SOME SERIES APPLIED TO THE THEORY OF STRUCTURES

By

J. M. KLITCHIEFF, Belgrade

1. Preliminary

The following series are used in this paper.

If $m-1$ lateral loads R_i $(i=1, 2, \ldots)$ act on a beam simply supported at both ends, the bending moment at a distance x from the left end can be expressed by the trigonometric series

$$M = -\frac{2}{\pi^2} l \sum_{i=1}^{m-1} R_i \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi\xi_i}{l} \sin \frac{n\pi x}{l}, \tag{1}$$

whene l denotes the span of the beam and ξ_i the distance of the load R_i from the left support.

The trigonometric polynomial

$$\sum_{i=1}^{m-1} \sin \frac{in\pi}{m} = \sin^2 \frac{n\pi}{2} \cot \frac{n\pi}{2m}, \tag{2}$$

in which n and m are integers, is equal to zero for even values of n and for values of it being multiples of m. For all other values of n it is equal to $\cot n\pi/2m$.

The trigonometric polynomial

$$2\sum_{i=1}^{m-1} \sin \frac{in\pi}{m} \sin \frac{ik\pi}{m} = \sum_{i=1}^{m-1} \cos \frac{i(n-k)\pi}{m} - \sum_{i=1}^{m-1} \cos \frac{i(n+k)\pi}{m}, \tag{3}$$

in which n, k and m are integers, is equal to m for $k = n + 2mj$

and it is equal to $-m$ for $k=-n+2mj$, $j=1,2,\ldots$ It becomes zero for all other values of k and also for values of n being multiples of m.

By differentiating the known series

$$\sum_{n=-\infty}^{\infty} \frac{1}{n\pi+\theta} = \cot\theta \tag{4}$$

it can be proved that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n\pi+\theta)^2} = 1+\cot^2\theta \tag{5}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n\pi+\theta)^4} = {}^1/_3(1+\cot^2\theta)(1+3\cot^2\theta). \tag{6}$$

By differentiating *with respect to the parameter k* the known series

$$\sum_{n=1}^{\infty} \frac{n\sin n\theta}{n^2-k^2} = \frac{\pi}{2}\,\frac{\sin k(\pi-\theta)}{\sin k\pi}, \quad 0<\theta<2\pi, \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n\sin n\theta}{n^2-k^2} = -\frac{\pi}{2}\,\frac{\sin k\theta}{\sin k\pi}, \quad -\pi<\theta<\pi, \tag{8}$$

and by replacing in the derivatives k by $k\sqrt{-1}$, it can be proved that

$$\sum_{n=1}^{\infty} \frac{n\sin n\theta}{(n^2+k^2)^2} = \frac{\pi}{4k}\,\frac{\theta\sinh k\pi\cosh k(\pi-\theta) - \pi\sinh k\theta}{\sinh^2 k\pi}, \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n\sin n\theta}{(n^2+k^2)^2} = \frac{\pi}{4k}\,\frac{\theta\sinh k\pi\cosh k\theta - \pi\cosh k\pi\sinh k\theta}{\sinh^2 k\pi}. \tag{10}$$

By differentiating the equations (9) and (10) *with respect* *to* θ and replacing θ by zero, it can be proved that

$$\sum_{n=1}^{\infty} \frac{n^2}{(n^2+k^2)^2} = \frac{\pi}{4k} \frac{\sinh k\pi \cosh k\pi - k\pi}{\sinh^2 k\pi}, \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n^2+k^2)^2} = \frac{\pi}{4k} \frac{\sinh k\pi - k\pi \cosh k\pi}{\sinh^2 k\pi}. \tag{12}$$

By resolving into partial fractions the general term of the series*)

$$\sum_{j=-\infty}^{\infty} \frac{1}{(2jm+n)^2[(2jm+n)^2-m^2]} = \frac{1}{2m^3} \sum_{j=-\infty}^{\infty} \frac{1}{2jm+n-m} -$$

$$- \frac{1}{2m^3} \sum_{j=-\infty}^{\infty} \frac{1}{2jm+n+m} - \frac{1}{m^2} \sum_{j=-\infty}^{\infty} \frac{1}{(2jm+n)^2}$$

and by applying the equations (4) and (5) it can be proved that

$$\sum_{j=-\infty}^{\infty} \frac{1}{(2jm+n)^2[(2jm+n)^2-m^2]} = - \frac{\pi^2}{4m^4 \sin^2 n\pi/2m}. \tag{13}$$

The same method of summation can be applied to the series

$$\sum_{j=-\infty}^{\infty} \frac{1}{[(j+k)^2+{}^1/_4\beta^2]^2 - {}^1/_4(j+k)^2(1+\beta^2)^2} =$$

$$= \frac{8\pi}{\beta^2(1-\beta^4)} \frac{\sin \pi\beta^2}{\cos 2k\pi - \cos \pi\beta^2}, \tag{14}$$

and to the series

$$\sum_{j=-\infty}^{\infty} \frac{1}{\lambda^2 - [1+(2jm+n)^2\gamma^2]^2} =$$

$$= \frac{\pi}{4m\gamma\lambda} \left\{ \frac{1}{\sqrt{\lambda+1}} \frac{\sinh \pi\sqrt{\lambda+1}/m\gamma}{\cosh \pi\sqrt{\lambda+1}/m\gamma - \cos n\pi/m} - \right.$$

$$\left. - \frac{1}{\sqrt{\lambda-1}} \frac{\sin \pi\sqrt{\lambda-1}/m\gamma}{\cos \pi\sqrt{\lambda-1}/m\gamma - \cos n\pi/m}. \right. \tag{15}$$

---

*) This method of summation of series was communicated to the author by Dr. *J. Karamata*.

2. Bending of a Beam on Elastic Supports

(Communicated September 14, 1949)

The problem (Fig. 1) of bending of a continuous girder simply supported at the ends on rigid supports and having $m-1$ intermediate elastic supports, is usually discussed by replacing the elastic supports by a continuous elastic medium. The use of this method is limited to the case of many equally spaced supports of equal rigidities. As it will be shown below, these restrictions become unnecessary, if the expression of the deflection curve of the beam in the form of trigonometric series is used.

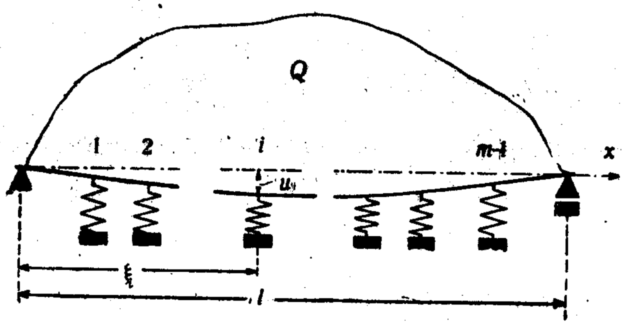

Fig. 1

The bending moment produced by any given concentrated or distributed load can be written in form of series

$$M' = -\pi^2 \frac{EI}{l^2} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}, \tag{16}$$

in which A_n are given coefficients. Denoting by u_i the deflection of the beam at the support number i and by α_i the „spring-constant" of the support, the bending moment produced by $m-1$ reactions $R_i = \alpha_i u_i$ of the elastic supports becomes [Equation (1)]

$$M'' = \frac{2}{\pi^2} l \sum_{i=1}^{m-1} \alpha_i u_i \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{k\pi\xi_i}{l} \sin \frac{k\pi x}{l}. \tag{17}$$

If we assume the deflection curve of the beam in the form

$$u=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \tag{18}$$

and introduce it into the equation (17), it becomes

$$M''=\frac{2}{\pi^2}\, l \sum_{i=1}^{m-1} \alpha_i \sum_{k=1}^{\infty} C_k \sin\frac{k\pi\xi_i}{l} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi\xi_i}{l} \sin\frac{n\pi x}{l}. \tag{19}$$

Introducing the expressions (16), (18) and (19) into the differential equation of the deflection curve

$$EI\frac{d^2u}{dx^2}=-(M'+M'').$$

where EI is the rigidity of the beam, a system of linear equations can be obtained of the kind

$$n^4 C_n+\frac{2}{\pi^4}B\sum_{k=1}^{\infty} C_k \sum_{i=1}^{m-1} \alpha_i \sin\frac{k\pi\xi_i}{l} \sin\frac{n\pi\xi_i}{l}=n^2 A_n, \tag{20}$$

in which $B=l^3/EI$. This system can be easily solved by successive approximations in every particular case.

In the special case of equally spaced supports of equal rigidities, α can be replaced by α, ξ_i by li/m. If we introduce the values of the expression (3) established above, the system of equations (20) becomes

$$\left.\begin{aligned}
n^4 C_n+\frac{m}{\pi^4}\alpha BS_n &= n^2 A_n,\\
(n-2m)^4 C_{|n-2m|}-\frac{m}{\pi^4}\alpha BS_n &= (n-2m)^2 A_{|n-2m|},\\
(n+2m)^4 C_{n+2m}+\frac{m}{\pi^4}\alpha BS_n &= (n+2m)^2 A_{n+2m},\\
(n-4m)^4 C_{|n-4m|}-\frac{m}{\pi^4}\alpha BS_n &= (n-4m)^2 A_{|n-4m|},\\
\dots\dots\dots\dots\dots\dots&\dots\dots
\end{aligned}\right\} \tag{21}$$

in which the following notation is introduced

$$S_n = C_n + C_{n+2m} + C_{n+4m} + \ldots - C_{|n-2m|} - C_{|n-4m|} - \ldots \quad (22)$$

By the elimination of S_n from the first pair of the equations (21) we have*)

$$C_{|n-2m|} = -\frac{n^4}{(n-2m)^4} C_n + \frac{n^4 A_n}{(n-2m)^4} + \frac{A_{|n-2m|}}{(n-2m)^2},$$

and in the same way obtain

$$C_{n+2m} = \frac{n^4}{(n+2m)^4} C_n - \frac{n^2 A_n}{(n+2m)^4} + \frac{A_{n+2m}}{(n+2m)^2},$$

$$C_{|n-4m|} = -\frac{n^4}{(n-4m)^4} C_n + \frac{n^2 A_n}{(n-4m)^4} + \frac{A_{|n-4m|}}{(n-4m)^2}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus, the equation (22) becomes

$$S_n = (n^4 C_n - n^2 A_n) \sum_{j=-\infty}^{\infty} \frac{1}{(n+2jm)^4} + \frac{A_n}{n^2} + \\ + \sum_{j=1}^{\infty} \frac{A_{n+2jm}}{(n+2jm)^2} - \sum_{j=1}^{\infty} \frac{A_{|n-2jm|}}{(n-2jm)^2},$$

and, from the first of the equations (21), using the expression (6), we obtain the final equation determining C_n:

$$\left(C_n - \frac{A_n}{n^2}\right)\left[1 + \frac{\alpha B}{48 m^4}\left(1 + \cot^2 \frac{n\pi}{2m}\right)\left(1 + 3\cot^2 \frac{n\pi}{2m}\right)\right] = \\ = \frac{m\alpha}{n^4 \pi^4} B \left[\sum_{j=1}^{\infty} \frac{A_{|n-2jm|}}{(n-2jm)^2} - \sum_{j=0}^{\infty} \frac{A_{n+2jm}}{(n+2jm)^2}\right].$$

3. Cross-Girders Supporting Several Beams

(Communicated September 14, 1949)

The same method, as above, can be applied to the case (Fig. 2) of several equally spaced and equally loaded beams of equal rigidity, supported by a cross-girder. The deflection u_i at

*) Dr. *M. Tomić* called attention of the writer to this method of solution of equations, similar to equations (21).

the joint number i can be written in the form

$$u_i = B(\gamma Q - \beta R_i), \tag{23}$$

where Q denotes the load acting on one beam, R_i the pressure exercised by the beam on the girder, and γ and β are coeffi-

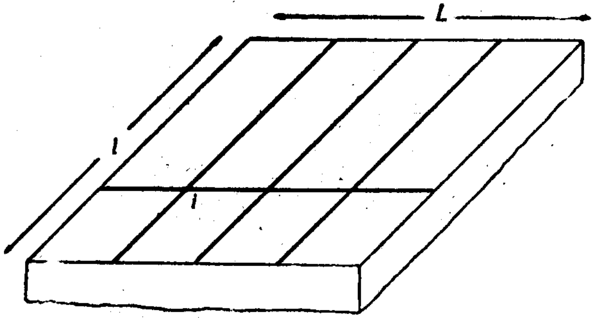

Fig 2

cients depending on the distribution of the load Q along the beam and on the position of the girder. For example, if the load is uniformly distributed and the girder bisects the beams $\gamma = 5/384$ and $\beta = 1/48$. From the equation (23) we have

$$R_i = \frac{\gamma}{\beta} Q - \frac{u_i}{\beta B} = \frac{\gamma}{\beta} Q - \frac{1}{\beta B} \sum_{k=1}^{\infty} C_k \sin \frac{k\pi i}{m}. \tag{24}$$

The bending moment produced by the pressures of $m-1$ beams is

$$M = \frac{2}{\pi^2} L \left\{ \frac{\gamma}{\beta} Q \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{i=1}^{m-1} \sin \frac{n\pi i}{m} \sin \frac{n\pi x}{L} - \right.$$

$$\left. - \frac{1}{\beta B} \sum_{k=1}^{\infty} C_k \sum_{i=1}^{m-1} \sin \frac{k\pi i}{m} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi i}{m} \sin \frac{n\pi x}{L} \right\}.$$

Introducing it into the differential equation of the deflection curve of the cross-girder and proceeding as in the case of elastic supports, we finally obtain

$$C_n + \frac{2}{\pi^4} \frac{B'}{\beta B} \frac{1}{n^4} \sum_{k=1}^{\infty} C_k \sum_{i=1}^{m-1} \sin \frac{k\pi i}{m} \sin \frac{k\pi i}{m} =$$

$$= \frac{2}{\pi^4} \frac{\gamma}{\beta} B'Q \frac{1}{n^4} \sum_{i=1}^{m-1} \sin \frac{n\pi i}{m},$$

where $B' = L^3/EJ, J$ being the moment of inertia of the cross-section of the girder. Using the equation (6) and introducing the established values of the expressions (2) and (3), this equation becomes

$$C_n\left[1+\frac{1}{48}\frac{B'}{\beta m^3 B}\left(1+\cot^2\frac{n\pi}{2m}\right)\left(1+3\cot^2\frac{n\pi}{2m}\right)\right]=$$

$$=\frac{2}{\pi^4}\frac{\gamma}{\beta}B'Q\frac{\cot n\pi/2m}{n^4}, \tag{25}$$

R_i being now determined by the equation (24).

The equations (25) and (24) can also be applied to the case of two symmetrically situated girders of equal rigidities. The coefficient β in that case must correspond to the deflection of the beam produced by two equal and symmetrically situated loads.

4. Beams Supported by Several Cross-Girders

(Communicated September 14, 1949)

Let us consider (Fig. 3) the case of 11 equally spaced beams bearing uniformly distributed loads Q and supported by 5 equally spaced cross-girders, the moment of inertia of the

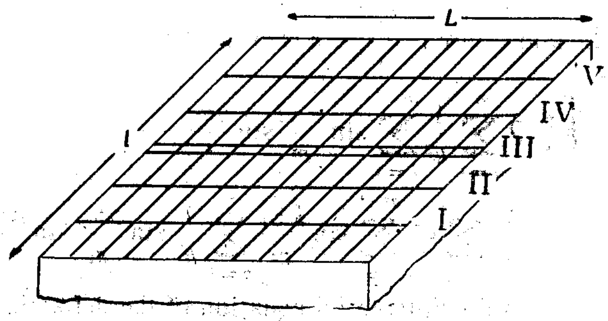

Fig. 3

central girder being twice as large as the moments of inertia of the side-girders. It could correspond, for example, to the bottom-structures of a large ship.

The problem contains 18 statically indeterminate reactions. They could be found by solving 18 equations, containing 8 unknown quantities each, or by using the approximate method introduced by *I. G. Boobnov**). Replacing the pressures exercised by the beams on the cross-girders by the pressure of a continuous elastic medium, this method reduces the problem to a system of 3 simultaneous differential equations of the fourth order. Such a system, in principle, represents no difficulties, but ih can be hardly expected that the mathematical work required to solve it, can be done by a designer. It will be shown that the application of trigonometric series reduces the problem to two systems of 3 linear algebraic equations each. In the general case, if there is no symmetry, the number of equations would be equal to the number of cross-girders.

We denote by R_i' the pressure exercised by the beam number i on the girders I and V, by R_i'' the pressures exercised by it on the girders II and IV, and by $2R_i'''$ the pressure on the central girder. The beam can be considered as being acted upon simultaneously by the given load Q, two equal reactions R_i' at the distances $^1/_6\, l$ from its ends, two equal reactions R_i'' at the distances $^1/_3\, l$ and two equal reactions R_i''' at the distances $^1/_2\, l$.

The deflections of the beam at its joints will be respectively

$$\left.\begin{aligned} u_i' &= (\gamma' Q - \beta_{11} R_i' - \beta_{12} R_i'' - \beta_{13} R_i'''), \\ u_i'' &= (\gamma'' Q - \beta_{21} R_i' - \beta_{22} R_i'' - \beta_{23} R_i'''), \\ u_i''' &= (\gamma''' Q - \beta_{31} R_i' - \beta_{32} R_i'' - \beta_{33} R_i'''). \end{aligned}\right\} \tag{23'}$$

This system of equatious corresponds to the equation (23).

The numerical values of the coefficient γ can be calculated from the known expression for the deflection of a simply supported beam, produced by the uniformly distributed load

$$\gamma = \frac{1}{24}\frac{x}{l}\left[1 - 2\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right].$$

Replacing in it x by $^1/_6\, l$, $^2/_6\, l$ and $^3/_6\, l$, respectively, we obtain

$$\gamma' = \frac{205}{24\cdot 6^4}, \quad \gamma'' = \frac{352}{24\cdot 6^4}, \quad \gamma''' = \frac{405}{24\cdot 6^4}.$$

*) „*Theory of Structures of Ships*", vol. 2, p. 395, St. Petersburg, 1914, (*Russ.*).

The values of the coefficients β can be calculated from a similar expression for the deflection produced by two equal concetrated loads at the distances $\xi \geqq x$ from its ends

$$\beta = \frac{1}{6}\frac{x}{l}\left[3\frac{\xi}{l}\frac{l-\xi}{l} - \left(\frac{x}{l}\right)^2\right],$$

namely

$$\beta_{11} = \frac{14}{6^4}, \quad \beta_{12} = \beta_{21} = \frac{23}{6^4}, \quad \beta_{13} = \beta_{31} = \frac{26}{6^4},$$

$$\beta_{22} = \frac{40}{6^4}, \quad \beta_{23} = \beta_{32} = \frac{46}{6^4},$$

$$\beta_{33} = \frac{54}{6^4}.$$

Thus, the equation (23′) becomes

$$\left.\begin{aligned} 14R_i' + 23R_i'' + 26R_i''' &= \frac{205}{24}Q - \frac{u_i'}{B}6^4, \\ 23R_i' + 40R_i'' + 46R_i''' &= \frac{352}{24}Q - \frac{u_i''}{B}6^4, \\ 26R_i' + 46R_i'' + 54R_i''' &= \frac{405}{24}Q - \frac{u_i'''}{B}6^4. \end{aligned}\right\} \qquad (23'')$$

Solving this system, we have

$$\left.\begin{aligned} R_i' &= \frac{1}{26}\left\{\frac{118}{24}Q - \frac{6^4}{B}\,(44u_i' - 46u_i'' + 18u_i''')\right\}, \\ R_i'' &= \frac{1}{26}\left\{\frac{100}{24}Q - \frac{6^4}{B}(-46u_i' + 80u_i'' - 46u_i''')\right\}, \\ R_i''' &= \frac{1}{26}\left\{\frac{53}{24}Q - \frac{6^4}{B}\,(18u_i' - 46u_i'' + 31u_i''')\right\}, \end{aligned}\right\} \qquad (24'')$$

corresponding to the equation (24), or if we replace

$$u' = \sum_{n=1}^{\infty} C_n' \sin\frac{n\pi x}{L}, \quad u'' = \sum_{n=1}^{\infty} C_n'' \sin\frac{n\pi x}{L}, \quad u''' = \sum_{n=1}^{\infty} C_n''' \sin\frac{n\pi x}{L}$$

it becomes

$$\left.\begin{aligned}
R_i' &= \frac{1}{26}\left\{\frac{118}{24}Q - \frac{6^4}{B}\sum_{k=1}^{\infty}(44\,C_k' - 46\,C_k'' + 18\,C_k''')\sin\frac{k\pi i}{12}\right\},\\
R_i'' &= \frac{1}{26}\left\{\frac{100}{24}Q - \frac{6^4}{B}\sum_{k=1}^{\infty}(-46\,C_k' + 80\,C_k'' - 46\,C_k''')\sin\frac{k\pi i}{12}\right\},\\
R_i''' &= \frac{1}{26}\left\{\frac{53}{24}Q - \frac{6^4}{B}\sum_{k=1}^{\infty}(18\,C_k' - 46\,C_k'' + 31\,C_k''')\sin\frac{k\pi i}{12}\right\}.
\end{aligned}\right\}\quad (24'')$$

Introducing these values into the differential equation of the first girder, we obtain, as above

$$C_n' = \frac{2\,B'}{\pi^4 n^4}\sum_{i=1}^{11} R_i' \sin\frac{n\pi i}{12} = \frac{B'}{26\,\pi^4 n^4}\left\{\frac{2\cdot 118}{24}\,Q\cot\frac{n\pi}{24} - \frac{12\cdot 6^4}{B}(44\,S_n' - 46\,S_n'' + 18\,S_n''')\right\},$$

where

$$S_n' = C_n' + C_{n+2m}' + C_{n+4m}' + \ldots - C_{|n-2m|}' - C_{|n-4m|}' - \ldots,$$
$$S_n'' = C_n'' + C_{n+2m}'' + C_{n+4m}'' + \ldots - C_{|n-2m|}'' - C_{|n-4m|}'' - \ldots,$$
$$S_n''' = C_n''' + C_{n+2m}''' + C_{n+4m}''' + \ldots - C_{|n-2m|}''' - C_{|n-4m|}''' - \ldots,$$

or

$$C_n' + \frac{12\cdot 6^4}{26\,\pi^4}\frac{B'}{B}(44\,C_n' - 46\,C_n'' + 18\,C_n''')\sum_{j=-\infty}^{\infty}\frac{1}{(24j+n)^4} = \frac{2\,QB'}{26\,\pi^4 n^4}\,\frac{118}{24}\cot\frac{n\pi}{24}.$$

Using the notations

$$K_n = \frac{12\cdot 6^4 B'}{26\,\pi^4 B}\sum_{j=-\infty}^{\infty}\frac{1}{(24j+n)^4} = \frac{6^4}{26\cdot 48\cdot 12^8}\,\frac{B'}{B}\left(1+\cot^2\frac{n\pi}{24}\right)\left(1+3\cot^2\frac{n\pi}{24}\right),$$

$$L_n = \frac{2}{24\cdot 26\,\pi^4}\,\frac{1}{n^4}\,\frac{B'}{B}\cot\frac{n\pi}{24},$$

it can be written in the form

$$(1+44K_n)\,C_n'-46K_n\,C_n''+18\,K_n\,C_n'''=118\,L_n\,QB. \qquad (25')$$

In a similar way we obtain for the girders II and III

$$\left.\begin{aligned}-46\,K_n\,C_n'+(1+80\,K_n)\,C_n''-46\,K_n\,C_n'''&=100\,L_n\,QB,\\ 18\,K_n\,C_n'-46\,K_n\,C_n''+(1+31\,K_n)\,C_n'''&=\;53\,L_n\,QB.\end{aligned}\right\} \qquad (25')$$

This system of 3 equations corresponds to the equation (25). Solving it, we get

$$C_n' = \frac{118+16\,744\,K_n+138\,500\,K_n^2}{1+155\,K_n+2\,808\,K_n^2+676\,K_n^3}\,L_n\,QB,$$

$$C_n'' = \frac{100+15\,365\,K_n+237\,952\,K_n^2}{1+155\,K_n+2\,808\,K_n^2+676\,K_n^2}\,L_n\,QB,$$

$$C_n''' = \frac{53+\;9\,048\,K_n+273\,780\,K_n^2}{1+155\,K_n+2\,808\,K_n^2+676\,K_n^3}\,L_n\,QB,$$

From the equations (24′) and (25′) the final expressions for reactions R_i can be obtained, giving the solution of the problem

$$R_i' = \frac{Q}{26}\left\{\frac{118}{24} - 6^4 \sum_{n=1,3\ldots}^{\infty} \frac{1546+192\,844\,K_n+79\,768\,K_n^2}{1+155\,K_n+2\,808\,K_n^2+676\,K_n^3}\,L_n \sin\frac{n\pi i}{12}\right\},$$

$$R_i'' = \frac{Q}{26}\left\{\frac{100}{24} - 6^4 \sum_{n=1,3\ldots}^{\infty} \frac{134+42\,848\,K_n+67\,600\,K_n^2}{1+155\,K_n+2\,808\,K_n^2+676\,K_n^3}\,L_n \sin\frac{n\pi i}{12}\right\},$$

$$R_i''' = \frac{Q}{26}\left\{\frac{53}{24} - 6^4 \sum_{n=1,3\ldots}^{\infty} \frac{-833-124956\,K_n+35\,828\,K_n^3}{1+155\,K_n+2\,808\,K_n^2+676\,K_n^3}\,L_n \sin\frac{n\pi i}{12}\right\},$$

5. Rectangular Plate with Clamped Edges

(Communicated June 8, 1949)

We start from the known expression for the deflection surface of a rectangular plate (Fig. 4) with simply supported edges, bent by a concentrated load Q applied at a point (ξ, η)

$$w_1 = \frac{4}{\pi^4}\,\frac{Q}{ab} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin m\pi\xi/a \cdot \sin n\pi\eta/b}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad (27)$$

where D, as usually, denotes the flexural rigidity of the plate.

If we apply at the same time a negative force $-Q$ (Fig. 5) at a point $(\xi+d\xi, \eta)$, the deflection produced by the simultaneous

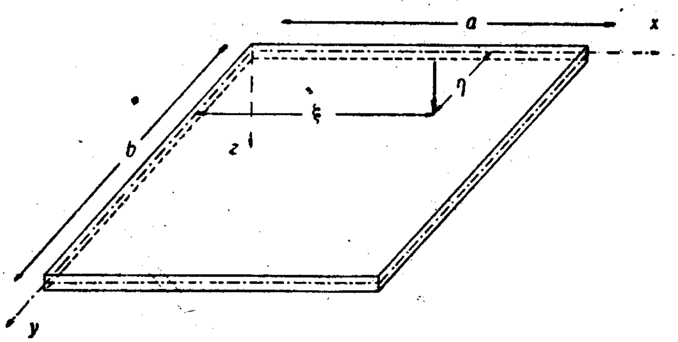

Fig. 4

action of the two forces, *i. e.* by the action of the couple $\mathfrak{M}' = Q\,d\xi$, will be

$$-w_1(\xi+d\xi, \eta) + w_1(\xi, \eta) = -\frac{\partial w_1}{\partial \xi}\,d\xi = -\frac{\mathfrak{M}'}{Q}\frac{\partial w_1}{\partial \xi}\,.$$

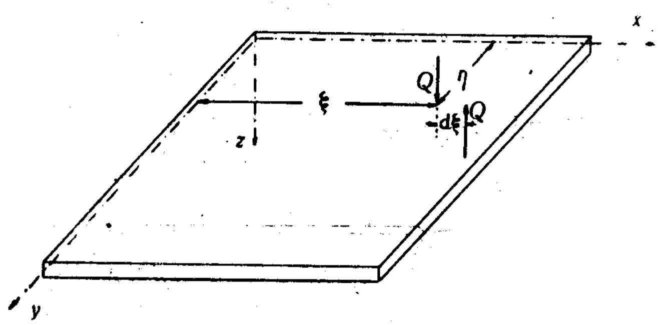

Fig. 5

The deflection, corresponding to a simultaneous action of *two* couples $\mathfrak{M}'$ and $\mathfrak{M}''$ applied at points $(0, \eta)$ and (a, η), respectively (Fig. 6) would be

$$-\frac{1}{Q}\left[\left(\mathfrak{M}'\frac{\partial w_1}{\partial \xi}\right)_{\xi=0} + \mathfrak{M}''\left(\frac{\partial w_1}{\partial \xi}\right)_{\xi=a}\right] \qquad (28)$$

Consider, now, such couples distributed (Fig. 7) along the edges $\xi=0$ and $\xi=a$, their intensities being represented by the series

$$\sum_{i=1}^{\infty} \mu_i' \sin\frac{i\pi\eta}{b}, \quad \sum_{i=1}^{\infty} \mu_i'' \sin\frac{i\pi\eta}{b}, \tag{29}$$

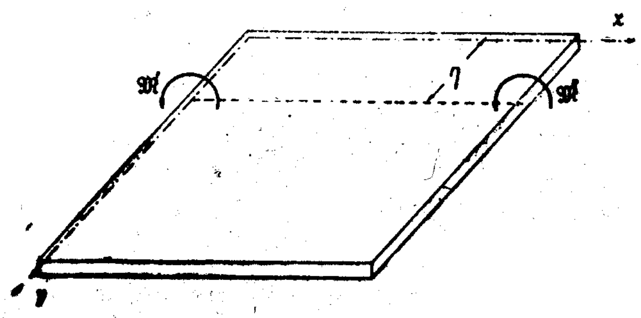

Fig. 6

respectively, in which μ_i' and μ_i'' are given coefficients. The corresponding deflection surface is determined by the equation (28), if we substitute into it

$$\mathfrak{M}' = \sum_{i=1}^{\infty} \mu_i' \sin\frac{i\pi\eta}{b}\, d\eta, \qquad \mathfrak{M}'' = \sum_{i=1}^{\infty} \mu_i'' \sin\frac{i\pi\eta}{b}\, d\eta, \tag{30}$$

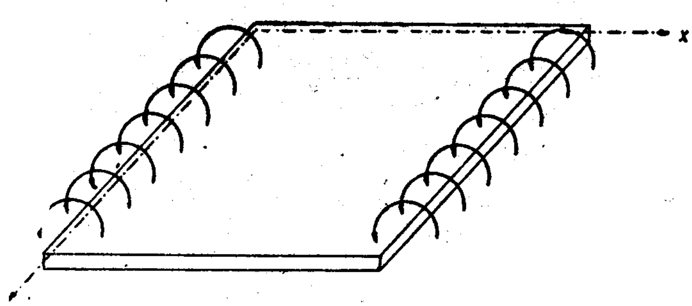

Fig. 7

replace w_1 by its expresion (27) and perform the integration of the expression obtained between the limits 0 and b, as

$$w_2 = -\frac{2}{\pi^3 D a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m[\mu'_n + (-1)^m \mu_n'']}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \quad (31)$$

In a similar manner the expression for the deflection surface produced by the couples

$$\sum_{j=1}^{\infty}\lambda_j'\sin\frac{j\pi\xi}{a}, \quad \sum_{j=1}^{\infty}\lambda_j''\sin\frac{j\pi\xi}{a}, \quad (32)$$

distributed along the edges $\eta = 0$ and $\eta = b$ can be obtained as

$$w_3 = -\frac{2}{\pi^3 D b^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n[\lambda_m' + (-1)^n \lambda_m'']}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \quad (33)$$

If the edges of the plate are clamped the coefficients λ', λ'', μ' and μ'' in the series (29) and (32) are determined by the condition that the rotations $\partial w/\partial x$ at the edges $x = 0$ and $x = a$, and $\partial w/\partial y$ at the edges $y = 0$ and $y = b$ are zero, $w = w_1 + w_2 + w_3$ being the sum of the expressions (27), (31) and (33), *i. e.* of the deflections produced by the given concentrated load Q and by the couples distributed along the edges.

The conditions at the edges $x = 0$ and $x = a$ lead to two equations

$$\frac{1}{a^2}\sum_{m=1}^{\infty}\frac{m^2[\mu_n' + (-1)^m\mu_n'']}{[m^2+(na/b)^2]^2} - \frac{n}{b^2}\sum_{m=1}^{\infty}\frac{m[\lambda_m' + (-1)^n\lambda_m'']}{[m^2+(na/b)^2]^2} = -\frac{2}{\pi}\frac{Q}{ab}\sin\frac{n\pi\eta}{b}\sum_{m=1}^{\infty}\frac{m\sin m\pi\xi/a}{[m^2+(na/b)^2]^2}, \quad (34)$$

$$\frac{1}{a^2}\sum_{m=1}^{\infty}\frac{m^2[(-1)^m\mu_n' + \mu_n'']}{[m^2+(na/b)^2]^2} - \frac{n}{b^2}\sum_{m=1}^{\infty}\frac{(-1)^m m[\lambda_m' + (-1)^n\lambda_m'']}{[m^2+(na/b)^2]^2} = -\frac{2}{\pi}\frac{Q}{ab}\sin\frac{n\pi\eta}{b}\sum_{m=1}^{\infty}\frac{(-1)^m m\sin m\pi\xi/a}{[m^2+(na/b)^2]^2}. \quad (35)$$

Two similar equations, corresponding to the conditions at the edges $y=0$ and $y=b$, can be deduced from the equations (34) and (35) by interchanging μ with $-\lambda$, m with n, a with b and ξ with η. These two equations, together with the equations (34) and (35), would represent a system of an infinite number of linear equations for calculating of coefficients λ and μ. It can be solved in every particular case by the method of successive approximations.

In the case of any kind of a distributed loading of intensity $q(\xi, \eta)$, the same equations can be used, but on the right-hand side of them Q must be replaced by $q\, d\xi\, d\eta$ and the expression integrated over the loaded surface of the plate.*)

Substituting na/b and $\pi\xi/a$ for k and θ, respectively, into the expressions (9) - (12), the series in the equations (34) and (35) can be replaced by their sums in a finite form. Thus, the equations established by *S. Timoshenko*,**) using a different method, for the particular cases of a uniformly distributed loading and of a load applied at the center of a plate, can be readily obtained.

5. Buckling of a Continuous Beam on Elastic Supports

(Communicated June 23, 1948)

It is sometimes necessary to select the rigidity of intermediate equidistant elastic supports of a beam (Fig. 8) so that they behave as though they were absolutely rigid, when the beam buckles on being subjected to longitudinal compresion by the forces P applied at its ends. The usual method of solving the problem***) leads to the tedious work of developing a determinant of $m-1$ order and of solving an equation of $m-1$ degree, m being the number of spans. Using the expression (18)

---

*) In this form the equations coresponding to the particular case of a uniformly loaded plate have been established by *J. A. Shimansky*. „Bending of Plates", Leningrad, 1936, *(Russ.)*.

**) „Plates and Shells", New York, 1940, p. 222.

***) *I. G. Boobnov*, l. c., vol. 1, p. 259.

for the deflection curve, the equation (39) can be established, which is much more convenient for practical purposes.

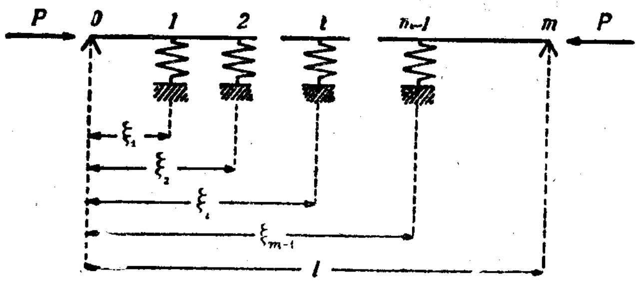

Fig. 8

We start from the differential equation for the deflection curve of a beam submitted to the simultaneous action of $m-1$ lateral reactions and of axial compressive forces P

$$EI\frac{d^2u}{dx^2} + M'' = -Pu,$$

where M'' is determined by the equation (17). Introducing the expression (18), we obtain as in **2.**

$$n^4 C_n + \frac{2}{\pi^4} B\alpha \sum_{k=1}^{\infty} C_k \sum_{i=1}^{m-1} \sin\frac{k i \pi}{m} \sin\frac{n i \pi}{m} = \frac{P}{P_e} n^2 C_n, \qquad (36)$$

where $P_e = \pi^2 EI/l^2$.

Substituting the established values of the expression (3), into the equation (36), we get for the values of n being multiples of m a group of independent equations of the form

$$P = n^2 P_e,$$

corresponding to buckling of the beam into m, $2m, \ldots$ half-waves, which is possible even in the case of absolutely rigid

Fig. 9

supports. The smallest value of the critical force, given by the first of them, corresponds to buckling into m half-wawes (Fig. 9).

For the values of n which are not multiples of m, we obtain $m-1$ systems of homogeneous linear equations in the coefficients C_n of the form

$$n^2\left(n^2-\frac{P}{P_e}\right)C_n+m\frac{B\alpha}{\pi^4}S_n=0,$$

$$(2m-n)^2\left[(2m-n)^2-\frac{P}{P_e}\right]C_{|n-2m|}-m\frac{B\alpha}{\pi^4}S_n=0, \qquad (37)$$

$$(2m+n)^2\left[(2m+n)^2-\frac{P}{P_e}\right]C_{n+2m}+m\frac{B\alpha}{\pi^4}S_n=0,$$

. . . .

. . . .

in which S_n is given by the equation (22).

Proceeding as in 2, we obtain

$$C_{|n-2m|}=-\left(\frac{n}{n-2m}\right)^2\frac{n^2-P/P_e}{(n-2m)^2-P/P_e}C_n,$$

$$C_{n+2m}=\left(\frac{n}{n+2m}\right)\frac{n^2-P/P_e}{(n+2m)^2-P/P_e}C_n,$$

. . . .

and introducing these values into the first of the equations (37), we have the equation

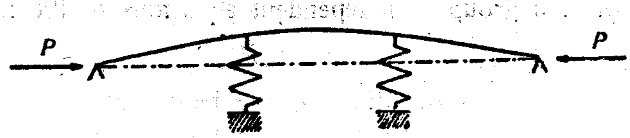

Fig. 10

$$1+m\frac{B\alpha}{\pi^4}\sum_{j=-\infty}^{\infty}\frac{1}{(2jm+n)^2\left[(2jm+n)^2-P/P_e\right]}=0$$

determining the values of the critical forces corresponding to the buckling of the beam into 1, 2,..., $(m-1)$ half-waves (Fig. 10 and 11).

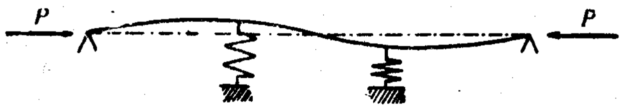

Fig. 11

If we impose the condition that no one of these values may be less, than the force $P_m = m^2 P_e$, required for buckling into m half-waves, and replace P/P_e by m^2 accordingly, the equation becomes[1])

$$1 + m\frac{B\alpha}{\pi^4} \sum_{j=-\infty}^{\infty} \frac{1}{(2jm+n)^2[(2jm+n)^2 - m^2]} = 1. \qquad (38)$$

Using the equation (13) this equation can be written in the form

$$\alpha = \frac{4\pi^2 m^3}{B} \sin^2 \frac{n\pi}{2m} .$$

The largest value of α, corresponding to $n = m-1$, thus, is determined by the equation

$$\alpha = \frac{4\pi^2 m^3}{B} \cos^2 \frac{\pi}{2m} .$$

7. Buckling of a Rectangular Plate Reinforced by Transverse Ribs[2])

The system of equations determining the critical pressure p uniformly distributed along the edges $x=0$ and $x=a$ of a rectangular plate stiffened by transverse ribs (Fig. 12), has been established by *S. Timoshenko*[3].) If the ribs are equal and equi-

---

[1]) The equation (38) can be deduced by a different method, s. author's paper in *Quart. Journ. Mech. and Appl. Math*, 1949, p. 257.

[2]) S. author's papers, *Journ. of Appl. Mech.*, 1949, p. 74 and *Publ. math.*, Belgrade, 1948, p. 53.

[3]) *„Elastic Stability"*, New-York, 1936, p. 378.

distant, these equations can be written in the following form

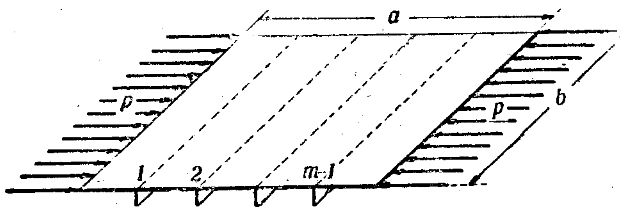

Fig. 12

$$C_n(n^2+m^2\beta^2)^2+2\tilde{\omega}\sum_{k=1}^{\infty}C_k\sum_{i=1}^{m-1}\sin\frac{in\pi}{m}\sin\frac{ik\pi}{m}=\lambda^2n^2C_n, \quad (39)$$

where $n=1,2,\ldots$, $\beta=a/mb$, $\tilde{\omega}=EI\beta^3/bD$, $\lambda^2=pa^2/\pi^2D$ and C_n are parameters.

Using the established values of the expression (3), we obtain for values of n being multiples of m from the equation (39) a group of independent equations of the form

$$m^2(n^2+\beta^2)^2=n^2\lambda^2, \quad (40)$$

which determine the critical pressure corresponding to buckling of the plate, when the ribs become nodal-lines of the buckled plate, as though they were absolutely rigid.

For values of n which are not multiples of m, a group of homogeneous linear equations are obtained from the equation (39). These equations can be divided into $m-1$ independent systems of the form

$$\begin{aligned}&C_n(n^2+\beta^2)^2+m\tilde{\omega}S_n=n^2\lambda^2C_n,\\&C_{|2m-n|}[(2m-n)^2+\beta^2]^2-m\tilde{\omega}S_n=(2m-n)^2\lambda^2C_{|2m-n|},\\&C_{2m+n}[(2m+n)^2+\beta^2]^2+m\tilde{\omega}S_n=(2m+)^2\lambda^2C_{2m+n},\\&\cdots\cdots\cdots\cdots\cdots\cdots\\&\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned} \quad (41)$$

similar to the equations (37). Proceeding in the same manner and eliminating the parameters C_n, a series of equations of the form

$$1+m\tilde{\omega}\sum_{j=-\infty}^{\infty}\frac{1}{[(2jm+n)^2+m^2\beta^2]^2-(2jm+n)\lambda^2}=0, \quad (42)$$

can be obtained which, with $n=1,2,\ldots,(m-1)$, determine the values of λ, *i.e.* of the pressure corresponding to buckling of the plate into $1,2,\ldots,(m-1)$ half-waves, which is possible with elastic ribs only.

It is required, usually, that the ribs must be rigid enough to become nodal-lines of the buckled plate, as if they were absolutely rigid. Introducing the smallest of the values of λ, given by the first[1]) of the equations (40) (with $n=1$), into the equation (42), it becomes

$$1+\frac{\tilde{\omega}}{16m^3}\sum_{j=-\infty}^{\infty}\frac{1}{\left[\left(j+\frac{n}{2m}\right)^2+\frac{1}{4}\beta^2\right]^2-\frac{1}{4}\left(j+\frac{n}{2m}\right)^2(1+\beta^2)^2}=0.$$

Using the equation (14) and summing up the series, the last equation can be written in the form

$$\frac{EI}{Db}=-\frac{2}{\pi}(1-\beta^4)\frac{\cos n\pi/m-\cos\pi\beta^2}{\beta\sin\pi\beta^2},$$

and the largest value of EI, determining the rigidity of ribs required, corresponding to $n=m-1$, is

$$\frac{EI}{Db}=\frac{2}{\pi}(1-\beta^4)\frac{\cos\pi/m+\cos\pi\beta^2}{\beta\sin\pi\beta^2}.$$

8. Buckling of a Rectangular Plate Reinforced by Longitudinal Ribs

(Communicated February 2, 1949)

The problem (Fig. 13) is quite similar to the foregoing one. We start again from the equations established by *S. Timoshenko*). In the case of equal and equidistant longitudinal ribs they become

$$C_n[(1+n^2\gamma^2)^2-\lambda^2]+ \\ +2(\Omega-\delta\lambda^2)\sum_{k=1}^{\infty}C_k\sum_{i=1}^{m-1}\sin\frac{in\pi}{m}\sin\frac{ik\pi}{m}=0, \qquad (43)$$

[1]) We assume that β is less than $\sqrt{2}$, which is the usual case in practice. Otherwise, the value of λ determined by some other of the equations (40) should be taken.

[2]) l. c, p. 372.

where $\Upsilon = a/b$, $\Omega = EI/Db$ and $\delta = A/bh$, A being the area of the cross-section of one rib.

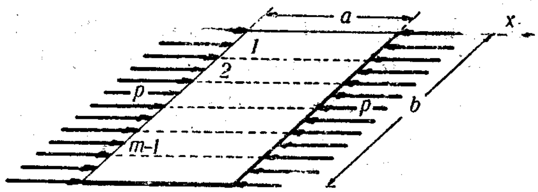

Fig. 13

Proceeding as in **7**, for values of n being multiples of m a group of independent equations can be obtained of the form

$$\lambda_n^2 = (1 + n^2\Upsilon^2)^2, \tag{44}$$

corresponding to buckling of the plate into m, $2m$, ... half-waves in the transverse direction. For the values of n, which are not multiples of m, we obtain $m - 1$ independent systems of linear homogeneous equations in C_n of the form

$$\begin{aligned} C_n[(1 + n^2\gamma^2)^2 - \lambda^2] + m(\Omega - \delta\lambda^2) S_n = 0, \\ C_{|2m-n|}\{[1 + (2m - n)^2 \gamma^2]^2 - \lambda^2\} - m(\Omega - \delta\lambda^2) S_n = 0, \\ C_{2m+n}\{[1 + (2m + n)^2 \gamma^2]^2 - \lambda^2\} + m(\Omega - \delta\lambda^2) S_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{45}$$

$n = 1, 2, \ldots, (m - 1)$, corresponding to buckling into 1, 2,.., $m - 1$ half-waves in the transverse direction. By elimination of the parameters C_n from the equations (45) we have, as in **7**

$$m(\Omega - \delta\lambda^2) \sum_{j=-\infty}^{\infty} \frac{1}{\lambda^2 - [1 + (2jm + n^2 \Upsilon^2]^2} = 1. \tag{46}$$

Using the equation (15), the series can be summed up and values of λ calculated, which would correspond to buckling of the plate into 1, 2,.., $(m - 1)$ half-waves in the transverse direction.

If it is required, as in **7**, that the ribs must become nodal-lines of the buckled plate, the smallest of the values of λ corresponding to absolutely rigid ribs, must be introduced into the equation (46), then, the largest of the values of Ω, determined by the equation, gives the rigidity of ribs required.

The most usual case in practice is $m\gamma < \sqrt{2}$. In that case the value of λ corresponding to buckling of the plate with absolutely rigid ribs is given by the first of the equations (44), *i. e.* with $n = m$, as

$$\lambda_1 = 1^2 + m^2 \gamma^2.$$

Thus, from the equation (15) the sum of the series becomes

$$\sum_{j=-\infty}^{\infty} \frac{1}{(1+m^2\gamma^2)^2 - [1+(2jm+n)^2\gamma^2]^2} =$$

$$= \frac{1}{4\, m\gamma\, (1+m^2\gamma^2)\sqrt{2+m^2\gamma^2}} \; \frac{\sinh \pi \sqrt{2+m^2\gamma^2}/m\gamma}{\operatorname{coh} \pi \sqrt{2+m^2\gamma^2}/m\gamma - \cos \pi n/m}.$$

With an error not exceeding 2½% it can be replaced by

$$\frac{\pi}{4\, m\gamma\, (1+m^2\gamma^2)\sqrt{2+m^2\gamma^2}}$$

and the rigidity of the ribs requested can he calculated from the equation

$$\Omega - \delta(1+m^2\gamma^2)^2 = \frac{4}{\pi}\gamma\,(1+m^2\gamma^2)\sqrt{2+m^2\gamma^2}.$$