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ON DEGREE SEQUENCES OF GRAPHS WITH GIVEN CYCLOMATIC NUMBER

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Abstract. Starting with the Criterion by Gutman and Ruch for graphical partitions, Gutman analyzed degree sequences of connected graphs with cyclomatic number c, for $c \leq 5$. In this paper, his results are revisited and, based on the Erdös-Gallai Criterion, extended to arbitrary values of c. Necessary and sufficient conditions are obtained for any partition to be the degree sequence of a connected graph with cyclomatic number c.

1. Introduction

The first characterization of *graphical partitions*, that is, partitions that occur as degree sequences of simple graphs, was given by Erdös and Gallai [**EG60**]:

Theorem 1.1. *Let m, n be positive integers. A partition $p = (p_1, \dots, p_n)$ of $2m$ is graphical if and only if*

$$\text{(EG)} \qquad \sum_{\nu=1}^{k} p_\nu \leq k(k-1) + \sum_{\nu=k+1}^{n} \min\{p_\nu, k\}$$

for all $k \in \{1, \dots, n\}$.

Other characterizations of graphical partitions are due to Hakimi [**Hak62**] and Gutman and Ruch [**GR79**].

It may be easily seen that any graphical partition $p = (p_1, \dots, p_n)$ of $2m$ is the degree sequence of a *connected* graph G if and only if $m \geq n-1$ (see [**GR79**]). In this case the cyclomatic number c of G is given by $c = m-n+1$. In other words, the set of degree sequences of connected graphs with m edges and cyclomatic number c is simply the set of degree sequences of graphs with m edges and $n = m-c+1$ vertices. These, of course, may be characterized by adding the condition $n = m - c + 1$ to the (EG)-inequalities in 1.1. But this is not the kind of characterization we are aiming at.

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For $0 \leq c \leq 5$, Gutman derived necessary conditions for a partition p to be the degree sequence of a connected graph with cyclomatic number c which are of a different type. He considered universal upper bounds for (a small number of) partial sums of p [**Gut89**][1]. In this vein, for arbitrary values of c, we prove that degree sequences p of connected graphs with cyclomatic number c may indeed be characterized using those concepts in a general form (2.4). These upper bounds do neither depend on p nor on c (with the single obvious exception that $p_1 \leq m - c$ must hold). In particular, for $c \leq 5$, we obtain simplified and (for $c \geq 3$) corrected versions of Gutman's results (2.5). The crucial step in our approach is to derive a certain modification of 1.1 by means of a combinatorial line of reasoning (2.2).

2. Graphs with given cyclomatic number

Let $\mathbb{N}$ ($\mathbb{N}_0$, resp.) be the set of all positive (nonnegative, resp.) integers and

$$\underline{n} := \{\, k \in \mathbb{N} \,|\, k \leq n \,\}$$

for all $n \in \mathbb{N}_0$. Let $s \in \mathbb{N}$. An n-tuple $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a *partition* of s, if $p_1 + \cdots + p_n = s$ and $p_1 \geq \cdots \geq p_n$. The set of all partitions of s is denoted by P_s. Any partition $p \in P_s$ may be visualized by its *Ferrers diagram*, an array of s dots in n rows, with p_i dots in the i-th row for all $i \in \underline{n}$. For example, the Ferrers diagram of $p = (4, 3, 1, 1)$ is given by

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \\ \bullet & & & \end{array} .$$

Counting the number p'_j of dots in the j-th column of the Ferrers diagram for $j \in \underline{p_1}$, we obtain again a partition $p' = (p'_1, \ldots, p'_{p_1}) \in P_s$ which is called the *conjugate partition* of p. The number of dots in the main diagonal of the Ferrers diagram is called the *diagonal length* $d(p)$ of p. More formally, we have $p'_j = |\{\, i \in \underline{n} \,|\, p_i \geq j \,\}|$ for all $j \in \underline{p_1}$ and $d(p) = \max\{\, k \in \underline{n} \,|\, p_k \geq k \,\}$. Hence, in the above example, we have $(4, 3, 1, 1)' = (4, 2, 2, 1)$ and $d((4, 3, 1, 1)) = 2$. Note that $d(p) = d(p')$ for all $p \in P_s$.

Let k_c be the least nonnegative integer such that $(k_c + 1)k_c/2 \geq c$. These numbers will play an important role in the sequel. For small values of c, we obtain $k_0 = 0$, $k_1 = 1$, $k_2 = k_3 = 2$ and $k_4 = k_5 = 3$. Gutman observed that any connected graph with cyclomatic number c has at least

$$m_c := c + k_c + 1.$$

edges [**Gut89**, Lemma 4 and its proof].

To start with, we derive the following necessary conditions for any partition to be graphical by a simple edge-counting argument.

[1] It was stated that these conditions are also sufficient. This is incorrect for $c = 3, 4, 5$, and so is the strategy of proof described in the last section of [**Gut89**], as will be explained at the end of this paper.

Proposition 2.1. *Let $n, m \in \mathbb{N}$ and $p \in P_{2m}$ be graphical. Then we have the universal upper bounds*

$$\text{(UUB)} \qquad \sum_{\nu=1}^{k} p_\nu + \sum_{\nu=1}^{i} p_\nu \leq 2m + k(i-1)$$

for all $k, i \in \underline{n}$ such that $k \leq i$.

Proof. Let G be a graph with vertex set $X = \{x_1, \ldots, x_n\}$ such that, for all $i \in \underline{n}$, the degree of x_i in G is p_i. Let X_1, X_2, X_3 be pairwise disjoint vertex sets in G such that $X = X_1 \cup X_2 \cup X_3$. For $a, b \in \{1, 2, 3\}$, $a \leq b$, denote by e_{ab} the number of edges $\{x_\nu, x_\mu\}$ in G such that $x_\nu \in X_a$ and $x_\mu \in X_b$. As $m = e_{11} + e_{12} + e_{13} + e_{22} + e_{23} + e_{33}$, it follows that

$$\begin{aligned}
\sum_{\nu \in X_1} p_\nu + \sum_{\mu \in X_1 \cup X_2} p_\mu &= 2e_{11} + e_{12} + e_{13} + 2(e_{11} + e_{12} + e_{22}) + e_{13} + e_{23} \\
&\leq 2m + 2e_{11} + e_{12} \\
&\leq 2m + |X_1|(|X_1| - 1) + |X_1||X_2| \\
&= 2m + |X_1|(|X_1 \cup X_2| - 1)\,.
\end{aligned}$$

For the special choice $X_1 = \underline{k}$, $X_2 = \underline{i} \setminus \underline{k}$ and $X_3 = \underline{n} \setminus \underline{i}$, this yields the Proposition. □

Surprisingly, the (UUB) conditions in 2.1 are also sufficient for a partition p to be graphical. More precisely:

Theorem 2.2. *Let $d, m, n \in \mathbb{N}$ and $p = (p_1, \ldots, p_n) \in P_{2m}$ with diagonal length d. Then p is graphical if and only if*

$$\sum_{\nu=1}^{k} p_\nu + \sum_{\nu=1}^{i} p_\nu \leq 2m + k(i-1)$$

for all $k \in \underline{d}$, $i \in \{d, \ldots, n\}$.

Proof. The necessity part is covered by the above Proposition. In order to prove the sufficiency part, let $k \in \underline{d}$ and define $i := p'_k$. Then $i \geq d$ and therefore

$$\begin{aligned}
\sum_{\nu=1}^{k} p_\nu &\leq 2m + k(i-1) - \sum_{\nu=1}^{i} p_\nu \\
&= k(i-1) + \sum_{\nu=i+1}^{n} p_\nu \\
&= k(k-1) + \sum_{\nu=k+1}^{i} k + \sum_{\nu=i+1}^{n} \min\{p_\nu, k\} \\
&= k(k-1) + \sum_{\nu=k+1}^{n} \min\{p_\nu, k\}\,.
\end{aligned}$$

Hence the inequality (EG) in 1.1 holds for $k \le d$. But the particular inequality (EG) for $k = d$ implies all the remaining inequalities in 1.1. For, if $k > d$ and $j := k - d$, we have

$$\begin{aligned}
\sum_{\nu=1}^{k} p_\nu = \sum_{\nu=1}^{d} p_\nu + \sum_{\nu=d+1}^{d+j} p_\nu \\
&\le d(d-1) + \sum_{\nu=d+1}^{n} p_\nu + \sum_{\nu=d+1}^{d+j} p_\nu \\
&\le d(d-1) + 2dj + \sum_{\nu=d+j+1}^{n} p_\nu \\
&= k(k-1) - j(j-1) + \sum_{\nu=k+1}^{n} \min\{p_\nu, k\}\,.
\end{aligned}$$

Hence p is graphical, by 1.1. □

As the proof shows, it suffices to consider the inequalities (EG) for $k \le d(p)$ in 1.1. This observation is due to Gutman and Ruch [**GR79**, Theorem 2]. In order to cancel out the dependence on the diagonal length we need an additional auxiliary result.

PROPOSITION 2.3. *Let $m \in \mathbb{N}$, $c \in \mathbb{N}_0$ and $p = (p_1, \dots, p_n) \in P_{2m}$ such that $n = m - c + 1$ and $d(p) > k_c$. Then p is graphical.*

PROOF. We use the characterization of graphical partitions given in 2.2. Let $d := d(p)$, $k \in \underline{d}$ and $i \in \{d, \dots, n\}$. As $2c \le k_c(k_c + 1) \le (d-1)d$, we have

$$\begin{aligned}
\sum_{\nu=1}^{k} p_\nu + \sum_{\nu=1}^{i} p_\nu &= 2m - \sum_{\nu=k+1}^{n} p_\nu + 2m - \sum_{\nu=i+1}^{n} p_\nu \\
&\le 2m - (d-k)d - (n-d) + 2m - (n-i) \\
&= 2m - (d-k+1)d + i + 2c - 2 \\
&\le 2m - (d-k+1)d + i + (d-1)d - 2 \\
&= 2m + (k-2)d + i - 2 \\
&\le 2m + k(i-1)\,.
\end{aligned}$$

□

We are now in a position to state and prove our main result.

THEOREM 2.4. *Let $m \in \mathbb{N}$, $c \in \mathbb{N}_0$ and $p = (p_1, \dots, p_n) \in P_{2m}$. Then p is the degree sequence of a connected graph with cyclomatic number c if and only if*

$n = m - c + 1$, $p_1 \le m - c$ and the following conditions hold:

(a) $$\sum_{\nu=1}^{k} p_\nu \le m + k(k-1)/2 \quad \textit{for all } 2 \le k \le k_c + 1,$$

$$\sum_{\nu=1}^{k} p_\nu + \sum_{\nu=1}^{i} p_\nu \le 2m + k(i-1) \textit{for all } 2 \le k \le k_c - 1,\ k+3 \le i \le b_{k,c},$$

where

$$b_{k,c} := \begin{cases} c, & k = 2 \\ [k/2 + c/(k-1)] + 1, & k > 2 \end{cases}.$$

Proof. Concerning the necessity part, we observe that $p_1 \le n - 1 = m - c$ and (a) is (UUB) for $k = i$, while (b) is immediate from 2.1. For the proof of the sufficiency part we can assume that $d := d(p) \le k_c$, by 2.3. Let $k \in \underline{d}$ and $i \in \{k, \dots, n\}$. We have

(i) $$\sum_{\nu=1}^{i} p_\nu = 2m - \sum_{\nu=i+1}^{n} p_\nu \le 2m - (n-i) = m + c + i - 1\,.$$

Hence, for $k = 1$, the condition $p_1 \le m - c$ implies (UUB) for all i. Let $k > 1$ and $i > k/2 + c/(k-1) + 1$. Then, by (a) and (i), we have

$$\begin{aligned}\sum_{\nu=1}^{k} p_\nu + \sum_{\nu=1}^{i} p_\nu &\le m + k(k-1)/2 + m + c + i - 1 \\ &= 2m + k(i-1) + k(k-1)/2 + c - (k-1)(i-1) \\ &\le 2m + k(i-1)\end{aligned}$$

and (UUB) holds for k and i again. As $k \le k_c$, condition (a) for k and $k+1$ implies that

$$\sum_{\nu=1}^{k} p_\nu + \sum_{\nu=1}^{k+1} p_\nu \le m + k(k-1)/2 + m + k(k+1)/2 = 2m + k^2\,,$$

hence (UUB) for $i = k + 1$. If $i > k_c + 1$, it follows from (i) and $c \le k_c(k_c+1)/2$ that

(ii) $$\sum_{\nu=1}^{i} p_\nu \le m + i(i-1)/2\,.$$

Therefore, we have (even if $k = k_c$)

$$\sum_{\nu=1}^{k} p_\nu + \sum_{\nu=1}^{k+2} p_\nu \le m + k(k-1)/2 + m + (k+1)(k+2)/2 = 2m + k^2 + k + 1\,.$$

This means (UUB) for $i = k+2$ except for the case that equality holds. But, in this case, we have $p_{k+1} + p_{k+2} = 2k+1$, that is, $p_{k+1} \geq k+1$ and

$$\sum_{\nu=1}^{k+1} p_\nu \geq m + k(k-1)/2 + k + 1 = 2m + k(k+1)/2 + 1\,,$$

a contradiction. Finally, for $k > 2$, note that $\{k+3, \dots, b_{k,c}\} \neq \emptyset$ implies that $k + 3 \leq k/2 + c/(k-1) + 1$ and hence $k(k+1) \leq (k+4)(k-1) \leq 2c$, that is, $k \leq k_c - 1$. We checked (UUB) for all necessary values of k and i and are done by 2.2 unless $k = 2$, $i \in \{c+1, c+2\}$. But, for $k = 2$ and $i = c+2$, we have

$$p_1 + p_2 + \sum_{\nu=1}^{c+2} p_\nu \leq m + 1 + m + c + (c+2) - 1 = 2m + 2(c+1),$$

by (a) and (i). The same argument works for $i = c+1$ in the case of $p_1 + p_2 \leq m$. If $p_1 + p_2 = m+1$, then $p_1 + p_2 + p_3 \leq m+3$ implies that $p_3 \leq 2$ and

$$p_1 + p_2 + \sum_{\nu=1}^{c+1} p_\nu \leq 2(m+1) + 2(c-1) = 2m + 2c.$$

□

Note that, for $m \in \mathbb{N}$, $c \in \mathbb{N}_0$ and $p = (p_1, \dots, p_n) \in P_{2m}$ such that $n = m - c + 1$, the condition $p_1 \leq m - c$ implies that indeed $m \geq m_c$, Gutman's lower bound for the number of edges in connected graphs with cyclomatic number c. This may be seen as follows: We have $2m \leq np_1 \leq (m-c+1)(m-c)$ and therefore $(m-c-1)(m-c) \geq 2c$. It follows that $m - c - 1 \geq k_c = m_c - c - 1$.

For $c \leq 5$, the preceding theorem leads to the following criteria that may be worth mentioning explicitly.

COROLLARY 2.5. *Let $m \in \mathbb{N}$ and $p = (p_1, \dots, p_n) \in P_{2m}$.*

1 *p is degree sequence of a tree if and only if $n = m+1$.*
2 *p is degree sequence of a connected unicyclic graph if and only if $n = m$, $p_1 \leq m-1$ and $p_1 + p_2 \leq m+1$.*
3 *p is degree sequence of a connected bicyclic graph if and only if $n = m-1$, $p_1 \leq m-2$, $p_1 + p_2 \leq m+1$ and $p_1 + p_2 + p_3 \leq m+3$.*
4 *p is degree sequence of a connected tricyclic graph if and only if $n = m-2$, $p_1 \leq m-3$, $p_1 + p_2 \leq m+1$ and $p_1 + p_2 + p_3 \leq m+3$.*
5 *p is degree sequence of a connected tetracyclic graph if and only if $n = m-3$, $p_1 \leq m-4$, $p_1+p_2 \leq m+1$, $p_1+p_2+p_3 \leq m+3$ and $p_1+p_2+p_3+p_4 \leq m+6$.*
6 *p is degree sequence of a connected pentacyclic graph if and only if $n = m-4$, $p_1 \leq m-5$, $p_1+p_2 \leq m+1$, $p_1+p_2+p_3 \leq m+3$, $p_1+p_2+p_3+p_4 \leq m+6$ and $2p_1 + 2p_2 + p_3 + p_4 + p_5 \leq 2m+8$.*

PROOF. For $c = 0$, the conditions of 2.4 are given by $n = m+1$ and $p_1 \leq m$. But $n = m+1$ already implies that $p_1 = 2m - \sum_{\nu=2}^{m+1} p_\nu \leq 2m - m = m$. For $c > 0$, the listed conditions are those of 2.4. □

Except for the m_c-condition, in 2.5 (1),(2) and (3), we obtain exactly Theorems 1, 2 and 3 of [**Gut89**], while the sufficiency part of the last three Theorems in [**Gut89**] is wrong as is the strategy of proof described in the last section of [**Gut89**]: For, if p is not S-greater than any $q \in P_{2m}(c; \max)$, we cannot deduce in general that there exists a partition $q \in P_{2m}(c; \max)$ which is S-greater than p (the condition of Lemma 5 in [**Gut89**]). The partition $p := (4,4,4,2,1,1) \in P_{16}(3)$ indeed is a counter-example for Theorem 4 in [**Gut89**]. Similar counter-examples can be found for Theorems 5 and 6.

References

[EG60] P. Erdös and T. Gallai. *Graphs with given degree of vertices,* Mat. Lapok **11** (1960), 264–274 (in Hungarian)

[GR79] I. Gutman and E. Ruch. *The branching extent of graphs,* J. Comb. Inf. Syst. Sci. **4** (1979),285–295.

[Gut89] I. Gutman. *Vertex degree sequences of graphs with small number of circuits,* Publ. Inst. Math. (N.S.) (Beograd) **46(60)** (1989), 7–12.

[Hak62] S. L. Hakimi. *On realizability of a set of integers as degrees of the vertices of a linear graph I,* Siam J. Appl. Math. **10** (1962), 496–506.

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