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BEST λ-APPROXIMATION FOR ENTIRE FUNCTIONS OF FINITE ORDER

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Abstract. We investigate so-called BLAS problem for entire functions whose logarithm of maximum moduli is regularly varying in the sense of Karamata or de Haan . We also give an interesting application on Hadamard-type convolutions with regularly varying sequences of arbitrary index.

Introduction

For a given entire function $f(z) := \sum_{k=0}^{\infty} a_k z^k$ we define, as usual, its partial sums $S_n(z) := \sum_{k\le n} a_k z^k$ and maximum moduli $M_f(r) := \max |f(z)|_{|z|=r} = |f(re^{i\phi_0})| = |f(z_0)|$. The order ρ of $f(z)$ is $\rho := \limsup_{r\to\infty} \log\log M_f(r)/\log r$.

In [5], we gave a notion of best λ-approximating (BLAS) partial sums for functions analytic on the unit disc. This can be easily reformulated for entire functions (analytic on the whole complex plane) as:

If there is an integer-valued function $n := n(r,\lambda) \to \infty$ $(r\to\infty)$ such that

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} o(1), & 0<\lambda<1; \\ 1+o(1), & \lambda>1; \end{cases} \qquad (r\to\infty) \tag{I}$$

we call $S_{n(r,\lambda)}(z_0)$ the best λ-approximating partial sum (BLAS).

In this way, we are going to find the "shortest" partial sum which is well approximating $f(z)$ at the point(s) of maximal growth, for r sufficiently large.

Note that analogous to (I) is the relation between moduli of BLAS and $M_f(r)$.

An important role in measuring the growth of entire functions of order $\rho > 0$ have the class R_ρ consisting of regularly varying functions in the sense of Karamata; i.e., $g(x) \in R_\rho$ can be represented in the form $g(x) = x^\rho l(x)$, $x>0$, $\rho\in R$, where ρ is the index of regular variation and $l(x)\in R_0$ is a slowly varying function i.e., positive, measurable and satisfying $l(tx)/l(x) \sim 1$, $\forall t>0$ $(x\to\infty)$.

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An immediate consequence which we are going to use in the sequel is

$$g(x) \in R_\rho \iff g(tx)/g(x) \sim t^\rho, \quad \forall t > 0. \quad (x \to \infty) \tag{0.1}$$

For further information on regular variation we recomend [1] and [4]. In order to study entire functions of order zero we shall consider a subclass of R_0 i.e. de Haan's class Π_l,

$$h(x) \in \Pi_l \iff \frac{h(tx) - h(x)}{l(x)} \sim \log t, \quad \forall t > 0; \quad (x \to \infty) \tag{0.2}$$

where $l(x) \in R_0$ is called the auxiliary function and we can take $h(x) = l(x) + \int_1^x l(t)/t dt$ [1, pp. 160–165].

We are going to apply our BLAS results to entire functions with non-negative coefficients i.e., to determine the asymptotic behavior of Hadamard-type convolutions $T_f(r) := \sum n^\alpha l_n a_n r^n$, where (l_n) are slowly varying sequences; therefore improving our results from [6].

Results

Let $f(z)$, $M_f(r)$, $n(r,\lambda)$, ρ, z_0 be defined as above. Then we have the following

Theorem 1. *If* $\log M_f(r) \in R_\rho$, $\rho > 0$, *and*

$$n(r,\lambda) \sim \lambda\rho \log M_f(r). \qquad (r \to \infty) \tag{1}$$

Then

$$\frac{S_{n(r,\lambda)}(z_0)}{(z_0)} = \begin{cases} \epsilon_1(r,\lambda), & 0 < \lambda < 1; \\ 1 + \epsilon_2(r,\lambda), & \lambda > 1, \end{cases}$$

with

$$|\epsilon_i(r,\lambda)| \le \frac{1}{|\lambda^{1/\rho} - 1|} M_f(r)^{-(\lambda \log \lambda - \lambda + 1 + o(1))}, \; i = 1,2 \qquad (r \to \infty).$$

Proof. An implementation of Cauchy's Integral formula gives

$$\frac{1}{2\pi i} \int_C f(w) \frac{(z_0/w)^{n+1}}{w - z_0} dw = \begin{cases} -S_n(z_0), & z_0 \notin \operatorname{int} C, \\ f(z_0) - S_n(z_0), & z_0 \in \operatorname{int} C. \end{cases} \tag{2}$$

Let the contour C be a circle $w = r\lambda^{1/\rho} e^{i\phi}$. Since

$$|z_0| = r \begin{cases} > |w|, & 0 < \lambda < 1, \\ < |w|, & \lambda > 1; \end{cases}$$

from (2) we get

$$I := \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r\lambda^{1/\rho} e^{i\phi})}{f(re^{i\phi_0})} \frac{\lambda^{-n/\rho} e^{in(\phi_0 - \phi)}}{\lambda^{1/\rho} e^{i(\phi - \phi_0)} - 1} d\phi = \begin{cases} -\frac{S_n(z_0)}{f(z_0)}, & 0 < \lambda < 1, \\ 1 - \frac{S_n(z_0)}{f(z_0)}, & \lambda > 1. \end{cases} \tag{3}$$

Taking into account that $|f(z_0)| = M_f(r)$, a simple estimation of I gives

$$|I| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(r\lambda^{1/\rho}e^{i\phi})|}{|f(z_0)|}\frac{e^{-\frac{n}{\rho}\log\lambda}}{|\lambda^{1/\rho}e^{i(\phi-\phi_0)}-1|}\,d\phi$$

i.e.,

$$|I| \le \frac{M_f(r\lambda^{1/\rho})}{M_f(r)}\frac{e^{-\frac{n}{\rho}\log\lambda}}{|\lambda^{1/\rho}-1|} \tag{4}$$

Since, for sufficiently large r (cf. (0.1)),

$$\log M_f(r) \in R_\rho \Rightarrow \frac{\log M_f(r\lambda^{1/\rho})}{\log M_f(r)} = \lambda(1+o(1)),$$

putting in (4), $n = n(r,\lambda) = \lambda\rho\log M_f(r)(1+o(1))$, we finally obtain

$$|I| \le \frac{1}{|\lambda^{1/\rho}-1|}\exp(-\log M_f(r)(\lambda\log\lambda-\lambda+1+o(1))) \qquad (r\to\infty) \tag{5}$$

For $\lambda > 0$, $\lambda \ne 1$, $\lambda \mapsto \lambda\log\lambda - \lambda + 1$ is strictly positive, hence the assertion of Theorem 1 follows.

In the case of entire functions of order zero, we shall treat the subclass whose logarithm of the maximum modulus belongs to de Haan's class Π_l with unbounded auxilary function $l \in R_0$.

Taking in (2) the contour $C : |w| = \lambda r$; $\lambda > 0$, $\lambda \ne 1$, the estimation (4) can be rewritten as

$$|I| \le \frac{1}{|\lambda-1|}\exp\Big(l(r)\frac{\log M_f(\lambda r)-\log M_f(r)}{l(r)} - n\log\lambda\Big) \tag{6}$$

Putting there $n = n(r,\lambda) = \lambda l(r)(1+o(1))$ $(r\to\infty)$ and taking into account the definition (0.2), (6) yields

$$|I| \le \frac{1}{|\lambda-1|}\exp\big(-l(r)((\lambda-1)\log\lambda+o(1))\big) \qquad (r\to\infty).$$

Since $\lambda \mapsto (\lambda-1)\log\lambda$ is strictly positive for $\lambda > 0$, $\lambda \ne 1$, we obtain

THEOREM 2. *If $\log M_f(r) \in \Pi_l$ with auxilary function $R_0 \ni l(r) \to \infty$ $(r \to \infty)$, and $n(r,\lambda) \sim \lambda l(r)$ $(r\to\infty)$, then*

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} \mu_1(r,\lambda), & 0<\lambda<1 \\ 1+\mu_2(r,\lambda), & \lambda>1 \end{cases}$$

with

$$|\mu_i| \le \frac{1}{|\lambda-1|}e^{-l(r)((\lambda-1)\log\lambda+o(1))}, \quad i=1,2; \quad (r\to\infty).$$

It is easy now to derive, from the Theorems above, various estimation formulae for the moduli of BLAS. We need the following in the sequel:

PROPOSITION 1. *Under the conditions of the Theorem 1, for any $\sigma > 1$, $\rho > 0$ and $n_1(r,\sigma) \sim e\sigma\rho \log M_f(r)$ $(r \to \infty)$,*

$$\Big| \sum_{n > n_1(r,\sigma)} a_n z_0^n \Big| < C M_f(r)^{-e\sigma \log \sigma + o(1)}.$$

Proof. Applying Theorem 1 with $\lambda > 1$, we get

$$\sum_{n > n(r,\lambda)} a_n z_0^n = f(z_0) - S_{n(r,\lambda)}(z_0) = -f(z_0)\epsilon_2(r,\lambda),$$

i.e., since $|f(z_0)| = M_f(r)$,

$$\Big| \sum_{n > n(r,\lambda)} a_n z_0^n \Big| = M_f(r)|\epsilon_2(r,\lambda)| \le \frac{1}{\lambda^{1/\rho} - 1} M_f(r)^{\lambda(\log \lambda - 1) + o(1)}.$$

Putting there $\lambda = e\sigma$, $n(r,\lambda) = n_1(r,\sigma)$ we obtain the proof with $C = C(\rho,\sigma) = 1/(e\sigma)^{1/\rho - 1}$.

Now, we give some applications of our BLAS results. For a given entire function $f(r) := \sum_n a_n r^n$ with non-negative coefficients, there is a classical problem of estimating asymptotic behavior of Hadamard-type convolutions $T_f(r) := \sum_n c_n a_n r^n$ $(r \to \infty)$.

In the well-known book [3, pp. 20, 197, 198) this is solved in the case

$$c_n := n^\alpha, \ \alpha \in R; \quad \log f(r) \sim a r^\rho, \ \ a, \rho > 0 \quad (r \to \infty).$$

In [6] we obtain a result for regularly varying $c_n := n^\alpha l_n$, $\alpha \in R$, $c_0 := 1$ and $\log f(r) \in SR_\rho$, $\rho > 0$.

Here l_n are slowly varying sequences [2], for example:

$$\log^a 2n, \quad \log^b(\log 3n), \quad \exp\Big(\frac{\log n}{\log\log 3n}\Big), \quad \exp(\log^c 2n); \qquad a, b \in R; \ 0 < c < 1;$$

and $SR_\rho \subset R_\rho$ is the class of smoothly varying functions [1, pp. 44–47].

Using Theorem 1 and Lemmas 1 and 2 below, we are going to prove the next:

THEOREM 3. *Let an entire function $f(r) := \sum_n a_n r^n$, $a_n \ge 0$, of order $\rho > 0$, satisfy $\log f(r) \in R_\rho$. Then*

$$T_f(r) := \sum_n c_n a_n r^n \sim \rho^\alpha \ c_{[\log f(r)]} \ f(r) \qquad (r \to \infty),$$

for any regularly varying sequence (c_n) of arbitrary index $\alpha \in R$.

For the proof we need two lemmas.

Lemma 1. *Define*

$$S(\lambda, r) := \sum_{n \le \lambda n(r)} a_n r^n, \qquad a_n \ge 0, \quad n \in N,$$

where $n(r)$ increases to infinity with r, and an operator T acting on S:

$$TS(\lambda, r) := \sum_{n \le \lambda n(r)} c_n a_n r^n, \qquad n \in N,$$

where $(c_n)_{n \in N}$ is a regularly varying sequence of index $\alpha \in R$.

If there exist $g, g_1, g_2 : R^+ \to R^+$; $b_1 : (0,1) \to R^+$; $b_2 : (1, \infty) \to R^+$, and

$$\lim_{r \to \infty} \frac{\log n(r)}{g_i(r)} = 0, \qquad i = 1, 2;$$

such that

$$\frac{S(\lambda, r)}{g(r)} = \begin{cases} O(e^{-b_1(\lambda) g_1(r)}), & 0 < \lambda < 1, \\ A + O(e^{-b_2(\lambda) g_2(r)}), & \lambda > 1, \end{cases} \quad A \in R^+, \quad (r \to \infty),$$

then

$$\frac{TS(\lambda, r)}{g(r)} = \begin{cases} o(c_{[n(r)]}), & 0 < \lambda < 1, \\ c_{[n(r)]}(A + o(1)), & \lambda > 1; \end{cases} \qquad (r \to \infty).$$

In this form Lemma 1 is proved in [7] as the Theorem A.

Lemma 2. *For any regularly varying sequence (c_n) of index $\alpha \in R$,*

$$c_{[\lambda x]} \sim c_{[\lambda [x]]} \sim \lambda^\alpha c_{[x]} \qquad (x \to \infty).$$

This is a well-known fact [1, pp. 49–53].

Now, we are able to prove cited Theorem 3.

First of all, note that the condition $a_n \ge 0$ implies that on the circle $|z| = r$ we have

$$|f(z)| = \Big| \sum_n a_n z^n \Big| \le \sum_n a_n |z|^n = \sum_n a_n r^n = f(r).$$

Hence, $M_f(r) = f(r)$, $z_0 = r$ and, comparing the assertions from Theorem 1 and Lemma 1, we see that the conditions of Lemma 1 are satisfied with

$$g(r) := f(r); \; n(r) := \rho \log f(r); \; g_1(r) = g_2(r) := \log f(r); \; A := 1.$$

Write, in terms of Theorem 1,

$$T_f(r) := \sum_n c_n a_n r^n = \sum_{n \le n(r, 2e2^\rho)} c_n a_n r^n + \sum_{n > n(r, 2e2^\rho)} c_n a_n r^n = S_1 + S_2.$$

Applying Lemma 1 with $\lambda := 2e2^\rho > 1$ and Lemma 2, we obtain

$$S_1 \sim c_{[\rho \log f(r)]} f(r) \sim \rho^\alpha c_{[\log f(r)]} f(r), \quad \alpha \in R \quad (r \to \infty). \tag{3.1}$$

For estimating S_2, note that (0.1) implies $2^\rho \log f(r) \sim \log f(2r)$, $(r \to \infty)$, i.e., using Proposition 1 with $n(r, 2e2^\rho) = n_1(2r, 2)$, we get

$$\begin{aligned} S_2 &\leq \sup_n (2^{-n} c_n) \sum_{n > n(r, 2e2^\rho)} a_n (2r)^n = O(1) \sum_{n > n_1(2r,2)} a_n (2r)^n \\ &= O(1) e^{-(2e \log 2 + o(1)) \log f(2r)} = o(c_{[\log f(r)]} f(r)) \quad (r \to \infty). \end{aligned}$$

This, along with (3.1) yields the proof of Theorem 3.

In the same manner, using Theorem 2 and Lemmas 1 and 2 we can prove

THEOREM 4. *Under the conditions of Theorem 2, for a given entire function f of order zero,*

$$f(r) := \sum_n a_n r^n, \quad a_n \geq 0, \ n \in N,$$

we have

$$T_f(r) := \sum_n c_n a_n r^n \sim c_{[l(r)]} f(r) \quad (r \to \infty),$$

for any regularly varying sequence (c_n) of arbitrary index.

Finally, we shall give two examples. To illustrate the results from Theorems 1 and 3, we shall consider the Mittag-Leffler function $E_s(z)$,

$$E_s(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+ns)}, \quad s > 0.$$

Then, for $z = re^{i\phi}$,

$$z_0 = r, \quad M_E(r) = E_s(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(1+ns)}, \quad s > 0,$$

and (cf. [1, p. 329]),

$$E_s(r) \sim (1/s) e^{r^{1/s}}; \quad \log E_s(r) \sim r^{1/s} \quad (r \to \infty).$$

Hence $E_s(z)$ is an entire function of order $1/s$ and Theorem 1 gives:

PROPOSITION 2. *For the Mittag-Leffler function $E_s(z)$,*

$$n(r, \lambda) \sim (\lambda/s) r^{1/s}, \quad s > 0, \quad (r \to \infty)$$

and

$$S_{n(r,\lambda)}(r) := \sum_{n \leq n(r,\lambda)} \frac{r^n}{\Gamma(1+ns)} = o(E_s(r)) \quad for \ \ 0 < \lambda < 1;$$

$$S_{n(r,\lambda)}(r) \sim E_s(r) \quad for \quad \lambda > 1 \quad (r \to \infty).$$

Similarly, applying Theorem 3 and the properties of $E_s(r)$ mentioned above, we obtain

PROPOSITION 3. *For any slowly varying sequence (ℓ_n) and arbitrary $\alpha \in R$,*

$$T_E(r) := \sum_{n=1}^{\infty} \frac{n^\alpha \ell_n}{\Gamma(1+ns)} r^n \sim (1/s)^{\alpha+1} r^{\alpha/s} \ell(r^{1/s}) \exp(r^{1/s}) \qquad (r \to \infty).$$

For the next example we take the function $Q(z)$ of zero order,

$$Q(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{q^n}\right) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{(q-1)(q^2-1)\cdots(q^n-1)}, \quad q > 1.$$

(Euler, cf. [8, p. 32])

For $z = re^{i\phi}$, we have

$$z_0 = r; \qquad M_Q(r) = Q(r) = \prod_{n=1}^{\infty} (1 + r/q^n).$$

That $\log Q(r)$ belongs to de Haan's class Π_l follows from Hardy's result (cf. [9, p. 171]),

$$\log Q(r) = \frac{1}{2\log q}\left(\log r - \frac{1}{2}\log q\right)^2 + O(1) \qquad (r \to \infty).$$

Therefore, for $t > 0$,

$$\log Q(tr) - \log Q(r) \sim \frac{\log t}{\log q} \log r,$$

i.e.,

$$\frac{\log Q(tr) - \log Q(r)}{\log r / \log q} \to \log t, \quad \forall t > 0 \quad (r \to \infty)$$

According to (0.2), $\log Q(r) \in \Pi_l$ and we can take for the auxiliary function $l(r) = \frac{\log r}{\log q} \in R_0$.

Applying Theorem 2 we obtain

PROPOSITION 4. *For the function $Q(r)$ defined above,*

$$n(r,\lambda) \sim\sim (\frac{\lambda}{\log q} \sim) \log r \qquad (r \to \infty),$$

and

$$S_{n(r,\lambda)}(r) := 1 + \sum_{n \le n(r,\lambda)} \frac{r^n}{(q-1)(q^2-1)\cdots(q^n-1)} = o(Q(r)) \quad \textit{for} \ \ 0 < \lambda < 1;$$

$$S_{n(r,\lambda)} \sim Q(r) \quad \textit{for} \ \ \lambda > 1 \qquad (r \to \infty).$$

Theorem 4 also gives

PROPOSITION 5. *For any slowly varying sequence (ℓ_n), $n \in N$ and arbitrary real α we have (when $r \to \infty$).*

$$T_Q(r) := 1 + \sum_{n=1}^{\infty} \frac{n^\alpha \ell_n}{(q-1)(q^2-1)\cdots(q^n-1)} r^n \sim \frac{1}{\log^\alpha q} \log^\alpha r \ \ell(\log r) \ Q(r)$$

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