ON THE APPLICATION OF TRIGONOMETRIC SERIES IN THE ANALYSIS OF BEAMS ON ELASTIC FOUNDATION

By

MILAN ĐURIĆ

N. Kriloff [1] already pointed to the possibility of the application of the trigonometric series in the analysis of beams on elastic foundation, in the case of a simply supported beam. At the Second International Congress for Bridge and Structural Engineering M. Hetényi demonstrated [2] the method of finding solutions by means of trigonometric series and in the case of other end conditions as well. In his book „Beams on Elastic Foundation" [3] he gave a more detailed exposition of this procedure. In the application of this method on the beam with free ends Hetényi comes to an infinite system of linear equations, from which one can determine the coefficients of the trigonometric series representing the elastic line of the beam. We shall show that it is possible to avoid this system of linear equations and to get the coefficients of the series explicitly in the case of a beam with free ends as well. Besides this we shall ameliorate the convergence of the series representing only the difference of deflection ordinates between the elastic lines of beams with and without elastic foundation by means of the series as Kriloff used to do.

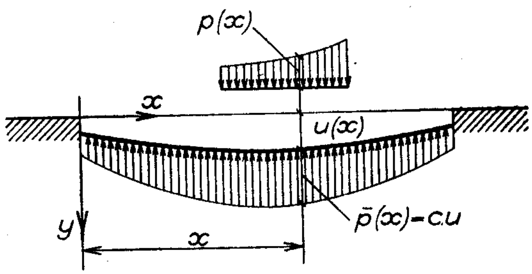

Fig. 1

Let us consider a beam whose length is l, with constant flexural rigidity EI, resting on an elastic foundation with the modulus of foundation c, acted upon by a distributed loading $p(x)$ and a distributed reaction $\bar{p}(x) = cu$. We can determine its deflection curve from the known differential equation:

$$EI\,u^{(IV)} + cu = p(x) \tag{1}$$

and from the end conditions which are

$$\begin{array}{lll} \text{for a simply supported end} & u=0 \ \text{and} \ u''=0, \\ \text{for a free end} & u''=0 \ \text{and} \ u'''=0. \end{array} \tag{2}$$

In the case of a beam simply supported on both ends, the upper conditions under (2) must be fulfilled for $x=0$ and $x=l$. These conditions are satisfied by the family of functions $\sin \frac{n\pi x}{l}$. Designating by u_0 the deflections under the load $p(x)$ of a simply supported beam of the same length, that does not rest on the elastic foundation, we can find the solution of the differential equation (1) in the form

$$u = u_0 + u_1 = u_0 - \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{l}. \tag{3}$$

Putting this expression into the equation (1) and taking into accont that $EI\,u_0^{(IV)} = p(x)$ we get

$$EI\,u_1^{(IV)} + cu_1 = cu_0$$

resp.

$$\sum_{n=1}^{\infty} \left[\frac{EI}{c}\left(\frac{n\pi}{l}\right)^4 + 1\right] \alpha_n \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} (\theta n^4 + 1)\,\alpha_n \sin \frac{n\pi x}{l} =$$

$$= \sum_{n=1}^{\infty} \mu_n \alpha_n \sin \frac{n\pi x}{l} = u_0, \tag{4}$$

where

$$\theta = \frac{EI\pi^4}{cl^4}; \quad \mu_n = \theta n^4 + 1. \tag{5}$$

For any given load we can represent u_0 in the form of a trigonometric series

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

converging in the most unconvenient cases better than $\frac{1}{n^3}$. From the equation (4) we get

$$\alpha_n = \frac{a_n}{\theta n^4 + 1} = \frac{a_n}{\mu_n}$$

and therefore (3) takes the form

$$u = u_0 - \sum_{n=1}^{\infty} \frac{a_n}{\mu_n} \sin \frac{n\pi x}{l}. \tag{6}$$

The reactive load is then $\bar{p}(x) = cu$. The slopes of the deflection curve, bending moments and shearing forces are given with

$$\varphi = u'(x) = \varphi_0 - \frac{1}{l} \sum_{n=1}^{\infty} \frac{a_n}{\mu_n} (n\pi) \cos \frac{n\pi x}{l},$$

$$M = -EI\,u''(x) = M_0 - \frac{EI}{l^2} \sum_{n=1}^{\infty} \frac{a_n}{\mu_n} (n\pi)^2 \sin \frac{n\pi x}{l},$$

$$Q = -EI\,u'''(x) = Q_0 - \frac{EI}{l^3} \sum_{n=1}^{\infty} \frac{a_n}{\mu_n} (n\pi)^3 \cos \frac{n\pi x}{l},$$

where φ_0, M_0 and Q_0 designate the corresponding values for the beam without elastic foundation.

In the case of the beam with free ends the lower conditions under (2) must be fulfilled for $x = 0$ and $x = l$. In this case we can find the solution in the form

$$u = u_{01} + u_{02} + u_{03} - \frac{a}{c} - \frac{b}{c}\left(1 - \frac{2x}{l}\right) - \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{l}, \tag{7}$$

where $u_{01}(x)$, $u_{02}(x)$ and $u_{03}(x)$ represent the elastic lines of a simply supported beam of the same length which does not rest on the elastic foundation, and which is acted upon by the loads: $p(x)$, uniformly distributed loading $a=\text{const.}$ and the loading distributed according to the law $b\left(1-\frac{2x}{l}\right)$. The constants a, b and α_n should be determined in such a way that the differential equation and the end conditions should be satisfied. Putting (7) into (1) and taking into account that $EI\,u_{01}^{(IV)}=p(x)$, $EI\,u_{02}^{(IV)}=a$ and $EI\,u_{03}^{(IV)}=b\left(1-\frac{2x}{l}\right)$ we get

$$\sum_{n=1}^{\infty} \mu_n \alpha_n \sin\frac{n\pi x}{l} = u_{01}+u_{02}+u_{03}, \tag{8}$$

where μ_n is given by (5). Let us represent u_{01}, u_{02} and u_{03} in the form of trigonometric series

$$u_{01}(x)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l},$$

$$EI\,u_{02}(x)=\frac{4al^4}{\pi^5}\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{l},$$

$$EI\,u_{03}(x)=\frac{4bl^4}{\pi^5}\sum_{n=2,4,6}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{l},$$

the equation (8) takes then the form

$$\sum_{n=1}^{\infty}\mu_n\alpha_n\sin\frac{n\pi x}{l}=$$

$$=\sum_{n=1,3,5}^{\infty}\left[a_n+\frac{1}{EI}\frac{4al^4}{(n\pi)^5}\right]\sin\frac{n\pi x}{l}+\sum_{n=2,4,6}^{\infty}\left[a_n+\frac{1}{EI}\frac{4bl^4}{(n\pi)^5}\right]\sin\frac{n\pi x}{l}.$$

This equation can be satisfied for all x only if

$$\begin{aligned}\alpha_n&=\frac{1}{\mu_n}\left[a_n+\frac{1}{EI}\frac{4al^4}{(n\pi)^5}\right] & n&=1,3,5\ldots\ldots\\ \alpha_n&=\frac{1}{\mu_n}\left[a_n+\frac{1}{EI}\frac{4bl^4}{(n\pi)^5}\right] & n&=2,4,6\ldots\ldots\end{aligned} \tag{9}$$

The expression for the deflection curve (7) is chosen in such a manner that the condition: that the bending moments vanish at the ends of the beam ($u''=0$) is already fulfilled. The remaining constants a and b should be determined in such a way that the shearing forces vanish at the ends of the beam (the condition $u'''=0$). Let us put

$$-EI(u_{01}''')_{x=0}=Q_{01}{}^{l}; \quad -EI(u_{01}''')_{x=l}=Q_{01}{}^{d},$$

$$-EI(u_{02}''')_{x=0}=EI(u_{02}''')_{x=l}=\frac{al}{2},$$

$$-EI(u_{03}''')_{x=0}=-EI(u_{03}''')_{x=l}=\frac{bl}{6},$$

then according to (7) the end conditions become

$$Q_{01}{}^{l}+\frac{al}{2}+\frac{bl}{6}-\frac{EI}{l^3}\sum_{n=1}^{\infty}(n\pi)^3\alpha_n=0,$$

$$Q_{01}{}^{d}-\frac{al}{2}+\frac{bl}{6}-\frac{EI}{l^3}\sum_{n=1}^{\infty}(-1)^n(n\pi)^3\alpha_n=0.$$

Adding and substracting these equations and taking into account (9) we get

$$a=-\frac{\dfrac{Q_{01}{}^{l}-Q_{01}{}^{d}}{2l}-\dfrac{EI}{l^4}\displaystyle\sum_{n=1,3,5}^{\infty}(n\pi)^3\frac{a_n}{\mu_n};}{\dfrac{1}{2}-4\displaystyle\sum_{n=1,3,5}^{\infty}\frac{1}{(n\pi)^2\mu_n}},$$

$$b=-\frac{\dfrac{Q_{01}{}^{l}+Q_{01}{}^{d}}{2l}-\dfrac{EI}{l^4}\displaystyle\sum_{n=2,4,6}^{\infty}(n\pi)^3\frac{a_n}{\mu_n}}{\dfrac{1}{6}-4\displaystyle\sum_{n=2,4,6}^{\infty}\frac{1}{(n\pi)^2\mu_n}}.$$

In this way the constants a and b are determined, thereafter the coefficients α_n are also determined by means of (9), and then the solution for the case of a beam with free ends is given by the expression (7).

This solution can readily be interpreted in the sense of the Structural analysis. It consists of the superposition of the solution for a beam with ends simply supported under the given loading, and the solutions for a beam with free ends loaded at the ends with concentrated forces symmetrically resp. antimetrically disposed. The solutions for a beam with free ends loaded in this manner are easily obtained from the solution for a simply supported beam loaded with the symmetrical uniformly distributed loading resp. with the antimetrical loading $b\left(1-\frac{2x}{l}\right)$.

[1] А. Н. Крылов, О расчёте балок, лежащих на упругом основании.

2] M. Hetényi, Analysis of Bars on Elastic Foundation. Second Congress for Bridge and Structural Engineering, Final Report, Berlin 1938, 849

4] M. Hetényi, Beams on Elastic Foundation, Michigan 1946, 69.