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ORTHOGONAL POLYNOMIALS AND REGULARLY VARYING SEQUENCES

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Abstract. We introduce a method of estimating asymptotic behaviour of polynomials $Q_n^{(\alpha)}(x) := \sum_{k\le n} c_k a_{nk} x^k$, $n\to\infty$, related to a given polynomial $Q_n(x) := \sum_{k\le n} a_{nk}x^k$, where (c_k), $k\in N$ is any regularly varying sequence of index α in the sense of Karamata. Then we apply our results to classical orthogonal polynomials as relevant examples.

Preliminaries

Slowly varying functions $L(x)$ (s.v.f.) in Karamata's sense are defined on the positive part of real axis, positive, measurable and satisfying: $\lim\limits_{x\to\infty} L(\lambda x)/L(x) = 1$ for each $\lambda > 0$. Examples of s.v.f. are:

$$\ln^a x,\ \ln^b(\ln x),\quad \exp(\ln^c x),\quad \exp\left(\frac{\ln x}{\ln\ln x}\right),\quad a,b\in R,\quad 0<c<1,\quad \text{etc.}$$

A regularly varying function $R_\alpha(x)$ (r.v.f.) of index α is defined as $R_\alpha(x) := x^\alpha L(x)$, $\alpha\in R$. An excellent survey of properties, characterization, representation, etc. connected with regular variation is given in [1] and [2]; therefore, we suppose the reader is familiar with it.

In [3] we defined a class L^* of analytic slowly varying functions, with which we deal afterwards. Namely, for any slowly varying $L(x)\in \mathrm{Loc}(L)$ (i.e., set of locally bounded functions with a property $L(0^+) = O(1)$), we define another s.v.f. $L^*(x)\in L^*$ by:

$$L^*(x) := x\int_0^\infty e^{-xt} L(1/t)\,dt,$$

satisfying $L^*(x)\in C^\infty$; $L^*(x)\sim L(x)$, $x\to\infty$. Another remarkable property is the possibility of analytic continuation of $L^*(x)$ on the right complex half-plane without loss of regularity mode, i.e.:

$$L^*(z)\sim L^*(|z|)\sim L(|z|),\quad |z|\to\infty,\quad \operatorname{Re} z>0.$$

We need here two more propositions from [3];

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PROPOSITION L4. *If* $a(s) \to \infty$, $s \to \infty$; $a(s) \sim b(s)$, $s \to \infty$, *then* $R^*_\alpha(a(s)) \sim R^*_\alpha(b(s))$, $\quad s \to \infty$.

PROPOSITION L5. (smooth variation) *The derivatives of analytic regularly varying functions satisfy:*

$$\frac{s^m (R^*_a(s))^{(m)}}{R^*_a(s)} \to a(a-1)(a-2)\cdots(a-m+1), \quad s \to \infty,\ m \in N.$$

Also, we have to introduce an operator, defined for polynomials $Q_n(x)$ with positive coefficients and $x > 0$, namely: $\widehat{Q}_n(x) := xQ'_n(x)/Q_n(x)$.

A sequence $(c_n), n \in N$, of positive numbers is regularly varying if:

$$\lim_{n\to\infty} c_{[\lambda n]}/c_n = \psi(\lambda) \in (0,\infty), \quad \text{for each } \lambda > 0.$$

In 1973 Bojanić and Seneta unified the theory of regularly varying functions and regularly varying sequences proving that:

(i) the above limit function $\psi(\lambda)$ is of the form λ^ρ for some $\rho \in \mathbb{R}$;

(ii) the function $f(x) := c_{[x]}$ varies regularly with index ρ.

Thus we can treat every regularly varying sequence as the integer values of some regularly varying function (the property $L \in \operatorname{Loc} L$ is obvious here). In the sequel we consider regularly varying sequences (c_k), $k \in N$ of index α, generated by an associated regularly varying function, i.e., $c_k := k^\alpha L(k)$, $\quad k \in N$, and, alternatively $c^*_k := k^\alpha L^*(k)$, $k \in N$, $\alpha \in \mathbb{R}$.

Results

Now, we are able to formulate a crucial theorem concerning asymptotic behaviour of $Q^*_n(x) := \sum_{k\le n} c^*_k a_{nk} x^k$, related to a given polynomial: $Q_n(x) := \sum_{k\le n} a_{nk} x^k$, with positive coefficients a_{nk}.

THEOREM A. *For any fixed* $x \in R^+$, *if*

$$\text{(I)} \qquad \sup_n \widehat{\widehat{Q}}_n(x) \le M < \infty,$$

where the constant M *does not depend on* x, *and*

$$\text{(II)} \qquad \lim_n \widehat{Q}_n(x)/\phi(n) = a(x) \neq 0$$

then for some ϕ *monotone increasing to infinity with* n, *then*

$$\text{(A)} \qquad Q^*_n(x) \sim a^\beta(x) c^*_{[\phi(n)]} Q_n(x), \quad n \to \infty,$$

for all regularly varying sequences $(c^*_k), k \in N$, *of index* β, $\beta \le -1$.

Then we show, under some supposition about the distribution of zeros of $Q_n(x)$, that (A) is valid for arbitrary $\beta \in \mathbb{R}$ (Proposition B2). Finally, using a form of Toeplitz Limits Preservance Theorem we prove that Theorem A is true for any regularly varying sequence (c_k), $k \in N$.

For the proof of the theorem we need some more lemmas.

Lemma A1. *$\widehat{\widehat{Q}}_n(x) > 0$ for $x \in \mathbb{R}^+$.*

Proof. Since $\widehat{\widehat{Q}}_n(x) = \dfrac{1}{\widehat{Q}_n(x)}\Big(\dfrac{x(xQ_n'(x))'}{Q_n(x)} - \widehat{Q}_n^2(x)\Big)$ and

$$x(xQ_n'(x))' = \sum_{k\le n} k^2 a_{nk} x^k, \quad \widehat{Q}_n(x) = \frac{\sum_{k\le n} k a_{nk} x^k}{Q_n(x)},$$

we have $\widehat{\widehat{Q}}_n(x) = \dfrac{1}{Q_n(x)\widehat{Q}_n(x)} \displaystyle\sum_{k\le n} (k - \widehat{Q}_n(x))^2 a_{nk} x^k > 0$, $x \in \mathbb{R}^+$.

Lemma A2. *$\widehat{Q}_n(x)$ is monotone increasing with x.*

Proof. Simple consequence of Lemma A1.

Lemma A3. *Under the condition (I) of Theorem A, for any $x, t \in R^+$,*

$$\frac{Q_n(xe^{-t})}{Q_n(x)} \le \exp\Big(\frac{e^{-Mt} - 1}{M}\widehat{Q}_n(x)\Big).$$

Proof. Condition (I) is equivalent with:

$$\text{(A3.1)} \qquad \frac{d(\widehat{Q}_n(s))}{\widehat{Q}_n(s)} \le \frac{M}{s}\, ds, \quad s > 0.$$

Integrating (A3.1) over $s \in [xe^{-u}, x]$, $u \ge 0$, we get: $\ln \widehat{Q}_n(x) - \ln \widehat{Q}_n(xe^{-u}) \le Mu$, $u \ge 0$, i.e.,

$$\text{(A3.2)} \qquad \widehat{Q}_n(xe^{-u}) \ge \widehat{Q}_n(x) e^{-Mu}.$$

Integrating (A3.2) over $u \in [0, t]$, we come to the conclusion of the lemma.

Lemma A4. *Regularly varying sequences (c_k^*) of index $-(a+1)$, $a \ge 0$, have the following integral representation:*

$$c_k^* := \frac{L^*(k)}{k^{a+1}} = \int_0^\infty e^{-kt} u(a,t)\, dt, \quad k \in N,$$

where $u(a,t)$ is given by $u(a,t) = \begin{cases} \frac{1}{\Gamma(a)} \int_0^t (t-u)^{a-1} L(1/u)\, du, & a > 0, \\ L(1/t), & a = 0. \end{cases}$

Proof. Case $a = 0$ is valid by definition (L1). For $a > 0$ and the Convolution Theorem for Laplace transform, we get:

$$\begin{aligned} \int_0^\infty e^{-kt} u(a,t)\, dt &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-kt} t^{a-1}\, dt \cdot \int_0^\infty e^{-kt} L(1/t)\, dt \\ &= \frac{1}{k^a} \frac{L^*(k)}{k} = \frac{L^*(k)}{k^{a+1}} = c_k^*, \quad k \in N. \end{aligned}$$

Now we are able to give

Proof of Theorem A. Using the expression for (c_k^*) of index $\beta \le -1$ from Lemma A4, we have:

$$\begin{aligned}\frac{Q_n^*(x)}{Q_n(x)} &= \frac{1}{Q_n(x)} \sum_{k \le n} c_k^* a_{nk} x^k = \int_0^\infty u(-\beta-1,t) \frac{Q_n(xe^{-t})}{Q_n(x)}\, dt = \int_0^{\xi_n} (\cdot) + \int_{\xi_n}^\infty (\cdot) \\ &= U + W,\end{aligned}$$

where $\xi_n = \xi_n(x) := \phi(n)^{-1/2}/a(x)$.

For the estimation of expression U we shall use the following identity:

$$\text{(A.1)} \qquad \ln \frac{Q_n(xe^{-t})}{Q_n(x)} + t\,\widehat{Q}_n(x) = \int_0^t w\,\widehat{Q}_n(a)\widehat{\widehat{Q}}_n(a)\, dw, \quad a := xe^{w-t};$$

which is easy to check by double partial integration. According to Lemma A1 and condition (I) from Theorem A, we have: $0 < \widehat{\widehat{Q}}_n(a) \le M < \infty$, where the constant M does not depend on a or n. Also, since $a \le x$, from Lemma A2 follows $\widehat{Q}_n(a) \le \widehat{Q}_n(x)$; hence

$$0 < \int_0^t w\,\widehat{Q}_n(a)\widehat{\widehat{Q}}_n(a)\, dw \le M\widehat{Q}_n(x) \int_0^t w\, dw = \frac{M}{2}\widehat{Q}_n(x)\, t^2.$$

Therefore, from (A.2) we get:

$$\ln \frac{Q_n(xe^{-t})}{Q_n(x)} = \widehat{Q}_n(x)(-t + O(t^2)), \quad x \in R^+,\ t \ge 0;$$

where the absolute constant in O does not depend on x or t. Now,

$$U = \int_0^{\xi_n} \exp\Big(\ln \frac{Q_n(xe^{-t})}{Q_n(x)}\Big) u(-\beta-1,t) dt = \int_0^{\xi_n} u(-\beta-1,t) e^{-t\widehat{Q}_n(x)} e^{O(t^2\widehat{Q}_n(x))} dt.$$

Since $e^B = 1 + O(Be^B)$, $B > 0$ with a constant in O independent of B and, for $t \in (0, \xi_n)$, $O(\widehat{Q}_n(x)t^2) = O(1)$; from (A.3) we get:

$$\begin{aligned} U &= \int_0^{\xi_n} u(-\beta-1,t) e^{-t\widehat{Q}_n(x)}\, dt + \int_0^{\xi_n} u(-\beta-1,t) e^{-t\widehat{Q}_n(x)} O(t^2\widehat{Q}_n(x))\, dt \\ &= \int_0^\infty u(-\beta-1,t) e^{-t\widehat{Q}_n(x)}\, dt - \int_{\xi_n}^\infty u(-\beta-1,t) e^{-t\widehat{Q}_n(x)}\, dt \\ &\quad + O(\widehat{Q}_n(x)) \int_0^\infty t^2 u(-\beta-1,t) e^{-t\widehat{Q}_n(x)}\, dt = U_1 + U_2 + U_3. \end{aligned}$$

Now, taking into account Proposition L4, condition (II) of Theorem A and elementary properties of regularly varying sequences (cf. [2, pp. 49–54]), we see that:

$$U_1 \sim a^\beta(x) c^*_{[\phi(n)]}, \quad n \to \infty;$$

and, evidently:

$$
\begin{aligned}
|U_2| &= \int_{\xi_n}^{\infty} e^{-t} u(-\beta-1,t) e^{-t(\widehat{Q}_n(x)-1)} dt \\
&= O(e^{-\xi_n(\widehat{Q}_n(x)-1)}) \int_0^{\infty} e^{-t} u(-\beta-1,t) dt = O(e^{-1/2\phi^{1/2}(n)}),
\end{aligned}
$$

for n sufficiently large.

Using Proposition we get

$$
\begin{aligned}
U_3 &= O(Q_n(x)) \cdot \frac{d^2}{ds^2}(c^*(s))_{(s=Q_n(x))} = O(Q_n(x)) \cdot O\Big(\frac{c^*(s)}{s^2}\Big)_{(s=Q_n(x))} \\
&= O\Big(\frac{c^*(\phi(n))}{\phi(n)}\Big).
\end{aligned}
$$

Therefore, we see that: $U \sim U_1 \sim a^{\beta}(x) c^*_{[\phi(n)]}$, $n \to \infty$. Estimating W, we consider the polynomial $P_n(x) := Q_n(x)/x$. Since $\widehat{Q}_n(x) = 1 + \widehat{P}_n(x)$, from the condition (I) we obtain $\widehat{\widehat{P}}_n(x) \le 2M$ for n sufficiently large; hence, Lemma A3 gives:

$$
\frac{P_n(xe^{-t})}{P_n(x)} \le \exp\Big(\frac{e^{-2Mt}-1}{2M}\widehat{P}_n(x)\Big).
$$

So

$$
\begin{aligned}
W = \int_{\xi_n}^{\infty} e^{-t} u(-\beta-1,t) \frac{P_n(xe^{-t})}{P_n(x)} dt &\le \int_{\xi_n}^{\infty} e^{-t} u(-\beta-1,t) \exp\Big(\frac{e^{-2Mt}-1}{2M}\widehat{P}_n(x)\Big) dt \\
< \exp\Big(\frac{e^{-2M\xi_n}-1}{2M}\widehat{P}_n(x)\Big) \int_0^{\infty} e^{-t} u(-\beta-1,t)\, dt &= O(e^{-\frac{1}{2}\phi(n)^{1/2}}), \quad n \to \infty.
\end{aligned}
$$

Hence, we see that $Q_n^*(x)/Q_n(x) = U + W \sim a^{\beta}(x) c^*_{[\phi(n)]}$, $n \to \infty$, and Theorem A is proved.

Analysing Theorem A, we see that condition (I) is the most ambiguous one. It happens that fulfilment of this condition essentially depends on the distribution of the zeros of polynomial $Q_n(x)$. In this article we are satisfied with the next two propositions:

PROPOSITION B1. *If all zeros of $Q_n(x)$ belong to left complex half-plane (including the imaginary axis) then, for all $x \in R^+$, the condition (I) in Theorem A is satisfied with $M = 2$.*

Proof. If $(-z_{nk})$, $k \le n$, are the zeros of $Q_n(x)$, then

$$
Q_n(x) = a_{nn} \prod_{k \le n} (x + z_{nk}), \quad \operatorname{Re} z_{nk} \ge 0, \ z_{n1} = 0;
$$

i.e.,

$$\widehat{Q}_n(x) = \sum_{k \le n} \frac{x}{x + z_{nk}} = \sum_{k \le n} u_{nk}; \qquad x \frac{d}{dx} \widehat{Q}_n(x) = \sum_{k \le n} \frac{x z_{nk}}{(x + z_{nk})^2} = \sum_{k \le n} v_{nk}.$$

Since $\operatorname{Im} \widehat{Q}_n(x) = 0$, we obtain: $\widehat{Q}_n(x) = \operatorname{Re}(\sum_{k \le n} u_{nk}) = \sum_{k \le n} \operatorname{Re} u_{nk}$ and, analogously, $x \frac{d}{dx} \widehat{Q}_n(x) = \sum_{k \le n} \operatorname{Re} v_{nk}$. But, since for $x \in R^+$,

$$0 < \frac{\operatorname{Re} v_{nk}}{\operatorname{Re} u_{nk}} = \frac{\operatorname{Re} z_{nk}(x + \operatorname{Re} z_{nk})^2 + \operatorname{Im}^2 z_{nk}(2x + \operatorname{Re} z_{nk})}{(x + \operatorname{Re} z_{nk})((x + \operatorname{Re} z_{nk})^2 + \operatorname{Im}^2 z_{nk})} < 2;$$

we get

$$x \frac{d}{dx} \widehat{Q}_n(x) = \sum_{k \le n} \operatorname{Re} u_{nk} \frac{\operatorname{Re} v_{nk}}{\operatorname{Re} u_{nk}} < 2 \sum_{k \le n} \operatorname{Re} u_{nk} = 2\widehat{Q}_n(x),$$

i.e., Proposition B1 is proved.

Remark. Because of the nature of Laplace transform, the proof of theorem A holds if the index β of regularly varying sequences (c_k^*), $k \in N$, satisfies the condition $\beta + 1 \le 0$.

But, for the special distribution of the zeros of $Q_n(x)$ mentioned above, we are able to prove that theorem A is valid for all finite values of β, i.e.:

PROPOSITION B2. *If all zeros of polynomial $Q_n(x)$ belong, as before, to the left complex half-plane, then Theorem A is valid for any value of indexes of sequences (c_k^*).*

Proof. We consider the polynomial $R_n(x) := xQ_n'(x) = \sum_{k \le n} k a_{nk} x^k$. The zeros of $R_n(x)$ are, according to a well-known theorem of Gauss, not outside of the convex polygon determined by the zeros of $Q_n(x)$; so they are also in the left complex half-plane and condition (I) is satisfied.

LEMMA C2. *If $\lim_n \phi(n) = +\infty$, then for $x \in R^+$, then the following statements are equivalent*

$$\lim_n \frac{\widehat{Q}_n(x)}{\phi(n)} = a(x), \qquad \lim_n \frac{\widehat{R}_n(x)}{\phi(n)} = a(x),$$

Proof. It is easy to check that $\widehat{R}_n(x) - \widehat{Q}_n(x) = \widehat{\widehat{Q}}_n(x)$ i.e., (Lemma A1 and Proposition B1) $0 < \widehat{R}_n(x) - \widehat{Q}_n(x) < 2$, i.e.,

$$0 < \Big(\frac{\widehat{R}_n(x)}{\phi(n)} - a(x)\Big) - \Big(\frac{\widehat{Q}_n(x)}{\phi(n)} - a(x)\Big) < \frac{2}{\phi(n)}$$

wherefrom lemma follows.

Now we can apply Theorem A to the polynomial $R_n(x)$. Remark about zeros of $R_n(x)$ and Lemma C2 says that conditions (I) and (II) are satisfied, so:

$$\sum_{k\le n} kc_k^* a_{nk}x^k \sim c^*_{[\phi(n)]}a^\beta(x)R_n(x) = c^*_{[\phi(n)]}a^\beta(x)xQ_n'(x)$$
$$\sim \phi(n)c^*_{[\phi(n)]}a^{\beta+1}(x)Q_n(x), \quad n\to\infty,\ \beta\le -1;$$

i.e.,

$$\sum_{k\le n} kc_k^* a_{nk}x^k \sim [\phi(n)]c^*_{[\phi(n)]}a^{\beta+1}(x)Q_n(x), \quad n\to\infty.$$

The last relation shows that Theorem A is valid for regularly varying sequences (c_k^*) of index $\beta+1$. Applying the algorithm mentioned above to the polynomial $S_n(x) := xR_n'(x) = \sum_{k\le n} k^2 a_{nk}x^k$, etc. we come to the conclusion from Proposition B2.

Together, Propositions B1 and B2 produce

THEOREM B. *Let S denote the set of positive reals satisfying the condition (II) of Theorem A, $x_0\in S$, and let all zeros of the polynomial $Q_n(x)=\sum_{k\le n} a_{nk}x^k$, $k\in N$, belong to the left complex half-plane (including the imaginary axis). Then, for every fixed $x\in S$, $x>x_0$:*

$$Q_n^{(\alpha)}(x) := \sum_{k\le n} c_k a_{nk}x^k \sim a^\alpha(x)c_{[\phi(n)]}Q_n(x), \quad n\to\infty,\ \alpha\in\mathbb{R};$$

where (c_k) is any regularly varying sequence of index α.

Proof. Propositions B1 and B2 say that Theorem B is valid for the class of sequences (c_n^*), in particular

$$(\mathrm{B}_1)\qquad Q_n^*(x) = \sum_{k\le n} c_k^* a_{nk}x^k \sim a^\alpha(x)c^*_{[\phi(n)]}Q_n(x), \quad n\to\infty,\ \alpha\in R.$$

But $c_m^*\sim c_m$, $m\to\infty$ (L2 and L4) so, all we have to prove is $Q_n^\alpha(x)\sim Q_n^*(x)$, $n\to\infty$, $x_0<x\in S$. For this purpose we invoke an old proposition:

LEMMA B1. (O. Toeplitz (1911)) *Let the triangular matrix (p_{nk}), $k\le n$, $n\in N$, consist of non-negative elements satisfying $\sum_{k\le n} p_{nk}=1$, and let (s_n), $n\in N$ is an arbitrary real sequence. Then necessary and sufficient condition for the implication $\lim_n s_n = s \Rightarrow \lim_n \sum_{k\le n} s_k p_{nk} = s$ is $\lim_n p_{nk}=0$, for every fixed k.*

We are going to use the lemma in the following way. Let

$$p_{nk} := \frac{c_k^* a_{nk}x^k}{Q_n^*(x)}; \quad s_n := \frac{c_n}{c_n^*}, \quad n\in N.$$

Then $\sum_{k\le n} p_{nk} = 1$; $\lim_n s_n = s = 1$, and

$$\lim_n \sum_{k\le n} s_k p_{nk} = \lim_n \frac{\sum_{k\le n} c_k^* a_{nk} x^k \frac{c_k}{c_k^*}}{Q_n^*(x)} = \lim_n \frac{Q_n^\alpha(x)}{Q_n^*(x)} = s = 1,$$

if and only if

$$\lim_n p_{nk} = \lim_n \frac{c_k^* a_{nk} x^k}{Q_n^*(x)} = 0. \tag{B$_2$}$$

To prove this, we recall (Lemma A2) that $\widehat{Q}_n(t)$ is monotone increasing with t, i.e., $\widehat{Q}_n(t) > \widehat{Q}_n(x_0)$ for each $t > x_0$, i.e.,

$$\frac{Q_n'(t)}{Q_n(t)} > \frac{\widehat{Q}_n(x_0)}{t}, \quad t > x_0.$$

Integrating the last expression for $t \in [x_0, x]$, we obtain:

$$Q_n(x) > Q_n(x_0)\ (x/x_0)^{\widehat{Q}_n(x_0)} > a_{nk} x_0^k\ (x/x_0)^{\widehat{Q}_n(x_0)}. \tag{B_3}$$

Since $x_0, x \in S$; $x/x_0 > 1$, using (B_1) and (B_3) for fixed k and sufficiently large n, we get:

$$p_{nk} = \frac{c_k^* a_{nk} x^k}{Q_n^*(x)} = O(\frac{c_k^* a_{nk} x^k}{a^\alpha(x) c^*_{[\phi(n)]} Q_n(x)}) = O(\phi^{2|\alpha|}(n)\ (x_0/x)^{\widehat{Q}_n(x_0)}) = o(1),$$

and Theorem B is proved.

Now, we shall give some examples concerning classical orthogonal polynomials which are good illustrations for our results.

Example 1: Laguerre polynomials. Laguerre polynomials $L_n^{(a)}(x)$ of index $a > -1$ are given in an explicit form by

$$L_n^{(a)}(x) = \sum_{k=0}^n \binom{n+a}{n-k} \frac{(-x)^k}{k!,}$$

and all their zeros are real and positive. So, we consider polynomials $L_n^{(a)}(-x)$, $x > 0$, whose zeros are real and negative; hence, they satisfy the conditions of Theorem B. Since

$$\begin{aligned}\frac{d}{dx}(L_n^{(a)}(-x)) &= \sum_{k=1}^n \binom{n+a}{n-k} \frac{x^{k-1}}{(k-1)!} = \sum_{k=1}^n \binom{n-1+(a+1)}{n-1-(k-1)} \frac{x^{k-1}}{(k-1)!} = \\ &= \sum_{k=0}^n \binom{n-1+(a+1)}{n-1-k} \frac{x^k}{k!} = L_{n-1}^{(a+1)}(-x),\end{aligned}$$

to obtain asymptotic behaviour of $\widehat{L}_n^{(a)}(-x)$, we use Perron's formula [4]

$$L_n^{(a)}(z) = 1/2\pi^{-1/2}e^{z/2}(-z)^{-a/2-1/4}n^{a/2-1/4}e^{2\sqrt{-nz}}(1+O(1/\sqrt{n})),$$

for any z in the complex plane cut along the positive part of the real axis. Now, for $z=-x$, $x\in R^+$, after some calculations, we obtain:

$$\widehat{L}_n^{(a)}(-x) = \frac{xL_{n-1}^{(a+1)}(-x)}{L_n^{(a)}(-x)} \sim \sqrt{nx}e^{2(\sqrt{(n-1)x}-\sqrt{nx})} \sim \sqrt{n}\sqrt{x}, \quad n\to\infty,\ x\in R^+,$$

that is: $\lim_n \widehat{L}_n^{(a)}(-x)/\sqrt{n} = \sqrt{x}$, $x>0$; so, we can apply Theorem B on $Q_n(x) = xL_{n-1}^{(a)}(-x)$ with $\phi(n)=\sqrt{n}$, $a(x)=\sqrt{x}$. It follows:

$$\sum_{k=1}^n k^\beta L_k \binom{n-1+a}{n-k}\frac{x^k}{(k-1)!} \sim xx^{\beta/2}[\sqrt{n}]^\beta L_{[\sqrt{n}]}\ L_{n-1}^{(a)}(-x), \quad n\to\infty;$$

i.e., putting: $a-1\to a$, $\beta+1\to\beta$:

$$\sum_{k=1}^n k^\beta L_k \binom{n+a}{n-k}\frac{x^k}{k!} \sim x^{\frac{\beta+1}{2}}[\sqrt{n}]^{\beta-1} L_{[\sqrt{n}]} L_{n-1}^{(a+1)}(-x), \quad n\to\infty$$

As we already showed $L_{n-1}^{(a+1)}(-x) \sim \sqrt{n/x}L_n^{(a)}(-x)$; $n\to\infty$. Hence

PROPOSITION C. *We have*

$$\text{(C)} \qquad \sum_{k\le n} c_k \binom{n+a}{n-k}\frac{x^k}{k!} \sim x^{\beta/2}c_{[\sqrt{n}]}L_n^{(a)}(-x), \quad x\in R^+,\ n\to\infty;$$

for any regularly varying sequence (c_k) *of index* $\beta\in R$.

EXAMPLE 2. JACOBI POLYNOMIALS. The Jacobi polynomials $P_n^{(a,b)}(t)$ are given by:

$$P_n^{(a,b)}(t) = \sum_{k=0}^n \binom{n+a}{n-k}\binom{n+b}{k}\Big(\frac{t-1}{2}\Big)^k\Big(\frac{t+1}{2}\Big)^{n-k}, \quad a,b>-1.$$

All their zeros are real and belong to the segment $[-1,1]$. We shall consider the associated class of polynomials $Q_n^{(a,b)}(x)$, defined as:

$$Q_n^{(a,b)} := \sum_{k=0}^n \binom{n+a}{n-k}\binom{n+b}{k}x^k = (1-x)^nP_n^{(a,b)}\Big(\frac{1+x}{1-x}\Big).$$

All their zeros are real and negative, so we can apply Theorem B. It is easy to show that:

$$\frac{d}{dx}Q_n^{(a,b)}(x) = (n+b)Q_{n-1}^{(a+1,b)}(x)$$

Therefore,

$$\lim_n \frac{\widehat{Q}_n^{(a,b)}(x)}{n} = \lim_n \frac{xQ_n^{(a+1,b)}(x)}{Q_n^{(a,b)}(x)}.$$

For the estimation of this last expression we use the Darboux formula for the asymptotic behaviour of Jacobi polynomials [4], valid for $t \notin [-1,1]$:

$$P_n^{(a,b)}(t) \sim (t-1)^{-a/2}(t+1)^{-b/2}[(t-1)^{1/2}+(t+1)^{1/2}]^{a+b} \cdot (2\pi n)^{-1/2}(t^2-1)^{-1/4}[t+(t^2-1)^{1/2}]^{n+1/2}, \quad n\to\infty.$$

Putting $t=(1+x)/(1-x)$, $x>0$, after some simplifications we get:

$$Q_n^{(a,b)}(x) = (1-x)^n P_n^{(a,b)}\left(\frac{1+x}{1-x}\right) \sim \frac{(\sqrt{x}+1)^{a+b+2n+1}}{2\sqrt{\pi n}\ x^{a/2+1/4}},\ n\to\infty.$$

Hence:

$$\lim_n \frac{\widehat{Q}_n^{(a,b)}(x)}{n} = \lim_n \frac{xQ_{n-1}^{(a+1,b)}(x)}{Q_n^{(a,b)}(x)} = \frac{\sqrt{x}}{1+\sqrt{x}}, \quad x>0,$$

i.e., the limit does not depend on the type of Jacobi polynomial $P_n^{(a,b)}(x)$.

Applying Theorem B to the polynomial $xQ_{n-1}^{(a+1,b)}(x)$, with: $a(x)=\sqrt{x}/(1+\sqrt{x})$; $\phi(n)=n$, we get (when $n\to\infty$)

$$\sum_{k=1}^n \binom{n+a}{n-k}\binom{n-1+b}{k-1}k^\alpha L_k x^k \sim x(1+x^{-1/2})^{-\alpha} n^\alpha L_n\, Q_{n-1}^{(a+1,b)}(x),\ x\in R^+,$$

or, since:

$$\frac{1}{k}\binom{n-1+b}{k-1} = \frac{1}{n+b}\binom{n+b}{k},\ k\geq 1;\quad x(1+x^{-1/2})Q_{n-1}^{(a+1,b)}(x)\sim Q_n^{(a,b)}(x),$$

with α instead of $\alpha+1$,

$$\text{(D)}\qquad \sum_{k=1}^n \binom{n+a}{n-k}\binom{n+b}{k}k^\alpha L_k x^k \sim n^\alpha L_n\ (1+x^{-1/2})^{-\alpha}\ Q_n^{(a,b)}(x),$$

for any slowly varying sequence (L_k) and any $\alpha\in R$. Since (D) is valid for every $x>0$, putting $x=(t-1)/(t+1)$, $t\notin[-1,1]$ and multiplying by $((t+1)/2)^n$, we obtain asymptotic behavior in terms of Jacobi polynomials $P_n^{(a,b)}(t)$:

Proposition E. *We have for* $n \to \infty$

$$\text{(E)}\quad \sum_{k=1}^{n} c_k \binom{n+a}{n-k}\binom{n+b}{k}\Big(\frac{t-1}{2}\Big)^k\Big(\frac{t+1}{2}\Big)^{n-k} \sim c_n\Big(1+\sqrt{\frac{t+1}{t-1}}\Big)^{-\alpha} P_n^{(a,b)}(t),$$

for any regularly varying sequence (c_k) *of index* $\alpha \in \mathbb{R}$, *and* $t \notin [-1,1]$.

Analogous formulae for ultraspherical polynomials $P_n^{(\lambda)}(\cdot)$, Legendre's $(P_n(\cdot))$ and Hermite's $(H_n(\cdot))$ polynomials can be deduced from (C), (D) and (E) by using identities:

$$P_n^{(\lambda)}(x) = P_n^{(\lambda-1/2,\lambda-1/2)}(x);\quad P_n(x) = P_n^{(0,0)}(x)$$
$$H_{2n}(i\sqrt{x}) = (-1)^n 2^{2n} n!\, L_n^{(-1/2)}(-x);\ H_{2n+1}(i\sqrt{x}) = (-1)^n 2^{2n+1} n!\, x L_n^{(1/2)}(-x).$$

For example:

$$\sum_{k=1}^{n} c_k \binom{n}{k}^2 x^k \sim c_n(1+x^{-1/2})^{-\alpha}(1-x)^n P_n\Big(\frac{1+x}{1-x}\Big),\ x \in \mathbb{R}^+,\quad n \to \infty,$$

for every regularly varying sequence (c_k) of index $\alpha \in \mathbb{R}$.

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