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COMMENTS ON ULTRAPRODUCTS OF FORCING SYSTEMS

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Abstract. We discuss the conditions under which the "Łoś theorem" holds for ultraproducts of forcing systems.

1. Preliminaries

The notion of reduced product of forcing systems was introduced in [**6**]. The aim of this paper is to contribute a bit to the examination of the properties of such products (more precisely of ultraproducts). Since the results are mostly either of "negative" or illustrative character, we will have (counter)examples instead of lemmas and theorems.

Throughout the article L is a first order finitary language. The basic logical symbols are $\neg$ (negation), $\wedge$ (conjunction) and $\exists$ (existential quantifier) (the others are defined by these basic ones). By a theory of the language L we mean a consistent, deductively closed set of sentences. The notation is more or less standard and the notions are more or less well known. However, for the reader's convenience we will repeat some of the basic facts.

$AT(L)$ and $SENT(L)$ are the set of atomic and of all sentences of the language L respectively. The notion of forcing relation and forcing system has been taken from [**8**].

Definition 1.1. Let $(C, \leq, 0)$ be a partial order with the least element 0 and let L be a language with at least one constant. The (unary) relation $\Vdash$ on $C \times SENT(L)$ is a forcing relation iff the following conditions are fulfilled:

(1) *Compatibility condition(s)*

for all $p, q \in C$ and for each $\phi \in AT(L)$ holds: if $p \Vdash \phi$ and $p \leq q$, then $q \Vdash \phi$;

The next two compatibility conditions are included if L is a language with equality:

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$(1a)$ for any $p \in C$ and any closed term t of the language L there exists a condition q (an element from C) satisfying $p \leq q \Vdash t = t$;

$(1b)$ for any closed terms t_1, t_2, and for any atomic formula $\phi(v)$ (of the language L) with at most one free variable and for any condition p there exists a condition $q \geq p$ such that either $p \Vdash t_1 = t_2$ or $p \Vdash \phi(t_1)$ is not true or $q \Vdash \phi(t_2)$.

(2) $p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi$ and $p \Vdash \psi$;

(3) $p \Vdash \neg\phi$ iff no condition greater than p forces ϕ $(\forall q \geq p\ \neg(q \Vdash \phi))$;

(4) $p \Vdash \exists v\, \phi(v)$ iff there exists a closed term t (of L) such that $p \Vdash \phi(t)$.

Definition 1.2. A *forcing system* is a triple $(C, \Vdash, L)$, where C is a partial ordering $(\leq)$ with the least element, L a language with at least one constant and $\Vdash$ is a forcing relation on $C \times SENT(L)$.

Definition 1.3. [**6**] The standard reduced product of the family of forcing systems $\{F_i\ (= (C_i, \Vdash_i, L)) \mid i \in I\}$ for a given filter U over I is the forcing system $F = (\prod_U C_i, \Vdash_U, L)$, where the relation $\Vdash_U$ $(\subseteq \prod_U C_i \times SENT(L))$ is defined for $p_U \in \prod_U C_i$ and $\phi \in AT(L)$ by: $p_U \Vdash_U \phi$ iff $\{i \in I \mid p(i) \Vdash_i \phi\} \in U$ (and like any forcing relation in other cases). We will write $F = \prod_U F_i$.

The language extended reduced product of the given family of forcing systems $\{F_i \mid i \in I\}$, in notation $F' = (\prod_U C_i, \Vdash'_U, L')$, is defined in a similar way. The difference is that in this product instead of the language L we have the language L' having in common with the language L the sets of function and relation symbols but with the set of constants $\prod_U T$, where T is the set of closed terms of the language L; consequently, for a condition p_U and an atomic sentence ϕ $(\in AT(L'))$ we define: $p_U \Vdash'_U \phi$ iff $\{i \in I \mid p(i) \Vdash_i \phi_i\} \in U$, where ϕ_i is a formula of the language L obtained by replacing each constant d_U of L' by $d(i)$ $(d \in \prod_i T)$.

This time we will be particularly interested in standard (nonprincipal) ultraproducts of forcing systems (as it was shown that an ultraproduct of forcing systems for a principal ultrafilter does not offer anything new). In order to simplify notation we will denote the set $\{i \in I \mid p(i) \leq_i q(i)\}$ by X_{p_U,q_U} (hence $p_U \leq q_U$ iff $X_{p_U,q_U} \in U$). Analogously, for a sentence ϕ of the language L, $X_{p_U,\phi}$ will be the set $\{i \in I \mid p(i) \Vdash_i \phi\}$. In [**6**] it was proved that

If the set of closed terms T of the language L is of infinite cardinality λ and if the ultrafilter U is λ^+- complete (that is if it is closed under intersection of any family of its elements of cardinality $\leq \lambda$), then the "Łoś theorem" holds: $p_U \Vdash_U \phi$ iff $X_{p_U,\phi} \in U$.

In particular, the forcing companion of the forcing system F, in notation T^C $(\stackrel{\text{def}}{=} \{\phi \in SENT(L) \mid 0_U \Vdash_U \neg\neg\phi\})$ is the "ultraproduct" of the forcing companions T^{C_i}, $i \in I$: $T^C = \{\phi \in SENT(L) \mid \{i \in I \mid \phi \in T^{C_i}\} \in U\}$ – we write $T^C = \prod_U T^{C_i}$.

In general, for a family of theories of the same language L, $\{T_i \mid i \in I\}$, and a filter U over I we put $\prod_U T_i = \{\phi \in SENT(L) \mid \{i \in I \mid \phi \in T_i\} \in U\}$ and call it the reduced product (of the theories T_i). It is obvious that $\prod_U T_i$ is itself a theory (in the sense given above). If U is an ultrafilter and if all theories T_i, $i \in I$, are complete, then $\prod_U T_i$ is a complete theory too; certainly, this does not hold in general for any filter. On the other hand, the reduced product of incomplete theories can be a complete theory. It is clear as well that if for each $i \in I$ $\mathbf{M}_i$ is a model of a theory T_i and if U is an ultrafilter, then $\prod_U \mathbf{M}_i \models \prod_U T_i$.

2. A word on "Łoś theorem"

The very strong condition of λ^+-completeness set on the ultrafilter U in the case when the set of closed terms is of infinite cardinality λ in order to provide the "Łoś theorem" cannot in general be omitted (that is one of the reasons why we introduced the language extended reduced products). For instance, if the index set I and the set of closed terms T are of the same infinite cardinality in some cases there is no help; no matter what nonprincipal ultrafilter is chosen the Łoś theorem will not hold. The following example illustrates this.

EXAMPLE 2.1. Let λ be an (arbitrary) infinite cardinal and let L be a language with equality, binary relation R and the set of constants $\{d_\alpha \mid \alpha < \lambda\}$. For each $\alpha \in \lambda$ let $\mathbf{M}_\alpha$ be a model of the language L whose domain is just the set of constants and which satisfies the sentences $d_\beta \neq d_\gamma$ for $0 \leq \beta < \gamma < \lambda$ and $R^{\mathbf{M}_\alpha}(d_0, d_\alpha)$ (that is $R^{\mathbf{M}_\alpha} = \{(d_0, d_\alpha)\}$). Let $\{\phi_\delta \mid 1 \leq \delta < \lambda\}$ be a well-ordering of the diagram of $\mathbf{M}_\alpha$, $D(\mathbf{M}_\alpha)$, and let C_α be the partial order $(\{p_\gamma \mid \gamma < \lambda\}, \subseteq)$, where $p_\gamma = \{\phi_\beta \mid 1 \leq \beta \leq \gamma\}$ (p_0 is supposed to be the empty set though we can do without it). The forcing relation $\Vdash_\alpha$ is defined for p_γ and $\phi \in AT(L)$ by: $p_\gamma \Vdash_\alpha \phi$ iff $\phi \in p_\gamma$. Finally, let $F_\alpha = (C_\alpha, \Vdash_\alpha, L)$ be the corresponding forcing system. If U is a nonprincipal ultrafilter over λ and $F = \prod_U F_\alpha$, then clearly for each α the following holds: $\emptyset \Vdash_\alpha \exists v \neg\neg R(d_0, v)$ (as well as $\emptyset \Vdash_\alpha \neg\neg\exists v\, R(d_0, v)$) while $\emptyset_U$ does not force either of the given sentences.

This example shows also that it would not do if we replaced the forcing relation by the weak forcing relation. By the way (and in connection with the example) let us just note that it holds trivially for any forcing system: if a condition forces the sentence of the form $\exists v \neg\neg\phi(v)$, then it forces the sentence $\neg\neg\exists v\, \phi(v)$ too. Certainly, we do not have always the inverse of this assertion. The next simple example illustrate this.

EXAMPLE 2.2. Let L be a language with equality, a binary relation symbol R and a constant d. If T is a theory of the language L which "says" that R is an irreflexive relation and that at least one element is in relation R with d and if A is an infinite set of new constants and $\Vdash$ Robinson's finite forcing relation, then $\emptyset \Vdash \neg\neg\exists v\, R(d, v)$, while $\emptyset$ does not force the sentence $\exists v \neg\neg R(d, v)$.

In the previous example we use the fact that we have at disposal infinitely many new constants "independent (enough)" of T and that's (generally) one of the main properties and advantages of Robinson's finite forcing (compare with the example 2.1). Another "nice" thing about Robinson's finite forcing (and its generalization – n-finite forcing, [**4**], [**5**]) is that the forcing companion is independent of the cardinality of the new set of constants. It is used in the next lemma which is mostly the reformulation of lemma 2.2 in [**6**].

LEMMA 2.3. *Let λ be an infinite cardinal, L a language with equality and let for each $\alpha < \lambda$ T_α be a theory of the language L. Further, let A be a set of new constants ($L \cap A = \emptyset$) of cardinality greater than λ and let for each $\alpha < \lambda$*

$\Vdash_{n_\alpha}$ $(n_\alpha \in \omega)$ *be Robinson's* n_α*-finite forcing (to be quite precise let us say that Robinson's finite forcing is "up to the forcing companion" what we call* $\Vdash_0$*-forcing – see Theorem* 2.10 *in* [**4**] *and the comment following it). If* U *is an ultrafilter over* λ *and* $F = (C, \Vdash_U, L(A))$ *the standard reduced product of the forcing systems* $(C_\alpha, \Vdash_{n_\alpha}, L(A))$ *(where* C_α *is the corresponding set of conditions, that is the set of finite sets of* $\Sigma_{n_\alpha} \cup \Pi_{n_\alpha}$ *sentences of the language* $L(A)$ *consistent with* T_α*), then for any* $p_U \in C$ $(= \prod_U C_\alpha)$ *and any sentence* ϕ *of the language* $L(A)$ *we have:*

$$p_U \Vdash_U \neg\neg\phi \quad \textit{iff} \quad X_{p_U,\neg\neg\phi} \in U.$$

Proof. The proof is by induction on the complexity of the formula ϕ. We will consider just the key step (which does not pass in general). Let $X_{p_U,\neg\neg\exists v\,\psi(v)} \in U$ and let us suppose that the proposition holds for all formulas of the complexity less than the complexity of the formula $\exists v\,\psi(v)$. For given $q_U \geq p_U$ and for each $\beta \in X_{p_U,q_U} \cap X_{p_U,\neg\neg\exists v\,\psi(v)}$ let r_β and t_β be, respectively, a condition (from C_β) and a closed term of the language $L(A)$ such that $q(\beta) \subseteq r_\beta \Vdash_{n_\beta} \psi(t_\beta)$. If a is a constant from A which appears neither in the sentences of any r_β nor in any of the terms t_β, then (for each β from the chosen subset of λ) $s_\beta = r_\beta \cup \{t_\beta = a\}$ is a condition and $s_\beta \Vdash_{n_\beta} \neg\neg\psi(a)$ (according to 2.11 from [**8**]; see also [**4**] and 2.12 in [**1**]). Therefore, if $s \in \prod_\alpha C_\alpha$ is defined by $s(\alpha) = \begin{cases} s_\alpha & \alpha \in X_{p_U,q_U} \cap X_{p_U,\neg\neg\exists v\,\phi(v)} \\ \emptyset & \text{otherwise} \end{cases}$, then $s_U \geq q_U \geq p_U$ and $s_U \Vdash_U \neg\neg\psi(a)$, whence $p_U \Vdash_U \neg\neg\exists v\,\psi(v)$.

If we put (in accordance with the notation from [**8**]) $T^C \overset{\text{def}}{=} \{\phi \in SENT(L(A)) \mid \emptyset_U \Vdash_U \neg\neg\phi\}$ and $T^{C_\alpha} \overset{\text{def}}{=} \{\phi \in SENT(L(A)) \mid \emptyset \Vdash_{n_\alpha} \neg\neg\phi\}$ (thus the forcing companion T^{f_α} is the set $T^{C_\alpha} \cap SENT(L)$), we have: $T^C = \prod_U T^{C_\alpha}$, $T^C \cap SENT(L) = \prod_U T^{f_\alpha}$. □

Again, the condition $|A| > \lambda$ cannot in general be omitted. The next example (which is similar to 2.1) shows it.

Example 2.4. Let λ be an infinite cardinal and let L be a language with equality, binary relation R and a set of constants $\{d_\alpha \mid \alpha < \lambda\}$. For each $\alpha < \lambda$ let T_α be the theory of the language L with the set of axioms: $\{\forall v\,(R(d_0, v) \iff v = d_\alpha)\} \cup \{d_\beta \neq d_\gamma \mid 0 \leq \beta < \gamma\ (< \lambda)\} \cup \{\forall u \forall v (u \neq d_0 \Rightarrow \neg R(u,v)\}$. Finally, let $A = \{a_\alpha \mid \alpha < \lambda\}$ be the set of new constants and let U be a regular ultrafilter over λ. If $F_\alpha = (C_\alpha, \Vdash_\alpha, L(A))$ is the Robinson's forcing system relative to the theory T_α and the set of constants A and if $F = \prod_U F_\alpha$ $(= (\prod_U C_\alpha, \Vdash_U, L(A))$, then $X_{\emptyset_U,\neg\neg\exists v\,R(d_0,v)} \in U$ but $\emptyset_U$ does not force $\neg\neg\exists v\,R(d_0, v)$.

Proof. Let $S_\omega(\lambda)$ be the set of all finite subsets of λ and let f be a bijective mapping of λ onto $S_\omega(\lambda)$ such that for each $\alpha < \lambda$ $X_\alpha \overset{\text{def}}{=} \{\beta < \lambda \mid \alpha \in f(\beta)\} \in U$. For each $\alpha < \lambda$ let $p_\alpha = \{\neg R(d_0, a_\gamma) \mid \gamma \in f(\alpha)\}$ and let $p \in \prod_\alpha C_\alpha$ be given by: $p(\alpha) = p_\alpha$. Let us suppose that there is some condition $q_U \geq p_U$ which forces $\exists v\,R(d_0, v)$. In that case there exists also some constant a_γ from A such that $q_U \Vdash R(d_0, a_\gamma)$. But then $X_{p_U,q_U} \cap X_{q_U,R(d_0,a_\gamma)} \cap X_\gamma = \emptyset \in U$, a contradiction $(\delta \in X_{p_U,q_U} \cap X_{q_U,R(d_0,a_\gamma)} \cap X_\gamma$ would give $\neg R(d_0, a_\gamma) \in p(\delta) \subseteq q(\delta) \Vdash_\delta R(d_0, a_\gamma))$. □

Let us remark that just the ω^+-incompleteness of an ultrafilter U does not allow that A be a countable set if we want the "Łoś theorem" for weak forcing. Namely, we have

EXAMPLE 2.5. Let the language L and the theories T_α, $\alpha < \lambda$, be as in the previous example and let A be a countable set $\{a_n \mid n \in \omega\}$ and U an ω^+-incomplete ultrafilter over λ. Then again $X_{\emptyset_U, \neg\neg\exists v\, R(d_0,v)} \in U$ while $\emptyset_U$ does not force $\neg\neg\exists v\, R(d_0, v)$.

PROOF. Let $\langle X_n \mid n \in \omega \rangle$ be a sequence of elements of U such that $X_k \supseteq X_m$ for $m \geq k$ and $\bigcap_{n\in\omega} X_n = \emptyset$. Let us put

$$p_\alpha = \begin{cases} \emptyset & \alpha \notin X_0 \\ \{\neg R(d_0, a_0), \dots, \neg R(d_0, a_n)\} & \alpha \in X_n \setminus X_{n+1} \end{cases}.$$

Now $\emptyset_U$ does not force $\neg\neg\exists v\, R(d_0, v)$ for there are no condition $q_U \geq p_U$ and a constant a_k from A such that $q_U \Vdash_U R(d_0, a_k)$. Really, if such condition (q_U) and constant (a_k) existed, we would obtain for $\alpha \in X_{p_U, q_U} \cap X_{q_U, R(d_0, a_k)} \cap X_k$: $\neg R(d_0, a_k) \in p(\alpha) \subseteq q(\alpha) \Vdash_\alpha R(d_0, a_k)$. □

Let L be a language with equality, $\{T_\alpha \mid \alpha < \lambda\}$ a family of theories of the language L, A a set of new constants of cardinality large enough and let for each $\alpha < \lambda$ $\Vdash_\alpha$ be the Robinson's finite forcing relative to the theory T_α and the set of constants A. We have just shown that for $(\prod_U C_\alpha, \Vdash_U, L(A))$, the standard ultraproduct of the forcing systems $(C_\alpha, \Vdash_\alpha, L(A))$, $\alpha < \lambda$, for a given ultrafilter U, holds: $T^C \cap SENT(L) = \prod_U T_\alpha^f$. However, the standard ultraproduct generally does not "close the circle"; by this we mean: the following need not hold $\prod_U T_\alpha^f = (\prod_U T_\alpha)^f$. Let us give examples for both possibilities.

EXAMPLE 2.6. Let L be a language with equality and a binary relation symbol R. For $n \in N$ $(= \omega \setminus \{0\})$ let $\mathbf{C}_n$ be a chain of n elements (in particular, each $\mathbf{C}_n$ is a model of the language L). If $T_n = Th(\mathbf{C}_n)$ $(\stackrel{\text{def}}{=} \{\phi \in SENT(L) \mid \mathbf{C}_n \models \phi\})$ and if U is a nonprincipal ultrafilter over N, then $(\prod_U T_n)^f \neq \prod_U T_n^f$.

PROOF. Each T_n being a complete, forcing complete theory ($T_n = T_n^f$ – see 4.10 in [**1**]) we have $\prod_U T_n^f = \prod_U T_n$, while $\prod_U T_n$ is a complete theory – we deal with the theory of infinite linear ordering with the minimal and maximal element satisfying $\forall v(\exists u(v < u) \Rightarrow \exists w(v < w \wedge \forall z(v < z \Rightarrow w \leq z))$ and $\forall v(\exists u(v > u) \Rightarrow \exists w(v > w \wedge \forall z(v > z \Rightarrow w \geq z))$ – which is not forcing complete (no model of this theory is existentially complete). □

EXAMPLE 2.7. Let $P = \{p_n \mid n \in \omega\}$ be the set of all primes ($p_0 = 2$, $p_1 = 3$ and so on) and for each $n \in \omega$ let $\mathbf{A}_n$ be the cyclic group of order $1 + \prod_{k=0}^n p_k$. If for each $n \in \omega$, $T_n = Th(\mathbf{A}_n)$ and if U is a nonprincipal ultrafilter over ω, then $\prod_U T_n^f = (\prod_U T_n)^f$.

PROOF. Again $T_n = T_n^f$ for each n and so $\prod_U T_n^f = \prod_U T_n$. On the other hand $\prod_U T_n$ is the complete theory of torsion-free divisible Abelian groups; in the group $\mathbf{A}_n$ for all $a \in A_n$ the equations $a = p_i x$, $i = 0, \dots, n$, (we use additive notation) have the unique solutions and if $\langle b_n \rangle_U \neq \langle 0 \rangle_U$, then for any p_k $p_k \langle b_n \rangle_U \neq \langle 0 \rangle_U$. Hence $\prod_U T_n$ is a model complete theory and $(\prod_U T_n)^f = \prod_U T_n^f$ $(= \prod_U T_n)$. $\square$

3. A word on generic models

In the sequel we show that in the case of ultraproducts of finite forcing systems we cannot in general expect the ultraproducts of generic models (when they exist) to satisfy the conditions of the definition of generic models (see 3.1 and 3.2 in [**1**]) with respect to the ultraproduct of corresponding theories and ultraproduct forcing relations.

Let T_α, $\alpha < \lambda$, be a family of theories of a language L and let for each $\alpha < \lambda$ $\mathbf{M}_\alpha$ be a T_α-generic model. Surely, if the language L is uncountable it can happen that we do not have any generic model; thus we simply presume their existence. Let U be a nonprincipal ultrafilter over λ and let $\mathbf{M} = \prod_U \mathbf{M}_\alpha$. Now we have

LEMMA 3.1. *Let A be an infinite set and let $\langle A, G \rangle$ be an assignment of constants to $\mathbf{M}$ (we recall: G is a mapping of A into M such that the set $\{G(a) \mid a \in A\}$ is a set of generators for $\mathbf{M}$; hence, any element from $\mathbf{M}$ is denoted by at least one closed term of $L(A)$). Let $G(a) = \langle a_\alpha \rangle_U$ $(a_\alpha \in M_\alpha)$ and let $\langle a_\alpha \rangle$ be the fixed "representative" of $G(a)$. We have:*

(1) *If we define $G_\alpha : A \longrightarrow M_\alpha$ by $G_\alpha(a) = a_\alpha$, then $X \overset{\text{def}}{=} \{\alpha < \lambda \mid \langle A, G_\alpha \rangle$ is an assignment of constants to $\mathbf{M}_\alpha\} \in U$;*

(2) (3.1 *in* [**1**]) $\mathbf{M}$ *is consistent with* $T = \prod_U T_\alpha$.

PROOF. (1) If we suppose $\overline{X} \in U$ and if we chose from each $\mathbf{M}_\beta$, $\beta < \lambda$, an element b_β so that if $\beta \in \overline{X}$, then b_β is not in the closure of the set $\{a_\beta \mid a \in A\}$, then the element $\langle b_\beta \rangle_U$ would not be denoted by any term of $L(A)$.

Let us note that in this item we did not use at all the fact that the models $\mathbf{M}_\alpha$, $\alpha < \lambda$, are generic; in other words this result holds for an ultraproduct of any family of models of the language L.

(2) We are to show that $T \cup D_{\langle A,G \rangle}(\mathbf{M})$ is consistent ($D_{\langle A,G \rangle}(\mathbf{M})$ is the set of basic sentences of $L(A)$ true in $\mathbf{M}$). Let $p = \{\phi_1, \dots, \phi_k\}$, $\phi_i \equiv \phi_i(a_1^i, \dots, a_{n_i}^i)$, be any finite subset of $D_{\langle A,G \rangle}(\mathbf{M})$. Since we have $\mathbf{M} \models \phi_i^{\mathbf{M}}[G(a_1^i), \dots, G(a_{n_i}^i)]$ (for each i), the sets $X_i \overset{\text{def}}{=} \{\alpha < \lambda \mid \mathbf{M}_\alpha \models \phi_i^{\mathbf{M}_\alpha}[G_\alpha(a_1^i), \dots, G_\alpha(a_{n_i}^i)]\}$ are elements of U. Thus $\{\alpha < \lambda \mid T_\alpha \cup p$ is consistent $\} \in U$ and consequently $T \cup p$ is consistent. $\square$

Unfortunately (or fortunately, it depends on how one looks at all this), but quite expected, we do not always have the second part of the definition of generic models, that is it does not have to hold (for sentences ϕ of $L(A)$):

$\mathbf{M} \models \phi$ *iff there exists a finite subset p of $D_{\langle A,G \rangle}(\mathbf{M})$ such that $p \Vdash_U \phi$;*
of course, it is understood that $p(\alpha) = p$ whenever $T_\alpha \cup p$ is consistent (and this will always happen when $p \subset D_{\langle A,G_\alpha \rangle}(\mathbf{M}_\alpha)$).

We prove the last claim indirectly. Namely, supposing that $|A| > \lambda$ and that the assertion above holds, we would have as well:

$\mathbf{M}$ $(= \prod_U \mathbf{M}_\alpha)$ *completes* $\prod_U T_\alpha^f$, (in contradiction to, for example, 2.6). Indeed, let $\mathbf{K}$ be any model of $\prod_U T_\alpha^f \cup D_{\langle A,G\rangle}(\mathbf{M})$ and let $\mathbf{M} \models \phi(\overline{a})$. If p ($\equiv p(\overline{a},\overline{b})$) is a finite subset of $D_{\langle A,G\rangle}(\mathbf{M})$ such that $p \Vdash_U \phi$, then $Y \overset{\text{def}}{=} \{\alpha < \lambda \mid p \Vdash_\alpha \neg\neg\phi\} \in U$. Hence $Z \overset{\text{def}}{=} \{\alpha < \lambda \mid T_\alpha^f \vdash \forall \overline{u}, \overline{v}(\bigwedge p(\overline{u},\overline{v}) \Rightarrow \phi(\overline{u}))\} \in U$ (for $Y \subseteq Z$ – see 2.20 in [**1**]) and so $\prod_U T_\alpha^f \vdash \forall \overline{u}, \overline{v}(\bigwedge p(\overline{u},\overline{v}) \Rightarrow \phi(\overline{u})$. It follows that $\mathbf{K} \models \phi(\overline{a})$ (because of $\mathbf{K} \models \bigwedge p(\overline{a},\overline{b})$).

In Example 2.7 we have that "the circle is closed" as well as that the ultraproduct of generic models is a generic model. Certainly, the later fact is not (necessarily) a consequence of the former.

Example 3.2. Let $T = Th(\mathbf{N})$, where $\mathbf{N}$ is the (standard) model of natural numbers in the language with equality and with the binary functions $+$ and $\times$ and constants $0, 1$. It is known that T is forcing complete and that $\mathbf{N}$ is, up to isomorphism, the only T-generic model [**1**]. It follows: if U is a nonprincipal ultrafilter over ω, then $(\prod_U T)^f = T^f = T$ ($= \prod_U T$), but $\prod_U \mathbf{N}$ is not a generic model. On the other hand, if F is an ω^+-complete ultrafilter over some set I, then $\prod_F \mathbf{N}$, being isomorphic to $\mathbf{N}$, is a generic model.

For the last proposition we will need

Lemma 3.3. *Let T be a theory of the language L with a constant c and let $C = \{c_\alpha \mid \alpha < \lambda\}$ (λ an arbitrary cardinal) be a set of constants not included in L. If $L_1 = L \cup C$, $E = \{c_\alpha = c \mid c_\alpha \in C\}$ and $T_1 = T \cup E$ (here we do not stick to our definition of a theory – what we mean exactly is, of course, that T_1 is the deductive closure of the "right side"; this remark will be tacitly assumed in the sequel too) and if A is an arbitrary infinite set of constants disjoint with L_1, then the following hold:*

(1) *If p is a condition of $L(A)$ relative to the theory T and if ϕ is a sentence of the language $L(A)$, then*

$$p \Vdash_T \neg\neg\phi \quad \textit{iff} \quad p \Vdash_{T_1} \neg\neg\phi,$$

where $\Vdash_T$ and $\Vdash_{T_1}$ are forcing relations with respect to the theories T and T_1 and the languages $L(A)$ and $L_1(A)$ respectively;

(2) *T_1^f is the deductive closure of $T^f \cup E$; consequently, $T^f = T_1^f \cap SENT(L)$;*

(3) *If $\mathbf{M}$ is a T-generic model, then $\mathbf{M}_1 = (\mathbf{M}, c_\alpha^{\mathbf{M}_1})_{\alpha<\lambda}$, where $c_\alpha^{\mathbf{M}_1} = c^{\mathbf{M}}$ for each $\alpha < \lambda$, is a T_1-generic model. On the other hand, if $\mathbf{K}$ is a T_1-generic model, then its reduction to the language L is a T-generic model.*

Proof. Everything is rather obvious. However, the next facts are even more obvious. If p is a finite set of basic sentences of the language $L(A)$, then $T \cup p$ is consistent iff $T_1 \cup p$ is consistent. If $p(c_{\alpha_1}, \dots, c_{\alpha_k})$ is a condition of T_1, then $p(c, \dots, c)$ is a condition of both T and T_1 (since for any formula ψ $T_1 \vdash \psi(c_{\beta_1}, \dots, c_{\beta_m}) \Leftrightarrow \psi(c, \dots, c)$); let us just say that we point out only the constants from C (the others are of no interest for the proof).

(1) The proof is by induction on the complexity of the formula ϕ.

Let ϕ be atomic and let $p \Vdash_T \neg\neg\phi$. Let us suppose that it does not hold that $p \Vdash_{T_1} \neg\neg\phi$. Then, for some condition $q(c_{\alpha_1}, \dots, c_{\alpha_k})$ of T_1, $p \subseteq q \Vdash_{T_1} \neg\phi$. Because of Lemma 2.3 in [**4**] and the previous remark, $p \subseteq q(c, \dots, c) \Vdash_{T_1} \neg\phi$. But for some condition r of T $q(c, \dots, c) \subseteq r \Vdash_T \phi$, whence also $r \Vdash_{T_1} \phi$, a contradiction.

If $p \Vdash_{T_1} \neg\neg\phi$ (ϕ is still atomic), but $p \subseteq q \Vdash_T \neg\phi$ (for some condition q of T), then for some condition $r(c_{\alpha_1}, \dots, c_{\alpha_k})$ of T_1 holds: $q \subseteq r(c_{\alpha_1}, \dots, c_{\alpha_k}) \Vdash_{T_1} \phi$. It follows that $q \subseteq r(c, \dots, c) \cup \{\phi\}$, contradictory to $q \Vdash_T \neg\phi$.

The case $\phi \equiv \psi \wedge \theta$ is trivial (let us just recall: $p \Vdash \neg\neg(\psi \wedge \theta)$ iff $p \Vdash \neg\neg\psi$ and $p \Vdash \neg\neg\theta$).

Let $\phi \equiv \neg\psi$ and let $p \Vdash_T \neg\psi$. If for some condition $q(c_{\alpha_1}, \dots, c_{\alpha_k})$ (of T_1), extending p, $q(c_{\alpha_1}, \dots, c_{\alpha_k}) \Vdash_{T_1} \psi$, then (again by Lemma 2.3 from [**4**]) $q(c, \dots, c) \Vdash_{T_1} \neg\neg\psi$ and by inductive hypothesis ($p \subseteq$) $q(c, \dots, c) \Vdash_T \neg\neg\psi$, a contradiction. The other direction is very trivial.

Finally, let $\phi \equiv \exists v\, \psi(v)$ and let $p \Vdash_T \neg\neg\exists v\, \psi(v)$. Let us presume that for some condition $q(c_{\alpha_1}, \dots, c_{\alpha_k})$ of T_1 we have: $p \subseteq q(c_{\alpha_1}, \dots, c_{\alpha_k}) \Vdash_{T_1} \neg\exists v\, \psi(v)$. But for some condition r of T and for some closed term of the language $L(A)$ it holds that $p \subseteq q(c, \dots, c) \subseteq r \Vdash_T \psi(t)$. By the inductive assumption $r \Vdash_{T_1} \neg\neg\psi(t)$. But if $r_1 = (r \setminus q(c, \dots, c)) \cup q(c_{\alpha_1}, \dots, c_{\alpha_k})$, then $T_1 \vdash \bigwedge r \Leftrightarrow \bigwedge r_1$ and hence $r_1 \Vdash_{T_1} \neg\neg\psi(t)$, a contradiction.

Let now $p \Vdash_{T_1} \neg\neg\exists v\, \psi(v)$. If a condition q of T extends p and forces, with respect to T, the sentence $\neg\exists v\, \psi(v)$, then, for some condition $r(c_{\alpha_1}, \dots, c_{\alpha_k})$ and some closed term $t(c_{\alpha_1}, \dots, c_{\alpha_k})$, $q \subseteq r(c_{\alpha_1}, \dots, c_{\alpha_k}) \Vdash_{T_1} \psi(t(c_{\alpha_1}, \dots, c_{\alpha_k}))$. Hence $q \subseteq r(c, \dots, c) \Vdash_{T_1} \neg\neg\psi(t(c_{\alpha_1}, \dots, c_{\alpha_k}))$, but also $r(c, \dots, c) \Vdash_{T_1} \neg\neg\psi(t(c, \dots, c))$ for $T_1^f[r] \vdash \psi(t(c_{\alpha_1}, \dots, c_{\alpha_k})) \Leftrightarrow \psi(t(c, \dots, c))$ (surely, all sentences $c_\alpha = c$, $\alpha < \lambda$, belong to T_1^f). The inductive hypothesis gives ($q \subseteq$) $r(c, \dots, c) \Vdash_T \neg\neg\psi(t(c, \dots, c))$, a contradiction again.

(2) and (3) follow directly from (1). For instance, let $\mathbf{M}$ be a T-generic model and let $\langle A, G\rangle$ be an assignment of constants to $\mathbf{M}$. Certainly, $\langle A, G\rangle$ is an assignment of constants to $\mathbf{M}_1 = (\mathbf{M}, c_\alpha^{\mathbf{M}_1})_{\alpha<\lambda}$ ($c_\alpha^{\mathbf{M}_1} = c^{\mathbf{M}}$ for each $\alpha < \lambda$) as well. It is clear also that $\mathbf{M}_1$ is consistent with T_1. Still we are to prove that for any sentence ϕ of the language $L_1(A)$ the following holds: $\mathbf{M}_1 \models \phi$ iff $\exists p \subset D_{\langle A,G\rangle}(\mathbf{M}_1)$ $p \Vdash_{T_1} \phi$. As it is known we should check only the case $\mathbf{M}_1 \models \neg\phi \Longrightarrow \exists p \subset D_{\langle A,G\rangle}(\mathbf{M}_1)$ $p \Vdash_{T_1} \neg\phi$. Let $\mathbf{M}_1 \models \neg\phi(c_{\alpha_1}, \dots, c_{\alpha_k})$. Then $\mathbf{M} \models \neg\phi(c, \dots, c)$, so for some condition $p \subset D_{\langle A,G\rangle}(\mathbf{M})$ $p \Vdash_T \neg\phi(c, \dots c)$. By (1) $p \Vdash_{T_1} \neg\phi(c, \dots, c)$, whence also $p \Vdash_{T_1} \neg\phi(c_{\alpha_1}, \dots, c_{\alpha_k})$.

In a similar way we prove that the reduction of T_1-generic model to the language L is a T-generic model. $\square$

In the end using an example of S. Shelah we give a family of theories such that each member of the family has (finitely) generic model, while the ultraproduct of the family is without generic models.

THEOREM 3.4. *Let L be a language with equality, binary operations addition and multiplication and individual constant $\overline{n}$ for each natural number n and let $T = Th(\mathbf{N})$, where $\mathbf{N}$ is the standard model of natural numbers (in the language L). Let $C = \{c_\alpha \mid \alpha < \aleph_1\}$ be the set of constants disjoint with L and let for each $\alpha < \aleph_1$ $T_\alpha = T \cup E_\alpha \cup D_\alpha$, where $E_\alpha = \{c_\beta = \overline{0} \mid \beta \geq \alpha\}$, $D_\alpha = \{c_\beta \neq c_\gamma \mid 0 \leq \beta < \gamma < \alpha\}$.*

If U is a uniform ultrafilter over $\aleph_1$, then $(\prod_U T_\alpha)^f = \prod_U T_\alpha^f$ and there is no $\prod_U T_\alpha$-generic model.

PROOF. We are working in the language $L_1 = L \cup C$. Let $\Gamma = \{\phi(v_1, \dots, v_k) \in FORM(L) \mid \mathbf{N} \models \phi[n_1, \dots, n_k]$ for all distinct natural numbers $n_1, \dots, n_k\}$. Due to Theorem 2.1 from [**9**] and the previous lemma we have (for each $\alpha < \aleph_1$):

$T_\alpha^f = T \cup \{\phi(c_{\beta_1}, \dots, c_{\beta_k}) \mid$ all indices $\beta_1, \dots, \beta_k$ are distinct and less than α and $\phi(v_1, \dots, v_k) \in \Gamma\} \cup E_\alpha$.

Since U is a uniform ultrafilter we obtain $\prod_U T_\alpha = T \cup \{c_\beta \neq c_\gamma \mid 0 \leq \beta < \gamma < \aleph_1\}$ and $\prod_U T_\alpha^f = T \cup \{\phi(c_{\beta_1}, \dots, c_{\beta_k}) \mid$ all indices $\beta_1, \dots, \beta_k$ are distinct and less than $\aleph_1$ and $\phi(v_1, \dots, v_k) \in \Gamma\} = (\prod_U T_\alpha)^f$ (thus we have the "closed circle").

Shelah showed in paper mentioned above that $\prod_U T_\alpha$ is without generic models, while due to previous lemma and the well known fact that theories of countable languages have (at least one) generic model [**1**], each T_α has generic model. $\square$

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