^4. In the sequel, we consider each of these cases.

$(b.2.1.1)$ $b_1 \neq 0$, $c_2 \neq 0$ (or $b_2 \neq 0$, $c_1 \neq 0$).

Let $b_1 = \rho\cos(p\varphi)$, $c_1 = \rho\sin(p\varphi)$, $b_2 = m\cos(p\theta)$, $c_2 = m\sin(p\theta)$, $\rho, m \in R_0$, $\varphi, \theta \in R$, $\varphi \neq \theta$, $p\varphi \neq \frac{\pi}{2} + k\pi$, $p\theta \neq k\pi$, $k \in Z$. Then the equations (2) and (4) imply that $b_3 = \frac{1}{12d_3}(m^2\cos(2p\theta) - \rho^2\cos(2p\varphi))$, $b_4 = \frac{1}{12d_3}(m^2\sin(2p\theta) - \rho^2\sin(2p\varphi))$. Consequently, α has the form

$$\begin{aligned}\alpha(s) = (&\rho\cos(p(\varphi - s)), m\cos(p(\theta - s)),\\ &\tfrac{m^2}{12d_3}\cos(p(2\theta + s)) - \tfrac{\rho^2}{12d_3}\cos(p(2\varphi + s)) + d_3\cos(3ps),\\ &\tfrac{m^2}{12d_3}\sin(p(2\theta + s)) - \tfrac{\rho^2}{12d_3}\sin(p(2\varphi + s)) + d_3\sin(3ps)).\end{aligned}$$

Putting $u = s - \varphi$ and $\omega = \theta - \varphi$, we get that

$$\begin{aligned}\alpha(u) = (&\rho\cos(pu), m\cos(p(\omega - u)), \tfrac{m^2}{12d_3}\cos(p(2\omega + u)) - \tfrac{\rho^2}{12d_3}\cos(pu)\\ &+ d_3\cos(3pu), \tfrac{m^2}{12d_3}\sin(p(2\omega + u)) - \tfrac{\rho^2}{12d_3}\sin(pu)\\ &+ d_3\sin(3pu))\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos(3p\varphi) & \sin(3p\varphi)\\ 0 & 0 & -\sin(3p\varphi) & \cos(3p\varphi)\end{bmatrix},\end{aligned}$$

where the equation (1) becomes $p^2(\frac{1}{2}(m^2-\rho^2)+(\frac{1}{12d_3})^2(m^4+\rho^4-2m^2\rho^2\cos(2p\omega))+9d_3^2) = \pm 1$. Up to isometries of E_1^4, α is a spacelike or a timelike curve lying fully in E_1^4, which gives form (6).

$(b.2.1.2)$ $b_1 = b_2 = 0$ (or $c_1 = c_2 = 0$).

The equation (4) implies $b_4 = 0$ and thus $b = (0,0,b_3,0)$, $c = (c_1,c_2,0,b_3)$. If $b_3 = 0$, then the equation (2) gives $c = (c_1,c_1,0,0)$, $c_1 \neq 0$. Therefore, α has the equation $\alpha(s) = (c_1\sin(ps), c_1\sin(ps), d_3\cos(3ps), d_3\sin(3ps))$, where the equation (1) becomes $(3pd_3)^2 = 1$. It follows that α is a spacelike curve, lying fully in a lightlike hyperplane of E_1^4. This gives us form (7). Next, suppose that $b_3 \neq 0$. If c is spacelike, let $c_1 = \rho\sinh(p\theta)$, $c_2 = \rho\cosh(p\theta)$, $\rho \in R_0$, $\theta \in R$. Then the equation (2) gives $\rho^2 + 12b_3d_3 = 0$ and α has the equation

$$\begin{aligned}\alpha(s) &= (\rho\sinh(p\theta)\sin(ps), \rho\cosh(p\theta)\sin(ps), b_3\cos(ps) + d_3\cos(3ps),\\ &\qquad b_3\sin(ps) + d_3\sin(3ps))\\ &= (0, \rho\sin(ps), b_3\cos(ps) + d_3\cos(3ps),\\ &\qquad b_3\sin(ps) + d_3\sin(3ps))\begin{bmatrix}\cosh(p\theta) & \sinh(p\theta) & 0 & 0\\ \sinh(p\theta) & \cosh(p\theta) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\end{aligned}$$

where the equation (1) becomes $p^2(b_3 - 3d_3)^2 = 1$. Up to isometries of E_1^4, α is a spacelike curve lying fully in a spacelike hyperplane of E_1^4, which gives form (8). If c is timelike, then α lies fully in a timelike hyperplane of E_1^4, which is a contradiction. Finally, if c is null, let $c_1 = b_3\cosh(p\theta)$, $c_2 = b_3\sinh(p\theta)$, $b_3 \in R_0$, $\theta \in R$. Then (2) gives $b_3 = 12d_3$, so α has the form

$$\begin{aligned}\alpha(s) = (&b_3\cosh(p\theta)\sin(ps), b_3\sinh(p\theta)\sin(ps), b_3\cos(ps) + d_3\cos(3ps),\\ &b_3\sin(ps) + d_3\sin(3ps))\\ = (&b_3\sin(ps), 0, b_3\cos(ps) + d_3\cos(3ps),\\ &b_3\sin(ps) + d_3\sin(3ps))\begin{bmatrix}\cosh(p\theta) & \sinh(p\theta) & 0 & 0\\ \sinh(p\theta) & \cosh(p\theta) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

Then the equation (1) becomes $(9pd_3)^2 = 1$ and up to isometries of E_1^4, α is a spacelike curve lying fully in a lightlike hyperplane of the space E_1^4, which gives form (9).

$(b.2.2)$ Take that $d = (d_1, d_1, 0, 0)$, $e = (e_1, e_1, 0, 0)$, i.e., that $d = \lambda e$, $\lambda \in R$, d_1 and e_1 are not both equal to zero. The equations (6) and (7) imply $(1 + \lambda^2)g(c, d) = 0$ and thus $g(c, d) = 0$. Then $g(b, d) = 0$, and thus $b = (b_1, b_1, b_3, b_4)$, $c = (c_1, c_1, c_3, c_4)$. We have the same vectors b, c, d, e as in the subcase (b.1.2), so we get the same form (5) of the curve α.

$(b.2.3)$ It follows that d and e are two mutually orthogonal timelike vectors in E_1^4, which is not possible.

Case (c). After using the equation $g(\alpha', \alpha') = \pm 1$, we get that arguments of the functions $\sinh(x)$ and $\cosh(x)$ are the numbers $\{2p, 2t, p+t, t-p\}$. Since $0 < p < t$, we distinguish two subcases: $(c.1)$ $t - p \neq 2p$; $(c.2)$ $t - p = 2p$.

$(c.1)$ $t - p \neq 2p$. The corresponding system of equations has the form:

$$(1)\qquad \tfrac{p^2}{2}(g(c,c) - g(b,b)) + \tfrac{t^2}{2}(g(e,e) - g(d,d)) = \pm 1,$$

$$(2)\qquad g(d,d) + g(e,e) = 0,$$

$$(3)\qquad g(b,b) + g(c,c) = 0,$$

$$(4)\qquad g(b,c) = g(b,d) = g(b,e) = g(c,d) = g(c,e) = g(d,e) = 0.$$

With respect to the causal character of vectors d and e, we distinguish three subcases: $(c.1.1)$ $g(e,e) = -g(d,d) > 0$; $(c.1.2)$ $g(e,e) = -g(d,d) < 0$; $(c.1.3)$ $g(e,e) = -g(d,d) = 0$.

$(c.1.1)$ Assume that $d = (e_4, 0, 0, 0)$, $e = (0, 0, 0, e_4)$, $e_4 \neq 0$. The equation (4) implies $b = (0, b_2, b_3, 0)$, $c = (0, c_2, c_3, 0)$, while the equation (3) gives $b = c = 0$, which is a contradiction.

$(c.1.2)$ Again we get a contradiction.

$(c.1.3)$ Assume that $d = (d_1, 0, 0, d_1)$, $e = (e_1, 0, 0, e_1)$, d_1 and e_1 are not both equal to zero. Then the equations (3) and (4) imply $b = c = 0$, which is a contradiction.

$(c.2)$ $t - p = 2p$. The corresponding system of equations has the form:

$$\begin{aligned}
&(1) && \tfrac{p^2}{2}(g(c,c) - g(b,b)) + \tfrac{t^2}{2}(g(e,e) - g(d,d)) = \pm 1,\\
&(2) && g(e,e) + g(d,d) = 0,\\
&(3) && \tfrac{p^2}{2}(g(b,b) + g(c,c)) + pt(g(c,e) - g(b,d)) = 0,\\
&(4) && p^2 g(b,c) + pt(g(c,d) - g(b,e)) = 0,\\
&(5) && g(d,e) = 0,\\
&(6) && g(b,d) + g(c,e) = 0,\\
&(7) && g(b,e) + g(c,d) = 0.
\end{aligned}$$

With respect to the causal character of vectors d and e, we distinguish three subcases: $(c.2.1)$ $g(e,e) = -g(d,d) > 0$; $(c.2.2)$ $g(e,e) = -g(d,d) < 0$; $(c.2.3)$ $g(e,e) = -g(d,d) = 0$.

$(c.2.1)$ We may assume that $e = (0,0,0,e_4)$, $d = (e_4,0,0,0)$, $e_4 \neq 0$. The equations (6) and (7) give $b = (b_1,b_2,b_3,b_4)$, $c = (b_4,c_2,c_3,b_1)$. If b, d, e (c, d, e) are linearly dependent vectors, then α lies fully in a timelike hyperplane of E_1^4, which is a contradiction. Next, comparing the numbers b_2 and c_2 as well as b_3 and c_3, the following possibilities may occur:

$(c.2.1.1)$ $b_2^2 > c_2^2$, $\quad b_3^2 > c_3^2$;
$(c.2.1.2)$ $b_2^2 < c_2^2$, $\quad b_3^2 > c_3^2$ $\quad$ (or $b_2^2 > c_2^2$, $\quad b_3^2 < c_3^2$);
$(c.2.1.3)$ $b_2 = c_2 \neq 0$, $\quad b_3^2 > c_3^2$ $\quad$ (or $b_2^2 > c_2^2$, $\quad b_3 = c_3 \neq 0$);
$(c.2.1.4)$ $b_2^2 < c_2^2$, $\quad b_3^2 < c_3^2$;
$(c.2.1.5)$ $b_2 = c_2 \neq 0$, $\quad b_3^2 < c_3^2$ $\quad$ (or $b_2^2 < c_2^2$, $\quad b_3 = c_3 \neq 0$);
$(c.2.1.6)$ $b_2 = c_2 \neq 0$, $\quad b_3 = c_3 \neq 0$.

In the sequel, we consider them separately.

$(c.2.1.1)$ Let $b_2 = \rho\cosh(p\varphi)$, $c_2 = \rho\sinh(p\varphi)$, $b_3 = m\cosh(p\theta)$, $c_3 = m\sinh(p\theta)$, $m,\rho \in R_0$, $\varphi,\theta \in R$, $\varphi \neq \theta$. Then the equations (3) and (4) imply $b_1 = -\frac{1}{12e_4}(\rho^2\cosh(2p\varphi) + m^2\cosh(2p\theta))$, $b_4 = \frac{1}{12e_4}(\rho^2\sinh(2p\varphi) + m^2\sinh(2p\theta))$. Thus α has the form

$$\begin{aligned}
\alpha(s) = (&\tfrac{-\rho^2}{12e_4}\cosh(p(2\varphi - s)) - \tfrac{m^2}{12e_4}\cosh(p(2\theta - s)) + e_4\cosh(3ps),\\
&\rho\cosh(p(\varphi + s)), m\cosh(p(\theta + s)),\\
&\tfrac{\rho^2}{12e_4}\sinh(p(2\varphi - s)) + \tfrac{m^2}{12e_4}\sinh(p(2\theta - s)) + e_4\sinh(3ps)).
\end{aligned}$$

Putting $u = s + \varphi$ and $\omega = \theta - \varphi$, we obtain that

$$\begin{aligned}
\alpha(u) = (&-\tfrac{\rho^2}{12e_4}\cosh(pu) - \tfrac{m^2}{12e_4}\cosh(p(2\omega - u)) + e_4\cosh(3pu), \rho\cosh(pu),\\
&m\cosh(p(u + \omega)), -\tfrac{\rho^2}{12e_4}\sinh(pu) + \tfrac{m^2}{12e_4}\sinh(p(2\omega - u))\\
&+ e_4\sinh(3pu))\begin{bmatrix} \cosh(3p\varphi) & 0 & 0 & -\sinh(3p\varphi)\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\sinh(3p\varphi) & 0 & 0 & \cosh(3p\varphi) \end{bmatrix},
\end{aligned}$$

where the equation (1) becomes $p^2((\frac{1}{12e_4})^2(\rho^4+m^4+2m^2\rho^2\cosh(2p\omega))-\frac{1}{2}(\rho^2+m^2)+9e_4^2)=\pm 1$. Up to isometries of E_1^4, α is a spacelike or a timelike curve lying fully in E_1^4, which gives form (10).

(c.2.1.2) Using the similar methods as in the previous subcase (c.2.1.1), we get that α has the form

$$\begin{aligned}\alpha(u)=&(\tfrac{-\rho^2}{12e_4}\cosh(pu)-\tfrac{m^2}{12e_4}\cosh(p(2\omega-u))+e_4\cosh(3pu),\rho\sinh(pu),\\&m\cosh(p(u+\omega)),\tfrac{-\rho^2}{12e_4}\sinh(pu)+\tfrac{m^2}{12e_4}\sinh(p(2\omega-u))+e_4\sinh(3pu)),\end{aligned}$$

where the equation (1) becomes $p^2((\frac{1}{12e_4})^2(\rho^4+m^4+2m^2\rho^2\cosh(2p\omega))+\frac{1}{2}(\rho^2-m^2)+9e_4^2)=\pm 1$. Hence α is a spacelike or a timelike curve lying fully in E_1^4, which gives form (11). In the case $b_2^2>c_2^2$, $b_3^2<c_3^2$, up to isometries we obtain the same form (11).

(c.2.1.3) Let $b_3=m\cosh(p\theta)$, $c_3=m\sinh(p\theta)$, $m\in R_0$, $\theta\in R$. Then the equations (3) and (4) imply $b_1=-\frac{1}{12e_4}(2b_2^2+m^2\cosh(2p\theta))$, $b_4=\frac{1}{12e_4}(2b_2^2+m^2\sinh(2p\theta))$. It follows that α has the equation

$$\begin{aligned}\alpha(s)=&(\tfrac{-b_2^2}{6e_4}e^{-ps}-\tfrac{m^2}{12e_4}\cosh(p(2\theta-s))+e_4\cosh(3ps),b_2e^{ps},m\cosh(p(\theta+s)),\\&\tfrac{b_2^2}{6e_4}e^{-ps}+\tfrac{m^2}{12e_4}\sinh(p(2\theta-s))+e_4\sinh(3ps)).\end{aligned}$$

Putting $u=s+\theta$, we get that

$$\begin{aligned}\alpha(u)=&(\tfrac{-b_2^2}{6e_4}e^{-p(u+2\theta)}-\tfrac{m^2}{12e_4}\cosh(pu)+e_4\cosh(3pu),b_2e^{p(u-\theta)},m\cosh(pu),\\&\tfrac{b_2^2}{6e_4}e^{-p(u+2\theta)}-\tfrac{m^2}{12e_4}\sinh(pu)\\&+e_4\sinh(3pu))\begin{bmatrix}\cosh(3p\theta)&0&0&-\sinh(3p\theta)\\0&1&0&0\\0&0&1&0\\-\sinh(3p\theta)&0&0&\cosh(3p\theta)\end{bmatrix},\end{aligned}$$

where the equation (1) becomes $p^2((\frac{m}{12e_4})^2(m^2+4b_2^2e^{-2p\theta})-\frac{m^2}{2}+9e_4^2)=\pm 1$. Up to isometries of E_1^4, α is a spacelike or a timelike curve lying fully in E_1^4, which gives form (12). In the case $b_2^2>c_2^2$, $b_3=c_3\neq 0$, (up to isometries) we obtain the same form (12).

(c.2.1.4) Using the similar methods as in the subcase (c.2.1.1), up to isometries of E_1^4 we get that α has the form

$$\begin{aligned}\alpha(s)=&(-\tfrac{\rho^2}{12e_4}\cosh(ps)-\tfrac{m^2}{12e_4}\cosh(p(2\omega-s))+e_4\cosh(3ps),\rho\sinh(ps),\\&m\sinh(p(s+\omega)),-\tfrac{\rho^2}{12e_4}\sinh(ps)+\tfrac{m^2}{12e_4}\sinh(p(2\omega-s))+e_4\sinh(3ps))\end{aligned}$$

where the equation (1) becomes $p^2((\frac{1}{12e_4})^2(\rho^4+m^4+2m^2\rho^2\cosh(2p\omega))+\frac{\rho^2+m^2}{2}+9e_4^2)=1$. Thus α is a spacelike curve lying fully in E_1^4, which gives form (13).

(c.2.1.5) Using the similar methods as in the subcase (c.2.1.3), up to isometries of E_1^4 we obtain that α has the equation

$$\alpha(u) = \big(-\tfrac{b_2^2}{6e_4}e^{-p(u+2\theta)} - \tfrac{m^2}{12e_4}\cosh(pu) + e_4\cosh(3pu), b_2 e^{p(u-\theta)},\\ m\sinh(pu), \tfrac{b_2^2}{6e_4}e^{-p(u+2\theta)} - \tfrac{m^2}{12e_4}\sinh(pu) + e_4\sinh(3pu)\big)$$

where the equation (1) becomes $p^2((\frac{m}{12e_4})^2(m^2+4b_2^2e^{-2p\theta}) + \frac{m^2}{2} + 9e_4^2) = 1$. Thus α is a spacelike curve lying fully in E_1^4, which gives form (14). In the case $b_2^2 < c_2^2$, $b_3 = c_3 \neq 0$ (up to isometries) we obtain the same form (14).

(c.2.1.6) The equations (3) and (4) give $b_1 = -\frac{1}{6e_4}(b_2^2 + b_3^2) = -b_4$. It follows that b, c, d, e are linearly dependent vectors and since d is timelike, α lies fully in a timelike hyperplane of E_1^4, which is a contradiction.

(c.2.2) This subcase is analogous to the subcase (c.2.1). Thus the similar computations give forms (15), (16), (17), (18) and (19) of the curve α.

(c.2.3) Since d and e are two linearly dependent null vectors, α lies in a 3-dimensional subspace of E_1^4. Let $d = (d_1, d_1, 0, 0)$, $e = (e_1, e_1, 0, 0)$, d_1 and e_1 are not both equal to zero. Then $d = \lambda e$, $\lambda \in R$, and the equations (6) and (7) give $(1-\lambda^2)\,g(c,d) = 0$. Hence we distinguish two subcases: (c.2.3.1) $g(c,d) = 0$; (c.2.3.2) $\lambda^2 = 1$.

(c.2.3.1) The equation (7) gives $g(b,e) = 0$ and thus $b = (b_1, b_1, b_3, b_4)$, $c = (c_1, c_1, c_3, c_4)$. Then $g(b,d) = g(c,e) = 0$, so the equations (3) and (4) give $b = c = 0$, which is a contradiction.

(c.2.3.2) The equation (6) implies $g(b+\lambda c, d) = 0$. Next, equations (3) and (4) give that d and $b - \lambda c$ are two linearly independent null vectors. On the contrary, if d and $b-\lambda c$ are two linearly dependent null vectors, then $b - \lambda c = \mu d$, $\mu \in R$. Thus (6) implies $g(\lambda c + \mu d, d) + g(c, \lambda d) = 0$, i.e., $g(c,d) = 0$, which gives a contradiction, as in subcase (c.2.3.1). Next, assume that $b - \lambda c = (-a_0, a_0, 0, 0)$, $a_0 \neq 0$. Then $b + \lambda c = (b_0, b_0, b_3, b_4)$, $b_4 \neq 0$ and the last two equations for $b - \lambda c$ and $b + \lambda c$ give $b = \frac{1}{2}(b_0 - a_0, b_0 + a_0, b_3, b_4)$, $c = \frac{\lambda}{2}(a_0 + b_0, b_0 - a_0, b_3, b_4)$. Let $m = \frac{b_0 - a_0}{2}$, $n = \frac{b_0 + a_0}{2}$. Consequently, α has the form

$$\alpha(s) = (m\cosh(ps) + \lambda n\sinh(ps) + d_1 e^{3\lambda ps}, n\cosh(ps) + \lambda m\sinh(ps) + d_1 e^{3\lambda ps},\\ \tfrac{b_3}{2}e^{\lambda ps}, \tfrac{b_4}{2}e^{\lambda ps}).$$

Finally, let $b_3 = \rho\cos(p\theta)$, $b_4 = \rho\sin(p\theta)$, $\rho \in R_0$, $\theta \in R$. Then we get that

$$\alpha(s) = (m\cosh(ps) + \lambda n\sinh(ps) + d_1 e^{3\lambda ps}, n\cosh(ps) + \lambda m\sinh(ps) + d_1 e^{3\lambda ps},\\ \tfrac{1}{2}\rho e^{\lambda ps}, 0)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(p\theta) & \sin(p\theta) \\ 0 & 0 & \sin(p\theta) & -\cos(p\theta) \end{bmatrix}$$

and thus α lies fully in a timelike hyperplane of E_1^4 which is a contradiction.

4. A classification of 2-type curves in the Minkowski space $\mathbf{E}_1^5$

Since a 2-type curve is contained in at most 4-dimensional subspace of the space E_1^n, we study the cases when α lies in a spacelike, timelike or lightlike hyperplane of E_1^5. The case when α lies fully in a timelike hyperplane of E_1^5 is equivalent to the case when α lies fully in E_1^4. This case has already been studied in the theorems 3.1 and 3.2.

THEOREM 4.1. *Let $\alpha(s)$ be a unit speed spacelike or timelike curve, with both eigenvalues of its Laplace operator Δ in decomposition (1.1) different from zero, lying fully in E_1^5 and not lying in its timelike hyperplane. Then up to isometries of E_1^5, α is a 2-type curve if and only if α is a part of one of the following curves:*

(1) $\alpha(s)=(0, m\cos(ps), m\sin(ps), n\sin(ts), n\cos(ts))$, $\quad p^2m^2+t^2n^2=1$, $p,t\in N$, $\quad t\neq 3p$, $\quad m,n\in R_0$;

(2) $\alpha(s)=(0, m\sin(ps), r\cos(p(s-\theta)), \frac{r^2}{12n}\cos(p(2\theta+s))-\frac{m^2}{12n}\cos(ps)$
$+n\cos(3ps), \frac{r^2}{12n}\sin(p(2\theta+s))-\frac{m^2}{12n}\sin(ps)+n\sin(3ps))$,
$p^2(\frac{r^2+m^2}{2}+(\frac{1}{12n})^2((r^2-m^2)^2+4r^2m^2\sin^2(p\theta))+9n^2)=1$, $\quad p\in N$,
$\theta\in R$, $\quad r,m,n\in R_0$;

(3) $\alpha(s)=(m\sin(p(s+\theta)), m\sin(p(s+\theta)), r\cos(ps), \frac{r^2}{12n}\cos(ps)+n\cos(3ps)$,
$\frac{r^2}{12n}\sin(ps)+n\sin(3ps))$,
$p^2(\frac{r^2}{2}+(\frac{r^2}{12n})^2+9n^2)=1$, $\quad p\in N$, $\quad \theta\in R$, $\quad r,m,n\in R_0$.

Proof. Suppose that $\alpha(s)$ satisfies the assumptions of the theorem and that it is a 2-type curve. Then proof of the theorem 3.2 implies that subcases $(b.1.1)$ and $(b.2.1)$ are now the only possible cases. With respect to the causal character of vectors b, c, d, e, it is easy to see that in all other subcases we get a contradiction. In the sequel, we consider subcases $(b.1.1)$ and $(b.2.1)$ separately.

$(b.1.1)$ $g(e,e)=g(d,d)>0$. If $g(b,b)=g(c,c)>0$, then b, c, d, e are mutually orthogonal spacelike vectors, so we may take $b=(0,b_2,0,0,0)$, $c=(0,0,b_2,0,0)$, $d=(0,0,0,d_4,0)$, $e=(0,0,0,0,d_4)$, $b_2\neq 0$, $d_4\neq 0$. Thus α has the form

$$\alpha(s)=(0, b_2\cos(ps), b_2\sin(ps), d_4\cos(ts), d_4\sin(ts)),$$

where the equation (1) becomes $p^2b_2^2+t^2d_4^2=1$. Up to isometries of E_1^5, α is a spacelike curve lying fully in a spacelike hyperplane of the space E_1^5, which gives form (1).

$(b.2.1)$ $g(e,e)=g(d,d)>0$. Take that $d=(0,0,0,d_4,0)$, $e=(0,0,0,0,d_4)$, $d_4\neq 0$. Further, vectors b and c belong to a spacelike or to a lightlike hyperplane of E_1^5. If b and c belong to a spacelike hyperplane, then they are all spacelike vectors. Let $b=m_1f+m_2d+m_3e$, $c=n_1h+n_2f+n_3d+n_4e$, where $f=(0,0,d_4,0,0)$, $h=(0,d_4,0,0,0)$, $m_1,m_2,m_3,n_1,n_2,n_3,n_4\in R_0$. It follows that $b=(0,0,b_3,b_4,b_5)$, $c=(0,c_2,c_3,c_4,c_5)$. Next, the equations (6) and (7) give $c=(0,c_2,c_3,-b_5,b_4)$ and the equations (2) and (4) give $b_4=\frac{1}{12d_4}(b_3^2-c_2^2-c_3^2)$, $b_5=\frac{1}{6d_4}b_3c_3$. Let

$b_3 = \rho\cos(p\theta)$, $c_3 = \rho\sin(p\theta)$, $\rho \in R_0$, $\theta \in R$. Then α has the form

$$\alpha(s) = (0, c_2\sin(ps), \rho\cos(p(s-\theta)), \tfrac{\rho^2}{12d_4}\cos(p(2\theta+s)) - \tfrac{c_2^2}{12d_4}\cos(ps) + d_4\cos(3ps), \tfrac{\rho^2}{12d_4}\sin(p(2\theta+s)) - \tfrac{c_2^2}{12d_4}\sin(ps) + d_4\sin(3ps)),$$

where the equation (1) becomes $p^2(\frac{\rho^2+c_2^2}{2} + \frac{1}{(12d_4)^2}((\rho^2-c_2^2)^2 + 4\rho^2c_2^2\sin^2(p\theta)) + 9n^2) = 1$. Up to isometries of E_1^5, α is a spacelike curve lying fully in a spacelike hyperplane of the space E_1^5, which gives form (2). Finally, if b and c belong to a lightlike hyperplane of E_1^5, let $b = m_1f+m_2d+m_3e$, $c = n_1h+n_2f+n_3d+n_4e$, where $f = (0,0,d_4,0,0)$, $h = (d_4,d_4,0,0,0)$, $m_1,m_2,m_3,n_1,n_2,n_3,n_4 \in R_0$. It follows that $b = (0,0,b_3,b_4,b_5)$, $c = (c_1,c_1,c_3,c_4,c_5)$. Further, equations (6) and (7) give $c = (c_1,c_1,c_3,-b_5,b_4)$ and from equations (2) and (4) follows $b_4 = \frac{1}{12d_4}(b_3^2 - c_3^2)$, $b_5 = \frac{1}{6d_4}b_3c_3$. Let $b_3 = \rho\cos(p\theta)$, $c_3 = \rho\sin(p\theta)$, $\rho \in R_0$, $\theta \in R$. Thus α has the form

$$\alpha(s) = (c_1\sin(ps), c_1\sin(ps), \rho\cos(p(s-\theta)), \tfrac{\rho^2}{12d_4}\cos(p(2\theta+s)) + d_4\cos(3ps), \tfrac{\rho^2}{12d_4}\sin(p(2\theta+s)) + d_4\sin(3ps)).$$

Putting $u = s-\theta$, we get that

$$\alpha(u) = (c_1\sin(p(u+\theta)), c_1\sin(p(u+\theta)), \rho\cos(pu), \tfrac{\rho^2}{12d_4}\cos(pu) + d_4\cos(3pu), \tfrac{\rho^2}{12d_4}\sin(pu) + d_4\sin(3pu)) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(3p\theta) & \sin(3p\theta) \\ 0 & 0 & 0 & -\sin(3p\theta) & \cos(3p\theta) \end{bmatrix},$$

where the equation (1) becomes $p^2(\frac{\rho^2}{2} + (\frac{\rho^2}{12n})^2 + 9n^2) = 1$. Therefore, α is a spacelike curve lying fully in a lightlike hyperplane of E_1^5, which gives form (3) and completes the proof of the theorem. □

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