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A SOLUTION OF AN OLD PROBLEM OF KARAMATA

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ABSTRACT. We give a complete solution of an approximation problem originally posed in 1949 by Jovan Karamata.

1. Introduction

In [2] Karamata proposed the following problems (by interpretation of Mitrinović in [5]):

PROBLEM A. *Determine those algebraic functions $A_n(x)$, $n = 2, 3, \dots$ which have the following properties:*

$$1.\ \ \frac{\log x}{x-1} \le A_n(x) \quad (x > 0); \qquad 2.\ \ A_n(x) \sim x^{-1/n} \quad (x \to 0^+);$$

$$3.\ \ xA_n(x) \sim x^{1/n} \quad (x \to +\infty); \qquad 4.\ \ A_n(x) - \frac{\log x}{x-1} \sim a_n(x-1)^{2n-2} \quad (x \to 1),$$

where a_n is independent of x.

PROBLEM B. *Find algebraic functions $B_n(x)$, $n = 2, 3, \dots$ such that for $x > 0$,*

$$\frac{\log x}{x-1} \le B_m(x) \quad and \quad B_m(x) \ge B_n(x) \quad for \quad 2 \le m \le n.$$

The forms of $A_n(x), B_n(x)$ were important for Karamata's efforts to solve some approximation problems of Ramanujan [3]. In [2] he indicated that

$$A_2(x) = \frac{1}{\sqrt{x}}; \qquad A_3(x) = \frac{1 + x^{1/3}}{x + x^{1/3}}.$$

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In [1] Blanuša has given a rather complicated expression for $A_4(x)$. In 1968 Mitrinović restated this as Problem 5626 in the American Mathematical Monthly, the leading journal for mathematical problems [4]. He also put it in his well-known book [5] with a comment that "so far no solution has been published to the above problems". Our recent inquiry to the editors of Monthly, R. Horn and B. Palka shows that, after 34 years, this notice is still valid. In this article we shall give an explicit form of $A_n(x)$ and two different expressions for $B_n(x)$. It is obvious that those solutions of 5626 are not unique.

2. Results

At the begining let us quote some simple lemmas we shall need in the sequel.

Lemma 1. $\tanh^m y \sim y^m \ (y \to 0); \quad 1-\tanh^m y \sim 2m\exp(-2y) \ (y \to +\infty)$.

Lemma 2. $\frac{1}{2}\log\frac{1+y}{1-y} - \sum_{k=1}^{n} \frac{y^{2k-1}}{2k-1} \sim \frac{y^{2n+1}}{2n+1} \quad (y \to 0)$.

Lemma 3. $y^{n+1} \le \frac{n(n-1)}{2} - (n^2-1)y + \frac{n(n+1)}{2}y^2, \ \ 0 < y \le 1; \ \ n \in N$.

Lemmas 1 and 2 are obvious and Lemma 3 can be proved by standard method.

Proposition 1. *The algebraic functions $A_n(x)$, $x > 0$, $n = 2,3,\dots$ which represents a solution of Problem A, are defined by the following expression*

$$A_n(x) := 12n\frac{\sum_{k=1}^{n}\frac{1}{2k-1}(X_n^{2k-1}(x) - Y_n^{2k-1}(x)) + \frac{8}{9}(n^2-1)X_n^{2n+1}(x)}{(x-1)(1-\frac{n(n-1)}{2}X_n^{2n+2}(x) + (n^2-1)X_n^{2n}(x) - \frac{n(n+1)}{2}X_n^{2n-2}(x))}$$

where $X_n(x) := \dfrac{x^{1/3n}-1}{x^{1/3n}+1}; \ Y_n(x) := \dfrac{x^{1/6n}-1}{x^{1/6n}+1}.$

Proof. We shall prove that $A_n(x)$ satisfy the conditions 1, 2, 3 and 4 of Problem A. To prove the validity of the assertion 1 put $x = e^t$, $t \in R$ and re-write the inequality from 1 in the form

(A.1)
$$\frac{t}{2\sinh(t/2)} \le 6n\frac{\sum_{k=1}^{n}\frac{1}{2k-1}(X_n^{2k-1} - Y_n^{2k-1}) + \frac{8}{9}(n^2-1)X_n^{2n+1}}{\sinh(t/2)(1-\frac{n(n-1)}{2}X_n^{2n+2} + (n^2-1)X_n^{2n} - \frac{n(n+1)}{2}X_n^{2n-2})},$$

where, for the sake of simplicity, we put

$$X_n = X_n(e^t) := \tanh(t/6n), \quad Y_n = Y_n(e^t) := \tanh(t/12n).$$

Now, for $1 > X_n \ge w \ge Y_n \ge 0$, integrating the identity

$$\text{(I)} \qquad \frac{1}{1-w^2} = \frac{w^{2n+2}}{1-w^2} + \sum_{k=0}^{n} w^{2k}$$

with respect to w, we get

$$\begin{aligned}\frac{t}{12n} &= \frac{1}{2}\log\Big(\frac{1+X_n}{1-X_n}\frac{1-Y_n}{1+Y_n}\Big) = \int_{Y_n}^{X_n}\frac{dw}{1-w^2}\\ &= \int_{Y_n}^{X_n}\frac{w^{2n+2}}{1-w^2}dw + \sum_{k=1}^{n+1}\frac{1}{2k-1}(X_n^{2k-1}-Y_n^{2k-1})\end{aligned} \tag{A.2}$$

We shall estimate the integral on the right-hand of $(A.2)$ by using Lemma 3 with $y := (w/X_n)^2$. Hence

$$\begin{aligned}\int_{Y_n}^{X_n}\frac{w^{2n+2}}{1-w^2}dw &= X_n^{2n+2}\int_{Y_n}^{X_n}\frac{(w/X_n)^{2n+2}}{1-w^2}dw\\ &\le X_n^{2n+2}\int_{Y_n}^{X_n}\Big(\frac{n(n-1)}{2}\frac{1}{1-w^2}-\frac{n^2-1}{X_n^2}\frac{w^2}{1-w^2}+\frac{n(n+1)}{2X_n^4}\frac{w^4}{1-w^2}\Big)dw\\ &= S_n\int_{Y_n}^{X_n}\frac{1}{1-w^2}dw + T_n,\end{aligned}$$

where

$$\begin{aligned}S_n = S_n(t) &:= \frac{n(n-1)}{2}X_n^{2n+2}-(n^2-1)X_n^{2n}+\frac{n(n+1)}{2}X_n^{2n-2},\\ T_n = T_n(t) &:= \int_{Y_n}^{X_n}((n^2-1)X_n^{2n}-\frac{n(n+1)}{2}X_n^{2n-2}(1+w^2))dw.\end{aligned}$$

Since $\int_{Y_n}^{X_n}\frac{1}{1-w^2}dw = t/12n$ and $1-S_n>0$ for $0\le X_n<1$, putting this in $(A.2)$ and dividing by $2\sinh(t/2)$ which is positive for $t>0$, we obtain

$$\frac{t}{2\sinh(t/2)} \le \frac{6n}{\sinh(t/2)(1-S_n)}\sum_{k=1}^{n+1}\frac{1}{2k-1}(X_n^{2k-1}-Y_n^{2k-1})+T_n.$$

Now,

$$\begin{aligned}&\frac{1}{2n+1}(X_n^{2n+1}-Y_n^{2n+1})+T_n\\ &\le \frac{1}{2n+1}X_n^{2n+1}+(X_n-Y_n)\Big((n^2-1)X_n^{2n}-\frac{n(n+1)}{2}X_n^{2n-2}\Big)\le\frac{8}{9}(n^2-1)X_n^{2n+1},\end{aligned}$$

and $(A.1)$ follows for $t>0$. Since $t,\ \sinh(t/2),\ X_n(t),\ Y_n(t)$ are odd functions, we see from the form of the inequality $(A.1)$ that it is also valid for negative values of t. Applying the first part of Lemma 1, it follows that both sides of $(A.1)$ approach 1 when t approaches zero. Hence, the assertion 1 of Problem A is proved.

Now we shall prove assertion 3 i.e.,

$$e^tA_n(e^t) = 12n\frac{e^t}{e^t-1}\frac{1}{1-S_n}\sum_{k=1}^{n}\frac{1}{2k-1}(X_n^{2k-1}-Y_n^{2k-1})+\frac{8}{9}(n^2-1)X_n^{2n+1}\sim e^{t/n}$$

for $t \to +\infty$. Using the second part of Lemma 1, for fixed n, we obtain

$$\sum_{k=1}^{n} \frac{1}{2k-1}(X_n^{2k-1} - Y_n^{2k-1}) + \frac{8}{9}(n^2-1)X_n^{2n+1} \to \frac{8}{9}(n^2-1) \quad (t \to +\infty).$$

Since $X_n = 1 - 2/(e^{t/3n}+1)$, an easy computation shows that

$$1 - S_n \sim \frac{32}{3}n(n^2-1)e^{-t/n} \quad (t \to +\infty)$$

and the proof is finished.

In the same manner, assertion 2 is also valid.

According to condition 4, applying Lemmas 1 and 2, we have

$$\begin{aligned} A_n(e^t) - \frac{t}{e^t-1} &= \frac{12n}{(e^t-1)(1-S_n)}\Bigg(\bigg(\frac{1}{2}\log\frac{1+Y_n}{1-Y_n} - \sum_{k=1}^{n}\frac{1}{2k-1}Y_n^{2k-1}\bigg) \\ &\qquad - \bigg(\frac{1}{2}\log\frac{1+X_n}{1-X_n} - \sum_{k=1}^{n}\frac{1}{2k-1}X_n^{2k-1}\bigg)\Bigg) + tS_n + O(X_n^{2n+1}) \\ &= \frac{\frac{n(n+1)}{2}tX_n^{2n-2} + O(X_n^{2n+1}) + O(Y_n^{2n+1})}{(e^t-1)(1-S_n)} \\ &\sim \frac{n(n+1)}{2}X_n^{2n-2} \sim \frac{n(n+1)}{2(6n)^{2n-2}}t^{2n-2} \quad (t \to 0). \end{aligned}$$

Hence we obtain a very precise approximation

$$A_n(x) - \frac{\log x}{x-1} \sim \frac{n+1}{2(36)^{n-1}n^{2n-3}}(x-1)^{2n-2} \qquad (x \to 1),$$

and the proof of Problem A is finished.

In the sequel we shall give two different solutions of Problem B and a form of $b_n(x)$ which is well-approximating $\frac{\log x}{x-1}$ from the left-hand side.

PROPOSITION 2. *Let p, $0 < p < 1$ be an arbitrary rational number. Then we can take*

$$B_n(x) = B_n^{(p)}(x) := \frac{\sum_{k=1}^{n-1}\frac{1}{2k-1}(\frac{x^{2p}-1}{x^{2p}+1})^{2k-1}}{p(x-1)(1-(\frac{x^{2p}-1}{x^{2p}+1})^{2n-2})} \quad n = 2, 3, \dots$$

Proof. To prove

$$\frac{\log x}{x-1} \le B_n(x), \qquad x > 0; \quad n = 2, 3, \dots \tag{B.1}$$

we proceed as before i.e., we put $x = e^t$, $t \in R$; $X_p = X_p(t) := \tanh(pt)$, and re-write (B.1) in the form

$$\text{(B.1')} \qquad \frac{t}{2\sinh(t/2)} \leq \frac{1}{2p\sinh(t/2)(1 - X_p^{2n-2})} \sum_{k=1}^{n-1} \frac{1}{2k-1} X_p^{2k-1}.$$

Integrating again (I) over w, $0 \leq w \leq X_p < 1$, we get

$$\text{(B.2)} \qquad pt = \int_0^{X_p} \frac{1}{1-w^2} dw = \sum_{k=1}^{n-1} \frac{1}{2k-1} X_p^{2k-1} + \int_0^{X_p} \frac{w^{2n-2}}{1-w^2} dw$$

Because

$$\int_0^{X_p} \frac{w^{2n-2}}{1-w^2} dw \leq X_p^{2n-2} \int_0^{X_p} \frac{1}{1-w^2} dw = ptX_p^{2n-2},$$

from $(B.2)$, by dividing with $2\sinh(t/2)$, $t > 0$, we obtain (B.1') for $t > 0$. Since both functions on the left and right-hand side of (B.1') are even, we see that this inequality is also valid for the negative values of t. Therefore (B.1) is proved.

We shall prove next that $B_n(x)$ are monotone decreasing functions of n. Evidently

$$\begin{aligned} &B_n(x) - B_{n+1}(x) \\ &= \frac{\sum_{k=1}^{n-1} X_p^{2k-1}/(2k-1)}{p(e^t-1)} \frac{X_p^{2n-2}(1-X_p^2)}{(1-X_p^{2n-2})(1-X_p^{2n})} - \frac{X_p^{2n-1}}{p(2n-1)(e^t-1)(1-X_p^{2n})} \\ &= \frac{X_p^{2n-2}(1-X_p^2)}{p(e^t-1)(1-X_p^{2n-2})(1-X_p^{2n})} \left(\sum_{k=1}^{n-1} \frac{1}{2k-1} X_p^{2k-1} - \frac{X_p}{2n-1} \frac{1-X_p^{2n-2}}{1-X_p^2} \right) > 0, \end{aligned}$$

and the proof of Proposition 2 is completed.

Remark 1. The functions $B_n^{(p)}(x)$ approximate $\frac{\log x}{x-1}$ very precisely around the point $x = 1$. Namely, computations show that

$$B_n^{(p)}(x) - \frac{\log x}{x-1} \sim \Big(\frac{2n-2}{2n-1} p^{2n-2} \Big) (x-1)^{2n-2} \quad (x \to 1).$$

One can also find $B_2^{(1/4)}(x) = \dfrac{1}{\sqrt{x}}$ and $B_3^{(1/6)} = \dfrac{1 + x^{1/3}}{x + x^{1/3}}$, i.e., Karamata's versions of $A_2(x)$ and $A_3(x)$. Indeed, the functions $B_n^{(1/2n)}(x)$ are "almost" solutions of Problem A. They satisfy conditions 1 and 4 (with $a_n = (2n-2)/(2n-1)(2n)^{2n-2}$), but

$$xB_n^{(1/2n)}(x) \sim b_n x^{1/n} \quad (x \to \infty); \qquad B_n^{(1/2n)}(x) \sim b_n x^{-1/n} \quad (x \to +0),$$

with $b_n = \frac{n}{2(n-1)} \sum_{k=1}^{n-1} \frac{1}{2k-1}$, $n = 2, 3, \ldots$.

At this point we are able to give a form of algebraic functions $b_n(x)$ which approximate $\frac{\log x}{x-1}$ from the left, i.e., PROPOSITION 3. *Let q, $0 < q < 1$, be a rational number and define*

$$b_n(x) = b_n^{(q)}(x) := \frac{1}{q(x-1)} \frac{\sum_{k=1}^{n-1} \frac{1}{2k-1}(\frac{x^{2q}-1}{x^{2q}+1})^{2k-1}}{1 - \frac{1}{2n-1}(\frac{x^{2q}-1}{x^{2q}+1})^{2n-2}}, \quad x > 0; \quad n = 2, 3, \dots$$

Then $b_n(x)$ is monotone increasing with n and $\dfrac{\log x}{x-1} \geq b_n(x)$, $x > 0$; $n = 2, 3, \dots$

The proof goes along the same lines as the previous one, exept that we estimate the integral on the right of $(B.2)$ by the means of Tchebychef's theorem [5, Theorem 8, p. 39]

$$\int_0^{X_q} \frac{w^{2n-2}}{1-w^2} dw \geq \frac{1}{X_q} \int_0^{X_q} w^{2n-2} dw \int_0^{X_q} \frac{1}{1-w^2} dw = \frac{qt}{2n-1} X_q^{2n-2}.$$

The class of algebraic functions $b_n^{(q)}(x)$ approximate $\frac{\log x}{x-1}$ even better than $B_n(x)$. Namely,

$$\frac{\log x}{x-1} - b_n^{(q)}(x) \sim \Big(\frac{4(n-1)}{3(4n^2-1)} q^{2n}\Big)(x-1)^{2n} \quad (x \to 1).$$

Combining the results from Propositions 2 and 3, we get

PROPOSITION 4. *For p, q, $B_n^{(p)}(x)$, $b_n^{(q)}(x)$ defined as above, we have*

$$b_2^{(q)}(x) \leq ... \leq b_m^{(q)}(x) \leq ... \leq \frac{\log x}{x-1} \leq ... \leq B_n^{(p)}(x) \leq ... \leq B_2^{(p)}(x).$$

Remark 2. Note that for $p = q = 1/2$ we obtain precise approximations of $\frac{\log x}{x-1}$ by *rational* functions.

From the manner in which we obtain our solutions of the Problems A and B it follows that they are in no way unique. A good example for this is another solution of Problem B which is not so precise as before but is of very simple form.

PROPOSITION 5. *Let (c_n), $c_n > 2/\sqrt{3}$ be any monotone decreasing sequence of rational numbers. Then we can take* $B_n(x) = \dfrac{1 + x^{c_n/4 - 1/3c_n}}{x^{1/2 + c_n/4} + x^{1/2 - 1/3c_n}}$.

Proof. The assertion we have to prove is a consequence of the following

LEMMA 4. *For any $c \neq 0$, $t \in R$ one has*

$$\text{(B.3)} \qquad \frac{t}{2\sinh(t/2)} \leq \frac{\cosh(1/6c - c/8)t}{\cosh(1/6c + c/8)t} := f(c, t),$$

and $f(c, t)$ is monotone increasing for $c > 2/\sqrt{3}$, $t \in R$.

Indeed, putting in (B.3) $t = \log x$, $c = c_n$ we obtain $\frac{\log x}{x-1} \leq B_n(x)$ and, since $f(c, \log x)$ is increasing and the sequence (c_n) is decreasing, it follows that $B_n(x)$ is decreasing. Therefore it can be taken as a solution of the Problem B.

We shall prove a little bit more than the assertion from Lemma 4 i.e., an inequality which is new to the best of our knowledge.

Lemma 5. *For any real A, B, $A \neq B$ we have*

$$\frac{\sinh A - \sinh B}{A - B} \geq \cosh \sqrt{\frac{A^2 + AB + B^2}{3}}.$$

Proof. Applying an integral variant of convex means [5, p. 12], we have

$$\frac{1}{A-B} \int_B^A x^{2n} dx \geq \Big(\frac{1}{A-B} \int_B^A x^2 dx \Big)^n, \quad n = 0, 1, 2, \dots$$

i.e.,

$$\frac{A^{2n+1} - B^{2n+1}}{(A-B)(2n+1)} \geq (\frac{1}{A-B} \frac{A^3 - B^3}{3})^n$$

i.e.,

$$\frac{1}{A-B} \Big(\frac{A^{2n+1}}{(2n+1)!} - \frac{B^{2n+1}}{(2n+1)!} \Big) \geq \frac{1}{(2n)!} \Big(\sqrt{\frac{A^2 + AB + B^2}{3}} \Big)^{2n}.$$

By summing the last expression for $n = 0, 1, 2, \dots$ we obtain the proof of Lemma 5.

Putting there $A := (1/2 + c/8 - 1/6c)t$; $B := (c/8 - 1/6c - 1/2)t$, we obtain the proof of Lemma 4.

Monotonicity of $f(c, t)$ can be proved in a standard way; therefore we can conclude that the assertion from Proposition 5 is valid.

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