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HOMOTOPY COLIMIT OF QUASI-JOIN DIAGRAMS

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Abstract. We calculate the homotopy colimits of toric quasi-join diagrams that naturally appear in the shelling process of toric varieties [**2**]. Our objective is to give a sufficiently complete description of these spaces with the emphasis on those quasi-join diagrams which resemble the diagrams associated with lens spaces. The central result is a "recognition" theorem which is intended to serve as a principal result in a homological description of more complex, shellable spaces.

The problem of describing the geometric and combinatorial structure of toric varieties has attracted a lot of attention both in algebraic geometry and combinatorics [**3**], [**10**], [**6**], [**4**]. In [**2**] a new approach was developed, an approach based on the so called shelling diagrams, diagram techniques and fragment calculations. The question of describing the fragments of spaces, appearing in the shelling process, leads to special diagrams defined over the face poset of a simplex.

One of the objectives of this paper is to emphasize the role of these, so called quasi-join diagrams, in understanding the structure of general toric varieties. For example, a corollary of the recognition theorem, allows us to offer another interpretation or explanation for some results of [**5**].

Let Σ and Σ' be complete fans in $\mathbf{R}^2$ generated by vectors $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,0)$, $v_4 = (0,-1)$ and $v_1' = (1,0)$, $v_2' = (0,1)$, $v_3' = (-1,1)$, $v_4' = (-1,-1)$. The fans Σ and Σ' are combinatorial isomorphic, but it was shown in the example 3.8 from [**5**] that cohomology algebras of the associated toric varieties X_Σ and $X_{\Sigma'}$ are not isomorphic. From our point of view, this phenomenon is a consequence of the fact that the varieties X_Σ and $X_{\Sigma'}$ are constructed from different diagrams over the same poset

$$X_\Sigma \simeq \mathrm{hocolim}(S^2 \vee S^2 \leftarrow S^3 \rightarrow pt) \quad \text{and}$$
$$X_{\Sigma'} \simeq \mathrm{hocolim}(S^2 \vee S^2 \leftarrow L(1,1;2) \rightarrow pt)$$

where $L(1,1;2)$ is the lens space.

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1. Introduction

The functor $X : P \to \mathrm{Top}$ from a poset (or a small category) P to the category Top of compactly generated spaces (or local Hausdorff spaces) is a *diagram* of spaces. This means that diagrams over poset P are objects in the categories Top^P, and morphisms are natural transformations. A *morphism* $X \to Y$ of diagrams $X : P \to \mathrm{Top}$ and $Y : Q \to \mathrm{Top}$ is a pair $(f, \alpha) = \Gamma$, where $f : P \to Q$ is a poset map and α a family of maps $\{\alpha_c : X_c \to Y_{fc} \mid c \in P\}$ such that for every arrow $c \xrightarrow{g} c'$ in P, $Y_{fg} \circ \alpha_c = \alpha_{c'} \circ X_g$. For diagrams $X, Y : P \to \mathrm{Top}$, the morphism $(Id_P, \alpha) \in \mathrm{Mor}_{\mathrm{Top}^P}(X, Y)$ is *isomorphism* X and Y if and only if there is a morphism (Id_P, β), $\in \mathrm{Mor}_{\mathrm{Top}^P}(Y, X)$ such that $(\forall c \in P)\,(\beta_c \circ \alpha_c = id_{X_c}$ and $\alpha_c \circ \beta_c = id_{Y_c})$.

Definition 1.1. Let $\mathcal{B}_n$ be the face poset of the $(n-1)$-simplex Δ^{n-1} (i.e., the poset $\mathbf{P}(\{1, \ldots, n\}) - \{\emptyset\}$ ordered by inclusion), and $X_1, \ldots, X_n$, topological spaces. A diagram $X : \mathcal{B}_n \to \mathrm{Top}$ defined by

$$X_a := \prod_{i \in a} X_i, \text{ for } a \in \mathcal{B}_n$$

is a *quasi-join diagram*. If $X_1 = \cdots = X_n = A$, then the diagram X is called an A-quasi-join diagram, or simply a quasi-join diagram. If for every $a \supseteq b$ the morphism $X_{a \overset{\supseteq}{\to} b} : \prod_{i \in a} X_i \to \prod_{j \in b} X_j$ is a projection, then X is called a *join diagram*.

The *homotopy colimit* of the diagram $X : P \to \mathrm{Top}$ is a quotient space

$$\mathrm{hocolim}_P\, X = \coprod_{p \in P} \Delta(P_{\leq p}) \times X_p / \sim$$

where $\sim$ is a "naturally defined" equivalence relation (see e.g. [**14**] for details). For example, if X is a join diagram [**12**], then $\mathrm{hocolim}_{\mathcal{B}_n} X = X_1 * \cdots * X_n$. Many interesting properties of the homotopy colimit construction and the important tools for its calculation can be found in [**1**], [**7**], [**12**], [**14**]. Now we formulate the central question of this article.

Problem 1.1. *What can be said about the homotopy colimit of a quasi-join diagram in general, and in particular when*

(*a*) $X_1 = \cdots = X_n = S^1$, *or*

(*b*) $X_1 = \cdots = X_n = K(G, n)$ *for G Abelian group and $n \geq 1$?*

2. Lens spaces as quasi-join diagrams

In this section we focus our attention to lens spaces and interpret them as homotopy colimits. This is the first place where quasi-join diagrams naturally appear.

2.1. Let $r_1, r_2, \ldots, r_n$ and m be integers such that $(r_i, m) = 1$ for every $i \in \{1, \ldots, n\}$. Using the m-th primitive root ω of 1 (for example, $\omega = e^{2\pi i/m}$), we can define $\mathbf{Z}/m$ action on $\mathbf{C}^n$ with

$$\mathbf{Z}/m \times \mathbf{C}^n \to \mathbf{C}^n; (l, (z_1, \ldots, z_n)) \longmapsto (\omega^{lr_1} z_1, \ldots, \omega^{lr_n} z_n).$$

The action can be restricted to the sphere $S(\mathbf{C}^n) \approx S^{2n-1}$, because $|z_1|^2 + \cdots + |z_n|^2 = 1$ imply $|\omega^{lr_1} z_1|^2 + \cdots + |\omega^{lr_n} z_n|^2 = |\omega^{lr_1}|^2 |z_1|^2 + \cdots + |\omega^{lr_n}|^2 |z_n|^2 = |z_1|^2 + \cdots + |z_n|^2 = 1$.

DEFINITION 2.1. The *lens space* $L(r_1, r_2, \ldots, r_n; m)$ is the space of $\mathbf{Z}/m$-orbits $S(\mathbf{C}^n)/(\mathbf{Z}/m)$.

The following properties are well known:

PROPOSITION 2.1. (i) *The lens spaces* $L(r_1, \ldots, r_n; m)$ *and* $L(s_1, \ldots, s_n; m)$ *are homotopy equivalent if and only if there exists a unit* $u \in Z/m$ *such that* $\prod r_i = \pm u^n \prod s_i$ *in* Z/m.

(ii) *If* $(\forall i \in \{1, \ldots, n\})$, $0 < r'_i < m$ *and* $r'_i = r_i \pmod m$, *then* $L(r_1, \ldots, r_n; m) \simeq L(r'_1, \ldots, r'_n; m)$.

(iii) *If* $d = (r_1, r_2, \ldots, r_n)$, *then* $L(r_1/d, \ldots, r_n/d; m) \simeq L(r_1, \ldots, r_n; m)$.

(iv) *The lens spaces* $L(r_1, \ldots, r_n; m)$ *and* $L(s_1, \ldots, s_n; m)$ *are homeomorphic if and only if there are numbers* a *and* $e_1, \ldots, e_n \in \{1, -1\}$ *such that* $(r_1, \ldots, r_n)$ *is a permutation of* $(e_1 a s_1, \ldots, e_n a s_n) \pmod p$.

A consequence is that if we are interested only in the homotopy type of the lens space $L(r_1, \ldots, r_n; m)$ then we can always assume that $(r_i, m) = 1$, $r_i \in \{1, \ldots, m-1\}$ and $(r_1, \ldots, r_n) = 1$.

2.2. Let $V_i = \{0\} \times \cdots \times \mathbf{C} \times \cdots \{0\}$ be the i-th coordinate space in $\mathbf{C}^n$. The spaces V_i are invariant under the $\mathbf{Z}/m$-action on $\mathbf{C}^n$ described above:

$$(l, (0, \ldots, 0, z_i, 0, \ldots, 0)) \longmapsto (\omega^{lr_1} \cdot 0, \ldots, \omega^{lr_i} z_i, \ldots, \omega^{lr_n} \cdot 0) \in V_i.$$

Then the sphere $S(\mathbf{C}^n)$ is $\mathbf{Z}/m$-homeomorphic to the wedge $S(V_1) * \cdots * S(V_n)$, where the action on spaces V_i induces an action on the wedge [**11**]. The sphere $S(V_1) * \cdots * S(V_n)$ can be seen [**12**], [**2**] as a homotopy colimit of a join diagram $S : \mathcal{B}_n \to \mathrm{Top}$ defined by

$$S_a = \prod_{i \in a} S(V_i), \text{ for } a \in \mathcal{B}_n, \quad \text{and} \quad S_{a \overset{\supseteq}{\to} b} : \prod_{i \in a} S(V_i) \to \prod_{j \in b} S(V_j) \text{ is a projection,}$$

where $\mathcal{B}_n$ is the poset $P(\{1, \ldots, n\}) - \{\emptyset\}$ ordered by inclusion. Since the sphere $S(V_1) * \cdots * S(V_n)$ has a $\mathbf{Z}/m$-action, we actually deal with a diagram $Q : \mathcal{B}_n \to \mathrm{Top}^{\mathbf{Z}/m}$ defined in the same way, but with all spaces equipped with the appropriate $\mathbf{Z}/m$-actions. This proves the following lemma:

LEMMA 2.1. (i) $\operatorname{hocolim}_{\mathcal{B}_n} Q$ *is* $\mathbf{Z}/m$*-homeomorphic with* $S(V_1) * \cdots * S(V_n)$.

(ii) $L(r_1, r_2, \ldots, r_n; m) \approx (\operatorname{hocolim}_{\mathcal{B}_n} Q)/(\mathbf{Z}/m) = \operatorname{colim}_{\mathbf{Z}/m} \operatorname{hocolim}_{\mathcal{B}_n} Q$.

Now, if $\operatorname{colim}_{\mathbf{Z}/m}$ and $\operatorname{hocolim}_{\mathcal{B}_n}$ are allowed to commute in the last expression of the preceding lemma, we would immediately obtain a description of the lens space $L(r_1, r_2, \ldots, r_n)$ as a homotopy colimit.

The following theorem says that this is always possible.

THEOREM 2.1. $L(r_1, r_2, \ldots, r_n; m) \approx \operatorname{hocolim}_{\mathcal{B}_n} \operatorname{colim}_{\mathbf{Z}/m} Q$.

Proof. Since

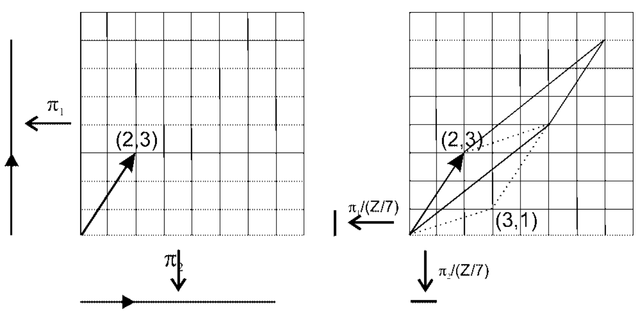
$$L(r_1, r_2, \ldots, r_n; m) \approx (\operatorname{hocolim}_{\mathcal{B}_n} Q)/(\mathbf{Z}/m) = \operatorname{colim}_{\mathbf{Z}/m} \operatorname{hocolim}_{\mathcal{B}_n} Q,$$

and homotopy colimit has the right adjoint [**7**], [**12**], the colimit can commute with the homotopy colimit. This follows from the known category theory lemma:

Let $T : \mathfrak{C} \to \mathfrak{D}$ and $S : \mathfrak{D} \to \mathfrak{C}$ be functors and S a right adjunct of T. Then the the functor T commutes with colimits. □

Thus, lens spaces are homotopy colimits of quasi-join diagrams over simplex. This motivated us to ask for a transparent description of the diagram $\operatorname{colim}_{\mathbf{Z}/m} Q$ of $\mathbf{Z}/m$-orbit spaces over $\mathcal{B}_n$.

2.3. Let $n = 2$. In order to describe the diagram $\operatorname{colim}_{\mathbf{Z}/m} Q$, we use the model $\mathbf{R}^2/(m\mathbf{Z})^2$ for torus $S^1 \times S^1$ and $\mathbf{R}/m\mathbf{Z}$ for circles S^1. For example, when $r_2 = 3$, $r_1 = 2$ and $m = 7$ a $\mathcal{B}_2$-diagram $S^1 \xleftarrow{\pi_1} S^1 \times S^1 \xrightarrow{\pi_2} S^1$ can be described by Figure 1, where the $\mathbf{Z}/7$-action on torus $S^1 \times S^1$ is encoded with vector $(2, 3)$.



If we pass to orbits, i.e., to the diagram $\operatorname{colim}_{\mathbf{Z}/7} Q$, we have to chose a new fundamental domain for the quotient torus. This can be done in many ways, for example $\{(2,3),(3,1)\}$ and $\{(2,3),(7,7)\}$ are two fundamental domains of the torus $S^1 \times S^1/\mathbf{Z}/7$. Now we can read off the maps $\pi_1/(\mathbf{Z}/7)$ and $\pi_2/(\mathbf{Z}/7)$ in both cases:

$$L(2,3;7) \approx \operatorname{hocolim}(S^1 \xleftarrow{(3,1)} S^1 \times S^1 \xrightarrow{(2,3)} S^1),$$
$$L(2,3;7) \approx \operatorname{hocolim}(S^1 \xleftarrow{(3,4)} S^1 \times S^1 \xrightarrow{(2,5)} S^1).$$

A question arises how one can conclude that this two different diagrams correspond to the same lens space?

2.4. Here the maps $\pi_1/(\mathbf{Z}/m)$ and $\pi_2/(\mathbf{Z}/m)$ and identified with a 1×2 matrix because $[S^1 \times S^1, K(\mathbf{Z}, 1)] \cong \operatorname{Hom}(\mathbf{Z}^2, \mathbf{Z}^1)$ and $K(\mathbf{Z}, 1) = S^1$. This means that every diagram $S^1 \xleftarrow{(a,b)} S^1 \times S^1 \xrightarrow{(c,d)} S^1$ is completely determined by the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. A natural question arises: *When two different matrices determine isomorphic diagrams?*

LEMMA 2.2. *Let* $S^1 \xleftarrow{(a,b)} S^1 \times S^1 \xrightarrow{(c,d)} S^1$, $S^1 \xleftarrow{(\alpha,\beta)} S^1 \times S^1 \xrightarrow{(\gamma,\delta)} S^1$ *be diagrams,* $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ *and* $\det A \neq 0$, $\det B \neq 0$. *Then*

$$\operatorname{hocolim}(S^1 \xleftarrow{(a,b)} S^1 \times S^1 \xrightarrow{(c,d)} S^1) \approx \operatorname{hocolim}(S^1 \xleftarrow{(\alpha,\beta)} S^1 \times S^1 \xrightarrow{(\gamma,\delta)} S^1)$$

if and only if there exists an integer matrix P such that $A = BP$ and $\det P = \pm 1$.

PROOF. There are two ways to approach the proof of lemma. If we use a geometric model, then the result follows from [**9**], and the matrix P is the base-changing matrix from A to B, where A and B are interpreted as bases of $\mathbf{R}^2$. The second way is to assume the existence of a diagram isomorphism

$$\begin{array}{ccc} S^1 & \xleftarrow{(a,b)} S^1 \times S^1 \xrightarrow{(c,d)} & S^1 \\ \downarrow^f & \downarrow^g & \downarrow^h \\ S^1 & \xleftarrow{(\alpha,\beta)} S^1 \times S^1 \xrightarrow{(\gamma,\delta)} & S^1 \end{array}$$

from where $f = \pm id$, $h = \pm id$ and $H^1(g)$ is the matrix P. Again, we use the main property of Eilenberg–MacLane spaces. □

3. Recognition Theorem

Here we give a detailed description of the diagram $\operatorname{colim}_{\mathbf{Z}/m} Q$ over $\mathcal{B}_n$ for every $n \in \mathbf{N}$. Also, we calculate the homotopy colimit of a general class of quasi-join diagrams $X : \mathcal{B}_n \to \mathrm{Top}$ where $X_a = (S^1)^{\operatorname{card} a}$, for every $a \subseteq \{1, \dots, n\}$.

3.1. Let $r_1, \dots, r_n \in \{1, \dots, m-1\}$, $(r_i, m) = 1$ and $(r_1, \dots, r_n) = 1$. Our objective is to describe the diagram $\operatorname{colim}_{\mathbf{Z}/m} Q$ associated with the lens space $L(r_1, \dots, r_n; m)$. Again, like in the previous example, we use $\mathbf{R}^n/(m\mathbf{Z})^n$ as a model of a torus and the translation by the vector $(r_1, \dots, r_n)$ for $\mathbf{Z}/m$-action. Thus, to identify the quotient torus $(\mathbf{R}^n/(m\mathbf{Z})^n)/\mathbf{Z}/m \cong \mathbf{R}^n/((m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n))$ we actually have to find a basis for the free Abelian group $(m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n)$. Since $(m\mathbf{Z})^n \subseteq (m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n) \subseteq \mathbf{Z}^n$, the rank of group $(m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n)$ is n.

LEMMA 3.1. *Let* $(r_1, \dots, r_n) \in \mathbf{Z}^n$, $m \in \mathbf{N}$, $r_i \in \{1, \dots, m-1\}$, $(r_i, m) = 1$ *and* $(r_1, \dots, r_n) = 1$. *If* $\{f_1, \dots, f_n\}$ *is a basis of the group* $(m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n)$, *then* $\det[f_1, \dots, f_n] = m^{n-1}$.

PROOF. The lemma follows from the fact that the $\mathbf{Z}/m$-action is free,

$$(\mathbf{R}^n/(m\mathbf{Z})^n)/\mathbf{Z}/m \cong \mathbf{R}^n/((m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n)),$$

and the vectors $f_1, \dots, f_n$ form a fundamental domain for the quotient torus. □

LEMMA 3.2. *Let* $(r_1, \dots, r_n) \in \mathbf{Z}^n$, $m \in \mathbf{N}$, $r_i \in \{1, \dots, m-1\}$, $(r_i, m) = 1$ *and* $(r_1, \dots, r_n) = 1$. *There exist a basis* $\{f_1, \dots, f_n\}$ *of the group* $\mathbf{Z}^n$ *such that* $\{f_1, m f_2, \dots, m f_n\}$ *is a basis of the group* $(m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n)$ *and every coordinate of a vector* $f_1 \in \mathbf{Z}^n$ *is relatively prime to* m.

PROOF. The group $(m\mathbf{Z})^n + \mathbf{Z}\cdot(r_1,\dots,r_n)$ is a subgroup of the free group $\mathbf{Z}^n$, so there exist a basis $\{f_1,\dots,f_n\}$ of $\mathbf{Z}^n$ and integers $a_1,\dots,a_n$ such that $\{a_1f_1,\dots,a_nf_n\}$ is a basis of $(m\mathbf{Z})^n + \mathbf{Z}\cdot(r_1,\dots,r_n)$ and $a_1 \mid \cdots \mid a_n$. Now, from Lemma 3.1 and the fact that $(m\mathbf{Z})^n$ is a subgroup of $(m\mathbf{Z})^n + \mathbf{Z}\cdot(r_1,\dots,r_n)$, we conclude that

$$a_1\cdot\dots\cdot a_n = m^{n-1} \text{ and } a_1 \mid m,\dots,a_n \mid m.$$

Since, $(r_1,\dots,r_n) \in (m\mathbf{Z})^n + \mathbf{Z}\cdot(r_1,\dots,r_n)$, there exist integers $\beta_1,\dots,\beta_n \in \mathbf{Z}$ with the property $(r_1,\dots,r_n) = \beta_1a_1f_1 + \dots + \beta_na_nf_n$. From $(r_i,m)=1$, $a_i \mid m$ for every $i \in \{1,\dots,n\}$ and $a_1\cdot\dots\cdot a_n = m^{n-1}$ we conclude that $a_1 = 1, a_2 = m,\dots,a_m = m$ and $r_1 = \beta_1f_{11} \pmod m,\dots,r_1 = \beta_1f_{1n} \pmod m$ where $f_1 = (f_{11},\dots,f_{1n})$. We observe that $(f_{11},m)=1,\dots,(f_{1n},m)=1$.

Note that the sequences $(r_1,\dots,r_n)$ and $(f_{11},\dots,f_{1n})$ satisfy the conditions of Proposition 2.1(iv). $\square$

3.2. Let $X : \mathcal{B}_n \to \mathrm{Top}$ be a diagram such that $X_a = (S^1)^{\operatorname{card} a}$ for every $a \subseteq \{1,\dots,n\}$. If $a \subseteq b \subseteq \{1,\dots,n\}$, then the map $X_{b\supseteq a} : (S^1)^{\operatorname{card} b} \longrightarrow (S^1)^{\operatorname{card} a}$ is, up to a homotopy, completely determined by its representation on $H^n((S^1)^{\operatorname{card} b},\mathbf{Z}^{\operatorname{card} a})$, because

$$\begin{aligned}[(S^1)^{\operatorname{card} b},(S^1)^{\operatorname{card} a}] &\cong [(S^1)^{\operatorname{card} b},K(\mathbf{Z},1)^{\operatorname{card} a}] \cong [(S^1)^{\operatorname{card} b},K(\mathbf{Z}^{\operatorname{card} a},1)]\\ &\cong H^1((S^1)^{\operatorname{card} b},\mathbf{Z}^{\operatorname{card} a}) \cong \operatorname{Hom}(H_1((S^1)^{\operatorname{card} b}),\mathbf{Z}^{\operatorname{card} a})\\ &\cong \operatorname{Hom}(\mathbf{Z}^{\operatorname{card} b},\mathbf{Z}^{\operatorname{card} a}).\end{aligned}$$

So, every map $X_{b\supseteq a} : (S^1)^{\operatorname{card} b} \longrightarrow (S^1)^{\operatorname{card} a}$ can be represented as $\operatorname{card} b\times\operatorname{card} a$ matrix $M_{b;a}(X)$. From naturality, if $a \subseteq b \subseteq c \subseteq \{1,\dots,n\}$, then $M_{c;b}(X)\cdot M_{b;a}(X) = M_{c;a}(X)$. This means that the diagram $X : \mathcal{B}_n \to \mathrm{Top}$ is determined by the diagram $H_1(X) : \mathcal{B}_n \to \mathrm{Ab}$.

DEFINITION 3.1. Let $a = \{i_1,\dots,i_{\operatorname{card} a}\} \subseteq \{1,\dots,n\}$ and $i_1 < \dots < i_{\operatorname{card} a}$. Then we define a $\operatorname{card} a\times\operatorname{card} a$ matrix $M(X)_a$ with $M_{a;\{i_1\}}(X),..,M_{a;\{i_{\operatorname{card} a}\}}(X)$ as its rows. The $n\times n$ $\mathbf{Z}$-matrix $M(X) := M(X)_{\{1,\dots,n\}}$ is called the *matrix of the diagram* X.

LEMMA 3.3. *Let* $X : \mathcal{B}_n \to \mathrm{Top}$, $Y : \mathcal{B}_n \to \mathrm{Top}$ *be diagrams such that* $X_a = Y_a = (S^1)^{\operatorname{card} a}$, $\det M(X) \neq 0$, $\det M(Y) \neq 0$ *and* $X_{a\supseteq b}$, $Y_{a\supseteq b}$ *are surjections for every* $a,b \in \mathcal{B}_n$. *Then there is an integer matrix* A *such that* $M(X) = M(Y)\cdot A$ *and* $\det A = \pm1$ *if and only if there is a morphism of diagrams* X *and* Y *which is a homotopy equivalence on every level.*

PROOF. $\Rightarrow$: First, we prove that (under the assumptions of the lemma) for every $a \in \mathcal{B}_n$ the matrices $M(X)_a$ (and $M(Y)_a$) are invertible as matrices over $\mathbf{Q}$, i.e., $\det M(X)_a \neq 0$ (and $\det M(Y)_a \neq 0$). It is sufficient to prove this in the case $a = \{1,\dots,n-1\}$. Let $f = M_{\{1,\dots,n\};\{1,\dots,n-1\}}(X)$, $g = M_{\{1,\dots,n\};\{n\}}(X)$, $N = M(X)_{\{1,\dots,n-1\}}$ and $M = M(X)_{\{1,\dots,n\}}$. Since f and g are surjections, then

$\ker f \cong \mathbf{Z}$ and $\ker g \cong \mathbf{Z}^{n-1}$. Furthermore, $\ker f \cap \ker g = \{0\}$, since

$$x \in \ker f \cap \ker g \Rightarrow N \circ f(x) = 0, \quad g(x) = 0 \Rightarrow M(x) = 0 \Rightarrow x = 0.$$

So, $\mathbf{Z}^n = \ker f \oplus \ker g$ and the associated restrictions $\bar{f} = f|_{\ker g}$ and $\bar{g} = g|_{\ker f}$ are isomorphisms. Now we can factor M in the following way

$$\mathbf{Z}^n = \ker f \oplus \ker g \xrightarrow{\bar{g}\oplus\bar{f}} \mathbf{Z} \oplus \mathbf{Z}^{n-1} \xrightarrow{id\oplus N} \mathbf{Z} \oplus \mathbf{Z}^{n-1}$$

i.e., $M = (id \oplus N) \circ (\bar{g} \oplus \bar{f})$. Thus, $\det N \neq 0$

Secondly, let us observe that every map $\Gamma_a : X_a \to Y_a$ is completely determined, up to a homotopy, by the $\operatorname{card} a \times \operatorname{card} a$ matrix $H_1(\Gamma_a) : H_1(X_a, \mathbf{Z}) \to H_1(Y_a, \mathbf{Z})$. If Γ_a is a homotopy equivalence, then $H_1(\Gamma_a)$ has to be an isomorphism. Now we will define an isomorphism $H_1(\Gamma)$ of diagrams $H_1(X, \mathbf{Z})$ and $H_1(Y_a, \mathbf{Z})$.

Let $H_1(\Gamma_{\{1,\dots,n\}}) : H_1(X_{\{1,\dots,n\}}, \mathbf{Z}) \to H_1(Y_{\{1,\dots,n\}}, \mathbf{Z})$ be the matrix A and, without losing any generality, we assume that $H_1(\Gamma_{\{i\}}) : H_1(X_{\{i\}}, \mathbf{Z}) \to H_1(Y_{\{i\}}, \mathbf{Z})$ are the identity maps. Now let us define $H_1(\Gamma_a)$, for every $a \subseteq \{1, \dots, n\}$. Actually we have to complete the following diagram

$$\begin{array}{ccc} \mathbf{Z}^n \cong H_1(X_{\{1,\dots,n\}}, \mathbf{Z}) & \xrightarrow{A} & H_1(Y_{\{1,\dots,n\}}, \mathbf{Z}) \cong \mathbf{Z}^n \\ \downarrow f & & \downarrow g \\ \mathbf{Z}^k \cong H_1(X_a, \mathbf{Z}) & \overset{?}{\dashrightarrow} & H_1(Y_a, \mathbf{Z}) \cong \mathbf{Z}^k \end{array}$$

where $\operatorname{card} a = k$, $f = M_{\{1,\dots,n\};a}(X)$ and $g = M_{\{1,\dots,n\};a}(Y)$. This can be done uniquely if and only if f is a surjection (which is one of the assumptions) and $A(\ker f) \subseteq \ker g$. Let $y \in H_1(X_{\{1,\dots,n\}}, \mathbf{Z})$. Then under the assumption of the lemma

$$M(Y)_a \circ g \circ A(y) = M(X)_a \circ f(y) = 0 \overset{\det M(Y)_a \neq 0}{\Longrightarrow} g \circ A(y) = 0 \Rightarrow y \in A(\ker f).$$

So, if $x \in H_1(X_a, \mathbf{Z})$, in order to find $H_1(\Gamma_a)(x)$ we need $y \in H_1(X_{\{1,\dots,n\}}, \mathbf{Z})$ with the property $M_{\{1,\dots,n\};a}(X)(y) = x$, and then we define

$$H_1(\Gamma_a)(x) := M_{\{1,\dots,n\};a}(Y)(A(y)).$$

It only remains to be proved that maps $H_1(\Gamma_a)$ form a morphism of diagrams $H_1(X, \mathbf{Z})$ and $H_1(Y_a, \mathbf{Z})$. For $a \subseteq b \subseteq \{1, \dots, n\}$, $\operatorname{card} a = k$, $\operatorname{card} b = l$, we can prove $M_{b;a}(Y) \circ H_1(\Gamma_b) = H_1(\Gamma_a) \circ M_{b;a}(X)$ by chasing through diagrams. Let $x \in H_1(X_b, \mathbf{Z})$ and $y \in H_1(X_{\{1,\dots,n\}}, \mathbf{Z})$, such that $M_{\{1,\dots,n\};a}(X)(y) = x$. Then,

$$\begin{aligned} M_{b;a}(Y) \circ H_1(\Gamma_b)(x) &= M_{b;a}(Y) \circ M_{\{1,\dots,n\};b}(Y)(A(y)) \\ &= M_{\{1,\dots,n\};a}(Y)(A(y)) \\ &= H_1(\Gamma_a) \circ M_{\{1,\dots,n\};a}(X)(y) \\ &= H_1(\Gamma_a) \circ M_{b;a}(X) \circ M_{\{1,\dots,n\};b}(X)(y) \\ &= H_1(\Gamma_a) \circ M_{b;a}(X)(x). \end{aligned}$$

$\Leftarrow$: If Γ is a morphism of diagrams X and Y which is a homotopy equivalence on every level, then $H_1(\Gamma_{\{1,\dots,n\}})$ is the matrix A. □

Remark 3.1. Let us observe that the proof of the preceding lemma is actually a proof of the following more general result:

Let $X : \mathcal{B}_n \to \mathrm{Top}$, $Y : \mathcal{B}_n \to \mathrm{Top}$ be diagrams such that $X_a = Y_a = (S^1)^{\mathrm{card}\, a}$, $\det M(X) \neq 0$, $\det M(Y) \neq 0$ and $X_{a\supseteq b}$, $Y_{a\supseteq b}$ are surjections for every $a, b \in \mathcal{B}_n$. Then for every map $X_{\{1,\dots,n\}} \xrightarrow{g} Y_{\{1,\dots,n\}}$ such that $M(X) = M(Y) \cdot H_1(g, \mathbf{Z})$ there is a diagram morphism $\Gamma : X \to Y$ with property $\Gamma_a = g$.

Corollary 3.1. *Let $X : \mathcal{B}_n \to \mathrm{Top}$, $Y : \mathcal{B}_n \to \mathrm{Top}$ be diagrams such that $X_a = Y_a = (S^1)^{\mathrm{card}\, a}$, $\det M(X) \neq 0$, $\det M(Y) \neq 0$ and $X_{a\supseteq b}$, $Y_{a\supseteq b}$ are surjections for every $a, b \in \mathcal{B}_n$. If there exists a matrix $A \in GL(n, \mathbf{Q})$ such that $M(X) = M(Y) \cdot A$, then there exists a sequence of isomorphisms of diagrams $\Gamma_n : H_n(X, \mathbf{Q}) \to H_n(Y, \mathbf{Q})$ for every n, such that $(\Gamma_1)_{\{1,\dots,n\}} = A$.*

Proof. Let $f : (S^1)^{k+m} \to (S^1)^m$. Then all the maps $H_n(f, \mathbf{Q})$, for $n > 1$, are completely determined by the map $H_1(f, \mathbf{Q})$. If we know $H_1(f, \mathbf{Q})$, then we know $H^1(f, \mathbf{Q})$ because $H^j((S^1)^l, \mathbf{Q}) \cong \mathrm{Hom}(H_j((S^1)^l, \mathbf{Z}), \mathbf{Q})$ for every $j \in \mathbf{N}$ and $l \in \mathbf{N}$. Since, the cohomology ring $H^*((S^1)^l, \mathbf{Q})$ is generated as a ring by the 1-dimensional classes, $H^1(f, \mathbf{Q})$ determines $H^n(f, \mathbf{Q})$ and consequently $H_n(f, \mathbf{Q})$, for every $n \in \mathbf{N}$.

So, we only have to construct an isomorphism $H\Gamma_1$. This construction is completely identical with the construction in the preceding theorem, starting with $(\Gamma_1)_{\{1,\dots,n\}} = A$. $\square$

3.3. Finally, we describe the diagram $\mathrm{colim}_{\mathbf{Z}/m}\, Q$ by detecting its matrix $M(\mathrm{colim}_{\mathbf{Z}/m}\, Q)$. Let $R_1 = (r_{11}, \dots, r_{1n}), \dots, R_n = (r_{n1}, \dots, r_{nn})$ be a fundamental domain of the quotient torus $(\mathbf{R}^n/m\mathbf{Z}^n)/(\mathbf{Z}/m)$, i.e., a basis of the free Abelian group $(m\mathbf{Z})^n + \mathbf{Z} \cdot (r_1, \dots, r_n)$. Then,

$$M(\mathrm{colim}_{\mathbf{Z}/m}\, Q) = \begin{bmatrix} f_{11} & mf_{21} \dots & mf_{n1} \\ \dots & \dots & \dots \\ f_{1n} & mf_{2n} \dots & mf_{nn} \end{bmatrix}.$$

where $f_1 = (f_{11}, \dots, f_{1n}), \dots, f_n = (f_{n1}, \dots, f_{nn})$ is the basis from Lemma 3.2. Note, that the diagram $\mathrm{colim}_{\mathbf{Z}/m}\, Q$ satisfies the conditions of Lemma 3.3. Thus, the following theorem is true.

Theorem 3.1 (Recognition Theorem). *Let $X : \mathcal{B}_n \to \mathrm{Top}$ be a diagram such that $X_a = (S^1)^{\mathrm{card}\, a}$, $X_{a\supseteq b}$ is a surjection for every $a, b \subseteq \{1, \dots, n\}$, $\det M(X) \neq 0$ and $g_1, \dots, g_n$ are the rows of the matrix $M(X)$. Let $f_1, \dots, f_n$ be a basis of $\mathbf{Z}^n$ and $a_1, \dots, a_n$ integers such that $\{a_1 f_1, \dots, a_n f_n\}$ is a basis of the group $\langle g_1, \dots, g_n\rangle$ where $a_1 \mid \cdots \mid a_n$.*

(i) If $|\det M(X)| = |a_1 \cdots a_n| = 1$, then $\mathrm{hocolim}\, X \simeq S^{2n+1}$.

(ii) If there exists $m \in \mathbf{Z}$ such that $|\det M(X)| = |a_1 \cdots a_n| = m^{n-1}$ and every coordinate of the vector $f_1 \in \mathbf{Z}^n$ is relatively prime to m, then $\mathrm{hocolim}\, X \simeq L(f_{11}, \dots, f_{1n}; m)$*, where $f_1 = (f_{11}, \dots, f_{1n})$.*

(iii) $\mathrm{hocolim}\, X$ *is always a $\mathbf{Q}$-sphere.*

Proof. (i) and (ii) are consequences of the $\mathrm{colim}_{\mathbf{Z}/m}\, Q$ matrix description.

(iii) Since there exists $A \in GL(n, \mathbf{Q})$ with property $E = M(X) \cdot A$ and E is a matrix of join-diagram $J : \mathcal{B}_n \to \mathrm{Top}$, then (from Corollary 3.1) there exists a sequence of isomorphisms of diagrams $\Gamma_n : H_n(J, \mathbf{Q}) \to H_n(Y, \mathbf{Q})$, for every $n \in \mathbf{N}$ such that $(\Gamma_1)_{\{1,\dots,n\}} = A$. Now, the statement follows from Proposition 3.5 of [**14**]:

Let $X : P \to \mathrm{Top}$ be a diagram. Then there exists a spectral sequence abutting on $\tilde{H}_*(\mathrm{hocolim}\, X)$ with E^2-term described by $E^2_{m,n} \cong \tilde{H}_m(\mathcal{H}_n(X))$. Here $H_n(X)$ is a composition of X and $H_n : \mathrm{Top} \to \mathrm{Ab}$, $U \mapsto H_n(U)$ functor and $E^2_{m,n}$ is the m^{th} homology with coefficients in this diagram. $\square$

COROLLARY 3.2. *Let $X : \mathcal{B}_2 \to \mathrm{Top}$ be a diagram such that $X_a = (S^1)^{\mathrm{card}\, a}$, $X_{a \supseteq b}$ is surjection for every $a, b \subseteq \{1, 2\}$, $\det M(X) \neq 0$. Then* $\mathrm{hocolim}\, X$ *is homeomorphic to a lens space.*

REMARK 3.2. The remaining problem is to analyze homotopy colimits of quasi-join diagrams when $X_1 = \cdots = X_n = K(G, n)$ for some Abelian group G and $n \geq 1$.

EXERCISE 3.1. Describe the homotopy colimit of quasi-join diagrams when $X_1 = \cdots = X_n = K(G, 0)$ for some group G.

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