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A NOTE ON A THEOREM OF I. VIDAV

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ABSTRACT. We give a new proof of an important result of Vidav.

We shall consider only bounded linear operators on a Hilbert space. Vidav [10] in the very interesting paper *On idempotent operators in a Hilbert space*, among other things, proved the following result:

THEOREM 1. (Vidav [10, Theorem 2]) *The equations*

$$FF^* = A, \quad F^*F = B \tag{1.1}$$

can be solved with an idempotent operator F if and only if A and B are selfadjoint operators satisfying the relations

$$ABA = A^2, \quad BAB = B^2. \tag{1.2}$$

F is uniquely determined.

Let us recall that Vidav gave two proofs of Theorem 1, that is, the first proof is geometric and the second proof is algebraically. The essential part in the both proofs is that (1.2) implies (1.1). The aim of this note is to give an alternative proof of Theorem 1, that is (1.2) implies (1.1). Our method of proof is new, and we use some properties of theory of generalized inverses.

Let us recall that Drazin [2] has introduced and investigated a generalized inverse (he called it *pseudoinverse*) in associative rings and semigroups, i.e., if S is an algebraic semigroup (or associative ring), then an element $a \in S$ is said to have a *Drazin* inverse if there exists $x \in S$ such that

$$a^m = a^{m+1}x \qquad \text{for some non-negative integer } m, \tag{1.3}$$

$$x = ax^2 \quad \text{and} \quad ax = xa. \tag{1.4}$$

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If a has Drazin inverse, then the smallest non-negative integer m in (1.3) is called the *index* $i(a)$ of a. It is well known that there is at most one x such that equations (1.3) and (1.4) hold. The unique x is denoted by a^D and called the *Drazin* inverse of a. For recent results on Drazin inverse see e.g. [3], [5], [6], [7], [9].

Let A denote a unital C^*-algebra. An element $a \in A$ is *Moore-Penrose invertible* if there exists $x \in A$ such that

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad \text{and} \quad (xa)^* = xa. \tag{1.5}$$

There is at most one element x satisfying (1.5); this is well known result of Penrose [7]. If a is Moore-Penrose invertible, the unique solution of (1.5) is called the *Moore-Penrose inverse* of a and is denoted by $a^\dagger$. For recent results on Moore-Penrose inverse see e.g. [3], [4], [6], [7], [9].

If $a \in A$, then $\sigma(a)$ and acc $\sigma(a)$ denote the spectrum and the set of all accumulation points of $\sigma(a)$, respectively; a is *quasipolar* if $0 \notin \operatorname{acc}\sigma(a)$, and *polar* if it is quasipolar and 0 is at most a pole of the resolvent $R(\lambda; a) = (\lambda - a)^{-1}$ of a. In particular, a is *simply polar* if 0 is at most a simple pole of $R(\lambda; a)$ [3], [5], [6], [7].

Let us recall that $a \in A$ is *positive* if $a^* = a$ and $\sigma(a) \subset [0, +\infty)$. Notation: $a \geq 0$ [1], [3].

THEOREM 2. *Let A denote a unital C^*-algebra. The equations*

$$ff^* = a, \quad f^*f = b \tag{2.1}$$

can be solved with an idempotent element f if and only if a and b are selfadjoint elements satisfying the relations

$$aba = a^2, \quad bab = b^2. \tag{2.2}$$

f is uniquely determined.

Proof. We shall only prove that (2.2) implies (2.1). Suppose that (2.2) holds, and that a and b are not invertible. Then

$$a(b-1)[a(b-1)]^* = a(b-1)^2 a = a^3 - a^2 \geq 0. \tag{2.3}$$

By the spectral mapping theorem 0 is isolated point of $\sigma(a)$. Hence a is Drazin invertible [5, Theorem 4.2] and by [6, Lemma 1.5] a is simply polar and $i\,(a) = 1$. Thus a is Moore-Penrose invertible [4, Theorem 6], [6, Theorem 2.8], and since $a = a^*$ we have $a^D = a^\dagger$ [6, Example 2.3]. Hence $aa^\dagger = a^\dagger a$. Let us define

$$f = a^\dagger ab. \tag{2.4}$$

Now, by (2.2) we have

$$f^2 = (a^\dagger ab)(a^\dagger ab) = (a^\dagger(aba)(a^\dagger b) = a^\dagger a^2 a^\dagger b = a^\dagger ab = f, \tag{2.5}$$

that is f is idempotent. Again by (2.2) we get

$$\begin{aligned} ff^* &= (a^\dagger ab)(a^\dagger ab)^* = a^\dagger a(b^2)aa^\dagger = a^\dagger a(bab)aa^\dagger \\ &= a^\dagger(aba)baa^\dagger = a^\dagger(a^2)baa^\dagger = (aba)a^\dagger = a^2a^\dagger = a. \end{aligned} \tag{2.6}$$

Let us remark that by (2.3) $\sigma(a) \subset \{0\} \cup [1, \infty)$. Hence $a \geq 0$. We can conclude that $b \geq 0$. Further

$$(ba^\dagger a)(ba^\dagger a)^* = (ba^\dagger a)(a^\dagger ab) = ba^\dagger ab \geq 0. \tag{2.7}$$

By (2.2)

$$\begin{aligned} (ba^\dagger ab)^2 &= ba^\dagger a(b^2)a^\dagger ab = ba^\dagger a(bab)a^\dagger ab = ba^\dagger(aba)ba^\dagger ab = \\ &= ba^\dagger(a^2)ba^\dagger ab = b(aba)a^\dagger b = b(a^2)a^\dagger b = bab = b^2. \end{aligned} \tag{2.8}$$

Since each positive element has a unique positive square root [1], [3] by (2.7) and (2.8) we get

$$ba^\dagger ab = b. \tag{2.9}$$

Finally, (2.9) implies

$$f^*f = (a^\dagger ab)^*(a^\dagger ab) = ba^\dagger ab = b.$$

Hence it is proved that (2.2) implies (2.1). □

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