PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 68(82) (2000), 92–104

INTEGRAL AVERAGING TECHNIQUES FOR OSCILLATION OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Communicated by Vojislav Marić

Abstract. New oscillation criteria are established for the second order nonlinear differential equation with a damping term

$$[a(t)\psi(x(t))x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = 0.$$

These criteria are obtained by using an integral averaging technique. Moreover, we give conditions which ensure that every solution $x(t)$ of the forced second order differential equation with a damping term

$$[a(t)\psi(x(t))x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = r(t)$$

satisfies $\liminf_{t\to\infty} |x(t)| = 0$.

1. Introduction

Consider the nonlinear differential equation with a damping term

$$(E) \qquad [a(t)\psi(x(t))x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = 0$$

where
(i) $a, p \in C^1([t_0,\infty))$, $a(t) > 0$ for $t \geq t_0$,
(ii) $q \in C([t_0,\infty))$ and it has no restriction on its sign,
(iii) $\psi \in C^1(\mathbb{R})$, $\psi(x) > 0$ for $x \neq 0$,
(iv) $f \in C^1(\mathbb{R})$ and

$$xf(x) > 0, \quad f'(x) \geq 0 \quad \text{for} \quad x \neq 0.$$

1991 *Mathematics Subject Classification.* Primary 34C10; Secondary 34C15.
Key words and phrases. Oscillation, Nonlinear differential equations, Integral averages.
Partially supported by Grant 04M03E of RFNS through Math. Inst. SANU

We restrict our attention only to the solutions of the differential equation (E) that exist on some ray $[t_0, \infty)$. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros, otherwise, it is said to be *nonoscillatory.* The equation (E) is called *oscillatory* if all solutions are oscillatory.

Some effective oscillation criteria involve the average behaviour of the integral of the alternating coefficient. For such averaging techniques for second order nonlinear oscillation, we refer to the papers [1]–[8] and [18]–[20].

We will present new oscillation criteria in the case where equation (E) is *strongly superlinear* in the sense that

$$(F_1) \qquad \int^{\infty} \frac{du}{f(u)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{du}{f(u)} < \infty,$$

as well as in the case where equation (E) is *strongly sublinear* in the sense that

$$(F_2) \qquad \int_{0+} \frac{du}{f(u)} < \infty, \quad \text{and} \quad \int_{0-} \frac{du}{f(u)} < \infty.$$

The special case $f(x) = |x|^{\alpha} \operatorname{sgn} x$ with $0 < \alpha < 1$ corresponds to the sublinear case and with $\alpha > 1$ corresponds to the superlinear case.

Investigation of the second order nonlinear oscillation in this work is motivated by the most recent contributions in the sphere of *weighted averages.* Namely, among numerous papers dealing with averaging techniques in the study of second order nonlinear oscillation majority involve positive, continuously differentiable function ϱ such that ϱ' is nonnegative and decreasing function and the function $(t-s)^{\alpha}$, for $\alpha \geq 1$ integer or real, as the weighted functions. It is therefore natural to ask if it is possible to use more extensive class of functions as the weighted functions. An affirmative answer to this question has been given for the first time by Ch. G. Philos [15], who has used averaging functions from a general class of parameter functions $H : \mathcal{D} = \{(t,s) \, : t \geq s \geq t_0\} \to \mathbb{R}$ and proved the following oscillation criterion for the linear differential equation:

Theorem A. *Let* $H : \mathcal{D} = \{(t,s) \, : t \geq s \geq t_0\} \to \mathbb{R}$ *be a continuous function, which is such that*

$$H(t,t) = 0 \quad \text{for } t \geq t_0, \qquad H(t,s) > 0 \quad \text{for all } (t,s) \in \mathcal{D}$$

and has a continuous and nonpositive partial derivative on $\mathcal{D}$ *with respect to the second variable. Moreover, let* $h : \mathcal{D} \to \mathbb{R}$ *be a continuous function with*

$$-\frac{\partial H}{\partial S}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in \mathcal{D}$$

Then, equation $x''(t) + q(t)x(t) = 0$ *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)q(s) - \frac{h^2(t,s)}{4} \Big] \, ds = \infty.$$

By using such averaging functions Grace [8] proved oscillation theorems for second order differential equations with damping, while Li and Yeh [10] proved oscillation criteria for the undamped differential equation. The purpose of this paper is to establish new criteria for the oscillation of the equation (E) by using averaging conditions of type introduced by Philos [15] and following the results of Grace and Lalli [2].

2. Main results

Let the functions $f(x)$ and $\psi(x)$ satisfy

$$(C_1) \qquad \int_{0+} \frac{\psi(u)}{f(u)}\, du < \infty, \quad \int_{0-} \frac{\psi(u)}{f(u)}\, du < \infty.$$

Furthermore, we define the functions

$$\Phi(x) = \begin{cases} \displaystyle\int_{0+}^{x} \frac{\psi(u)}{f(u)}\, du, & x > 0 \\ \displaystyle\int_{0-}^{x} \frac{\psi(u)}{f(u)}\, du, & x < 0 \end{cases},$$

$$F_1(x) = \begin{cases} \displaystyle\int_{0+}^{x} \frac{du}{f(u)}, & x > 0 \\ \displaystyle\int_{0-}^{x} \frac{du}{f(u)}, & x < 0 \end{cases}, \quad F_2(x) = \begin{cases} \displaystyle\int_{x}^{\infty} \frac{du}{f(u)}, & x > 0 \\ \displaystyle\int_{x}^{-\infty} \frac{du}{f(u)}, & x < 0 \end{cases}.$$

Also, following the idea of Philos (see for example [11]–[14] and [16], [17]) we introduce the constant $M_{f,\psi}$ defined by

$$M_{f,\psi} = \min\left\{ \frac{\inf\limits_{x>0} \frac{f'(x)\Phi(x)}{\psi(x)}}{1 + \inf\limits_{x>0} \frac{f'(x)\Phi(x)}{\psi(x)}}, \frac{\inf\limits_{x<0} \frac{f'(x)\Phi(x)}{\psi(x)}}{1 + \inf\limits_{x<0} \frac{f'(x)\Phi(x)}{\psi(x)}} \right\},$$

and suppose that the functions f and ψ are such that $0 < M_{f,\psi} < 1$.

Theorem 1. *Let the function $p(t)$ be nonnegative on $[t_0, \infty)$ and let the function f satisfies (F_1). Suppose that there exists a continuous function*

$$H : \mathcal{D} = \{ (t,s) \mid t \geq s \geq t_0 \} \to \mathbb{R}$$

such that

$$(H_1) \qquad H(t,t) = 0, \ t \geq t_0, \qquad H(t,s) > 0, \ (t,s) \in \mathcal{D}$$

$$(H_2) \qquad \frac{\partial H}{\partial s}(t,t) = 0, \ t \geq t_0, \quad \frac{\partial H}{\partial s}(t,s) \leq 0, \ (t,s) \in \mathcal{D}$$

$$(H_3) \qquad \frac{\partial^2 H}{\partial s^2}(t,s) \geq 0, \ (t,s) \in \mathcal{D}$$

$$(H_4) \qquad \liminf_{t\to\infty} \frac{\frac{\partial H}{\partial s}(t,s)}{H(t,s)} > -\infty, \ s \geq t_0.$$

Equation (E) is oscillatory if there exists a positive function $\varrho \in C^2([t_0,\infty))$*, such that for some* $\alpha \in [0, M_{f,\psi}]$

$$\left(\frac{p(t)\varrho^{\alpha}(t)}{a(t)}\right)' \le 0, \quad t \ge t_0, \tag{R_1}$$

$$\frac{a'(t)}{a(t)}\frac{\varrho'(t)}{\varrho(t)} \ge \frac{\alpha}{1-\alpha}\,\frac{\varrho''(t)}{\varrho(t)} + \frac{1}{4\alpha}\left(\frac{a'(t)}{a(t)}\right)^2, \quad t \ge t_0. \tag{R_2}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\varrho^{\alpha}(s)\frac{q(s)}{a(s)}\,ds = \infty. \tag{C_2}$$

Proof. Assume the conclusion is false. Then there is a nonoscillatory solution $x(t)$ of (E), with $x(t) \ne 0$ for $t \ge T$. If we define $w(t)$ by

$$w(t) = \varrho^{\alpha}(t)\Phi(x(t)), \quad t \ge T \tag{1}$$

for every $t \ge T$ we obtain

$$w'(t) = \alpha\,\frac{\varrho'(t)}{\varrho(t)}\,w(t) + \varrho^{\alpha}(t)\,\frac{\psi(x(t))x'(t)}{f(x(t))}$$

and consequently

$$\begin{aligned} w''(t) = {} & \alpha\frac{\varrho'(t)}{\varrho(t)}\,w'(t) + \alpha\left(\frac{\varrho'(t)}{\varrho(t)}\right)' w(t) + \frac{\varrho^{\alpha}(t)}{a(t)}\,\frac{\big(a(t)\psi(x(t))x'(t)\big)'}{f(x(t))} \\ & + \left(\frac{\varrho^{\alpha}(t)}{a(t)}\right)' \frac{a(t)}{\varrho^{\alpha}(t)}\,\frac{\varrho^{\alpha}(t)\psi(x(t)x'(t)}{f(x(t))} \\ & - \varrho^{\alpha}(t)\frac{\psi(x(t))f'(x(t))\big[x'(t)\big]^2}{f^2(x(t))}. \end{aligned}$$

Denote by $Q(t) = w'(t) - \alpha\,\dfrac{\varrho'(t)}{\varrho(t)}\,w(t)$, so that $x'(t) = \dfrac{Q(t)\,f(x(t))}{\varrho^{\alpha}(t)\,\psi(x(t))}$. Then, from the previous equality we get

$$\begin{aligned} w''(t) = {} & \alpha\frac{\varrho'(t)}{\varrho(t)}\,Q(t) + \alpha^2\left(\frac{\varrho'(t)}{\varrho(t)}\right)^2 w(t) + \alpha\left[\frac{\varrho''(t)}{\varrho(t)} - \left(\frac{\varrho'(t)}{\varrho(t)}\right)^2\right] w(t) \\ & - \varrho^{\alpha}(t)\frac{q(t)}{a(t)} - \varrho^{\alpha}(t)\frac{p(t)}{a(t)}\,\frac{x'(t)}{f(x(t))} + \left[\alpha\,\frac{\varrho'(t)}{\varrho(t)} - \frac{a'(t)}{a(t)}\right] Q(t) \\ & - \frac{\Phi(x(t))f'(x(t))}{\psi(x(t))w(t)}\,Q^2(t)\,. \end{aligned} \tag{2}$$

Using the definition of α, we have

$$\frac{f'(x(t))\Phi(x(t))}{\psi(x(t))} \ge \frac{\alpha}{1-\alpha}\,. \tag{3}$$

By the method of completing the square, the previous equality becomes

$$\begin{aligned}
w''(t) \le &- \varrho^\alpha(t)\frac{q(t)}{a(t)} - \varrho^\alpha(t)\frac{p(t)}{a(t)}\frac{x'(t)}{f(x(t))} \\
&\qquad - \frac{\alpha}{(1-\alpha)w(t)}\left[Q(t) - \frac{1-\alpha}{2\,\alpha}\,w(t)\,\left(2\alpha\frac{\varrho'(t)}{\varrho(t)} - \frac{a'(t)}{a(t)}\right)\right]^2 \\
&\qquad + (1-\alpha)\left[\frac{\alpha}{1-\alpha}\,\frac{\varrho''(t)}{\varrho(t)} + \frac{1}{4\alpha}\left(\frac{a'(t)}{a(t)}\right)^2 - \frac{a'(t)}{a(t)}\,\frac{\varrho'(t)}{\varrho(t)}\right] w(t)
\end{aligned}$$

By the condition (R_2), we get

$$w''(t) \le -\varrho^\alpha(t)\frac{q(t)}{a(t)} - \varrho^\alpha(t)\,\frac{p(t)}{a(t)}\frac{x'(t)}{f(x(t))}, \quad t \ge T,$$

and therefore

$$\begin{aligned}
&\int_T^t H(t,s)\varrho^\alpha(s)\frac{q(s)}{a(s)}\,ds \\
&\qquad \le -\int_T^t H(t,s)w''(s)\,ds - \int_T^t H(t,s)\varrho^\alpha(s)\,\frac{p(s)}{a(s)}\,\frac{x'(s)}{f(x(s))}\,ds.
\end{aligned} \tag{4}$$

Since

$$\begin{aligned}
-\int_T^t H(t,s)w''(s)\,ds &= H(t,T)w'(T) + \int_T^t \frac{\partial H}{\partial s}(t,s)w'(s)\,ds \\
&= H(t,T)w'(T) - \frac{\partial H}{\partial s}(t,T)w(T) \\
&\qquad - \int_T^t \frac{\partial^2 H}{\partial s^2}(t,s)w(s)\,ds,
\end{aligned}$$

condition (H_3) implies

$$-\int_T^t H(t,s)w''(s)\,ds \le H(t,T)w'(T) - \frac{\partial H}{\partial s}(t,T)w(T). \tag{5}$$

Using condition (R_1) and applying the Bonnet theorem, we conclude that for any fixed $s \ge T$ and for some $\xi \in [T,s]$

$$\begin{aligned}
-\int_T^s \varrho^\alpha(u)\,\frac{p(u)}{a(u)}\frac{x'(u)}{f(x(u))}\,du &= -\varrho^\alpha(T)\,\frac{p(T)}{a(T)}\int_T^\xi \frac{x'(u)}{f(x(u))}\,du \\
&= \varrho^\alpha(T)\,\frac{p(T)}{a(T)}\int_{x(\xi)}^{x(T)} \frac{d\tau}{f(\tau)}.
\end{aligned} \tag{6}$$

Since

$$\int_{x(\xi)}^{x(T)} \frac{d\tau}{f(\tau)} < \begin{cases} 0, & \text{if } x(\xi) > x(T) \\ \displaystyle\int_{0+}^{x(T)} \frac{d\tau}{f(\tau)}, & \text{if } x(\xi) \le x(T) \end{cases} \quad \text{for} \quad x > 0,$$

$$\int_{x(\xi)}^{x(T)} \frac{d\tau}{f(\tau)} < \begin{cases} 0, & \text{if } x(\xi) < x(T) \\ \displaystyle\int_{0-}^{x(T)} \frac{d\tau}{f(\tau)}, & \text{if } x(\xi) \ge x(T) \end{cases} \quad \text{for} \quad x < 0,$$

and $\varrho^\alpha(T)\,\frac{p(T)}{a(T)} \ge 0$, we obtain from (6) that

$$-\int_T^s \varrho^\alpha(u)\,\frac{p(u)}{a(u)}\,\frac{x'(u)}{f(x(u))}\,du \le \varrho^\alpha(T)\,\frac{p(T)}{a(T)} F_1[x(T)] = K_1 \quad \text{for all } s \ge T. \tag{7}$$

Now, using (7), we obtain

$$\begin{aligned} -\int_T^t & H(t,s)\varrho^\alpha(s)\,\frac{p(s)}{a(s)}\,\frac{x'(s)}{f(x(s))}\,ds \\ &= \int_T^t H(t,s) d\Big(-\int_T^s \varrho^\alpha(u)\,\frac{p(u)}{a(u)}\,\frac{x'(u)}{f(x(u))}\,du\Big) \\ &= -\int_T^t \frac{\partial H}{\partial s}(t,s)\left(-\int_T^s \varrho^\alpha(u)\,\frac{p(u)}{a(u)}\,\frac{x'(u)}{f(x(u))}\,du\right) ds \\ &\le K_1\left(-\int_T^t \frac{\partial H}{\partial s}(t,s)\,ds\right) = K_1 H(t,T). \end{aligned} \tag{8}$$

From (4), by (5) and (8), we obtain

$$\int_T^t H(t,s)\varrho^\alpha(s)\frac{q(s)}{a(s)}\,ds \le L_1\,H(t,T) - \frac{\partial H}{\partial s}(t,T)w(T).$$

where $L_1 = w'(T) + K_1$. Consequently,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\varrho^\alpha(s)\frac{q(s)}{a(s)}\,ds \le L_1 - w(T) \liminf_{t\to\infty} \frac{\frac{\partial H}{\partial s}(t,T)}{H(t,T)},$$

which together with condition (H_4) contradicts condition (C_2). □

THEOREM 2. *Let the function $p(t)$ be a nonpositive on $[t_0,\infty)$ and let the function f satisfies (F_2). Suppose that there exists a continuous function $H \in C(\mathcal{D},\mathbb{R})$ which satisfies conditions (H_1)–(H_4). The equation (E) is oscillatory if there exists a positive function $\varrho \in C^2([t_0,\infty))$, such that for some $\alpha \in [0, M_{f,\psi}]$ satisfies conditions (R_2),*

$$\left(\frac{p(t)\varrho^\alpha(t)}{a(t)}\right)' \ge 0, \quad t \ge t_0, \tag{R_3}$$

and (C_2).

Proof. We consider a nonoscillatory solution x on an interval $[T,\infty)$, $T \geq t_0$ of the differential equation (E) and as in the proof of Theorem 1, we observe that (4) and (5) hold for all $t \geq T$, where the function $w(t)$ is defined by (1).

Using the fact that the function $\varrho^{\alpha}(t)\frac{p(t)}{a(t)}$ is nonpositive and condition (R_3), by the Bonnet theorem we have for a fixed $s \geq T$ and for some $\xi \in [T,s]$

$$\begin{aligned}-\int_T^s \varrho^{\alpha}(u)\,\frac{p(u)}{a(u)}\,\frac{x'(u)}{f(x(u))}\,du &= -\varrho^{\alpha}(T)\,\frac{p(T)}{a(T)}\int_T^{\xi}\frac{x'(u)}{f(x(u))}\,du \\ &= -\varrho^{\alpha}(T)\,\frac{p(T)}{a(T)}\int_{x(T)}^{x(\xi)}\frac{d\tau}{f(\tau)}.\end{aligned} \tag{9}$$

Since $-\varrho^{\alpha}(T)\,\frac{p(T)}{a(T)} \geq 0$ and

$$\int_{x(T)}^{x(\xi)}\frac{d\tau}{f(\tau)} < \begin{cases} 0, & \text{if } x(\xi) < x(T) \\ \displaystyle\int_{x(T)}^{\infty}\frac{d\tau}{f(\tau)}, & \text{if } x(\xi) \geq x(T)\end{cases} \quad \text{for} \quad x > 0,$$

$$\int_{x(T)}^{x(\xi)}\frac{d\tau}{f(\tau)} < \begin{cases} 0, & \text{if } x(\xi) > x(T) \\ \displaystyle\int_{x(T)}^{-\infty}\frac{d\tau}{f(\tau)}, & \text{if } x(\xi) \leq x(T)\end{cases} \quad \text{for} \quad x < 0,$$

we have for $s \geq T$

$$-\int_T^s \varrho^{\alpha}(u)\,\frac{p(u)}{a(u)}\,\frac{x'(u)}{f(x(u))}\,du \leq -\varrho^{\alpha}(T)\,\frac{p(T)}{a(T)}F_2[x(T)] = K_2.$$

Hence, for $t \geq T$, we get

$$\begin{aligned}-\int_T^t & H(t,s)\varrho^{\alpha}(s)\,\frac{p(s)}{a(s)}\,\frac{x'(s)}{f(x(s))}\,ds \\ &= -\int_T^t \frac{\partial H}{\partial s}(t,s)\left(-\int_T^s \varrho^{\alpha}(u)\,\frac{p(u)}{a(u)}\,\frac{x'(u)}{f(x(u))}\,du\right)ds \\ &\leq K_2\left(-\int_T^t \frac{\partial H}{\partial s}(t,s)\,ds\right) = K_2 H(t,T).\end{aligned}$$

Thus, (4) becomes

$$\int_T^t H(t,s)\varrho^{\alpha}(s)\frac{q(s)}{a(s)}\,ds \leq L_2\,H(t,T) - \frac{\partial H}{\partial s}(t,T)w(T).$$

where $L_2 = w'(T) + K_2$. Then, we come to the contradiction as in the proof of Theorem 1. □

Next theorem is an oscillation criterion for the equation (E) when no restriction is imposed on the sign of the damping term $p(t)$.

We assume that

$$(\Psi) \qquad \psi(x) \geq c > 0 \quad \text{for all } x$$

and for arbitrary positive function $\varrho \in C^2([t_0,\infty))$ we define the functions

$$\begin{aligned} \gamma_1(t) &= \frac{a'(t)}{a(t)}\frac{\varrho'(t)}{\varrho(t)} - \frac{\alpha}{1-\alpha}\frac{\varrho''(t)}{\varrho(t)} - \frac{1}{4\alpha}\left(\frac{a'(t)}{a(t)}\right)^2 \\ \gamma_2(t) &= p(t)\left[\frac{1}{4c}\frac{p(t)}{a(t)} + \frac{a'(t)}{2a(t)} - \alpha\frac{\varrho'(t)}{\varrho(t)}\right]. \end{aligned}$$

THEOREM 3. *Let the function $H \in C(\mathcal{D};\mathbb{R})$ satisfies conditions (H_1)–(H_4). If there exists a positive function $\varrho \in C^2([t_0,\infty))$, such that for some $\alpha \in [0, M_{f,\psi}]$*

$$(R_4) \qquad \gamma_2(t) \geq 0 \quad \text{and} \quad \gamma_2(t) \leq \alpha\, c\, a(t)\gamma_1(t) \quad \text{for} \quad t \geq t_0$$

and condition (C_2) holds, then the equation (E) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (E), with $x(t) \neq 0$ for all $t \geq T$ and $w(t)$ defined by (1). Then, as in the proof of Theorem 1. we have that (2) holds. Since

$$\varrho^\alpha(t)\frac{p(t)}{a(t)}\frac{x'(t)}{f(x(t))} = \frac{p(t)}{a(t)}\frac{Q(t)}{\psi(x(t))},$$

using (3), (2) now becomes

$$\begin{aligned} w''(t) \leq &- \varrho^\alpha(t)\frac{q(t)}{a(t)} + \left[\alpha\frac{\varrho''(t)}{\varrho(t)} + \alpha(\alpha-1)\left(\frac{\varrho'(t)}{\varrho(t)}\right)^2\right] w(t) \\ &+ \left[2\alpha\frac{\varrho'(t)}{\varrho(t)} - \frac{a'(t)}{a(t)} - \frac{p(t)}{a(t)\,\psi(x(t))}\right] Q(t) - \frac{\alpha}{(1-\alpha)\,w(t)}\,Q^2(t). \end{aligned}$$

By the method of completing the square, we obtain

$$\begin{aligned} w''(t) \leq &-\varrho^\alpha(t)\frac{q(t)}{a(t)} + \left[\alpha\frac{\varrho''(t)}{\varrho(t)} + \alpha(\alpha-1)\left(\frac{\varrho'(t)}{\varrho(t)}\right)^2\right] w(t) \\ &- \frac{\alpha}{(1-\alpha)\,w(t)}\left[Q(t) - \frac{1-\alpha}{2\alpha}\,w(t)\,\mu(t)\right]^2 + \frac{1-\alpha}{4\alpha}\,\mu^2(t)\,w(t), \end{aligned} \tag{8}$$

where we set that

$$\mu(t) = 2\alpha\frac{\varrho'(t)}{\varrho(t)} - \frac{a'(t)}{a(t)} - \frac{p(t)}{a(t)\,\psi(x(t))}.$$

Further, using assumptions (Ψ) and (R_4), we have that

$$\begin{aligned}
&\alpha\,\frac{\varrho''(t)}{\varrho(t)}+\alpha\,(\alpha-1)\left(\frac{\varrho'(t)}{\varrho(t)}\right)^2+\frac{1-\alpha}{4\,\alpha}\left(2\,\alpha\frac{\varrho'(t)}{\varrho(t)}-\frac{a'(t)}{a(t)}-\frac{p(t)}{a(t)\,\psi(x(t))}\right)^2\\
&=-(1-\alpha)\gamma_1(t)+\frac{1-\alpha}{\alpha}\,\frac{1}{\psi(x(t))}\\
&\qquad\qquad\times\left[\left(\frac{p(t)}{a(t)}\right)^2\frac{1}{4\,\psi(x(t))}+\frac{p(t)}{a(t)}\left(\frac{a'(t)}{2\,a(t)}-\alpha\,\frac{\varrho'(t)}{\varrho(t)}\right)\right]\\
&\le(\alpha-1)\gamma_1(t)+\frac{1-\alpha}{\alpha}\,\frac{1}{\psi(x(t))}\\
&\qquad\qquad\times\left[\frac{1}{4\,c}\left(\frac{p(t)}{a(t)}\right)^2+\frac{p(t)}{a(t)}\left(\frac{a'(t)}{2\,a(t)}-\alpha\,\frac{\varrho'(t)}{\varrho(t)}\right)\right]\\
&=(\alpha-1)\gamma_1(t)+\frac{1-\alpha}{\alpha}\,\frac{\gamma_2(t)}{a(t)\,\psi(x(t))}\le\frac{1-\alpha}{c\,\alpha\,a(t)}\Big(\gamma_2(t)-c\,\alpha\,a(t)\,\gamma_1(t)\Big)\le 0\,.
\end{aligned}$$

Accordingly, from (8) we obtain

$$w''(t)\le-\varrho^{\alpha}(s)\frac{q(s)}{a(s)},\quad t\ge T.$$

Since the function H satisfies the same conditions as in Theorem 1, (3) holds for all $t\ge T$. Therefore, for all $t\ge T$, we have

$$\begin{aligned}
\int_T^t H(t,s)\varrho^{\alpha}(s)\frac{q(s)}{a(s)}\,ds&\le-\int_T^t H(t,s)w''(s)\,ds\\
&\le H(t,T)w'(T)-\frac{\partial H}{\partial s}(t,T)w(T),
\end{aligned}$$

which leads us to the contradiction to (C_2) by the application of condition (H_4), as in the proof of Theorem 1. ☐

Remark 1. Taking $H(t,s)=(t-s)^{\gamma}$ for some constant $\gamma>1$, which obviously satisfies the conditions $(H_1)-(H_4)$, Theorem 3 reduces to Theorem 2 in Grace, Lalli [2].

By choosing various specific functions $H(t,s)$, we can derive several useful corollaries. Let us consider the function H defined by

$$H(t,s)=\left(\int_s^t\frac{du}{\theta(u)}\right)^{\gamma}\quad\text{for}\quad t\ge s\ge t_0,$$

for some constant $\gamma>1$, where $\theta(t)$ is a positive continuous function on $[t_0,\infty)$ such that

$$(C_3)\qquad\qquad\int_{t_0}^{\infty}\frac{du}{\theta(u)}=\infty$$

Clearly,

$$H(t,t)=0 \quad \text{for} \quad t\ge t_0, \quad H(t,s)>0 \quad \text{for} \quad t>s\ge t_0$$

and

$$\frac{\partial H}{\partial s}(t,s)=-\frac{\gamma}{\theta(s)}\left(\int_s^t \frac{du}{\theta(u)}\right)^{\gamma-1}<0 \quad \text{for} \quad t\ge s\ge t_0,$$

$$\liminf_{t\to\infty}\frac{\dfrac{\partial H(t,s)}{\partial s}}{H(t,s)}=-\limsup_{t\to\infty}\frac{\gamma}{\theta(s)}\left(\int_s^t \frac{du}{\theta(u)}\right)^{-1}=0>-\infty.$$

Further, if the function $\theta(t)$ satisfies the condition

$$(C_4) \qquad \theta'(t)\int_s^t \frac{du}{\theta(u)}\ge 1-\gamma,$$

then the function H satisfies condition (H_3). Thus, we have the following corollary:

COROLLARY 1. *Let $\theta(t)$ be a positive continuous function on $[t_0,\infty)$ that satisfies conditions (C_3) and (C_4) for some constant $\gamma>1$. The equation (E) is oscillatory in the strongly sublinear case if $p(t)\ge 0$ for $t\ge t_0$ and there exists a positive function $\varrho\in C^2([t_0,\infty))$ that satisfies conditions (R_1), (R_2) and*

$$\limsup_{t\to\infty}\left(\int_{t_0}^t \frac{du}{\theta(u)}\right)^{-\gamma}\int_{t_0}^t\left(\int_s^t \frac{du}{\theta(u)}\right)^{\gamma}\varrho^{\alpha}(s)\frac{q(s)}{a(s)}\,ds=\infty\,,$$

for some $\alpha\in[0,M_{f,\psi}]$.

By similar arguments, we can formulate corollaries from Theorems 2 and 3.

Moreover, Li and Yeh have proved in [10] that the conditions (H_1)–(H_4) are also satisfied by the following functions:

$$H(t,s)=[A(t)-A(s)]^{\gamma}, \quad \text{for} \quad t\ge s\ge t_0,\ \gamma>1,$$

$$H(t,s)=\left(\log\frac{A(t)}{A(s)}\right)^{\gamma}, \quad \text{for} \quad t\ge s\ge t_0,\ \gamma>1,$$

where $A(t)$ is a positive differentiable function such that $A'(t)=\frac{1}{a(t)}$ and also the following functions

$$H(t,s)=\left(\ln\frac{A_1(s)}{A_1(t)}\right)^{\gamma}A_1(t), \quad \text{for} \quad t\ge s\ge t_0,\ \gamma>1$$

$$H(t,s)=\left(\frac{1}{A_1(t)}-\frac{1}{A_1(s)}\right)^{\gamma}A_1^2(s), \quad \text{for} \quad t\ge s\ge t_0,\ \gamma>1,$$

where

$$A_1(t)=\int_t^{\infty}\frac{ds}{a(s)}<\infty, \quad t\ge t_0.$$

Therefore, by Theorems 1, 2 and 3, we get many new oscillation criteria for the equation (E).

3. Asymptotic behavior of solutions of the forced differential equation

Let us consider the forced differential equation with a damping term

$$(E_1) \qquad [a(t)\psi(x(t))x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = r(t)$$

where $r \in C([t_0,\infty),\mathbb{R})$.

Theorem 4. *If in addition to the hypotheses of Theorem* 1, *we assume that*

$$(C_5) \qquad \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\, \frac{\varrho^\alpha(s)}{a(s)}\, |r(s)|\, ds < \infty, \quad \textit{for all } T \ge t_0,$$

then every solution $x(t)$ *of* (E_1) *satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Assume the conclusion is false. Then there is a solution $x(t)$ of (E_1) such that satisfies $\liminf_{t\to\infty}|x(t)| > 0$ and therefore, there exist $m > 0$, $M > 0$ and $T \ge t_0$ such that

$$|x(t)| > m \quad \text{and} \quad |f(x(t))| \ge M \quad \text{for} \quad t \ge T.$$

As in the proof of Theorem 1, we obtain for every $t \ge T$

$$\begin{aligned} w''(t) &\le \frac{\varrho^\alpha(t)}{a(t)} \left(\frac{r(t)}{f(x(t))} - q(t) \right) - \varrho^\alpha(t) \frac{p(t)}{a(t)} \frac{x'(t)}{f(x(t))} \\ &\le \frac{\varrho^\alpha(t)}{a(t)} \frac{|r(t)|}{M} - \varrho^\alpha(t) \frac{q(t)}{a(t)} - \varrho^\alpha(t)\frac{p(t)}{a(t)} \frac{x'(t)}{f(x(t))}. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_T^t H(t,s)\varrho^\alpha(s)\frac{q(s)}{a(s)}\, ds &\le -\int_T^t H(t,s) w''(s)\, ds \\ &+ \frac{1}{M}\int_T^t H(t,s)\, \varrho^\alpha(s)\, \frac{|r(s)|}{a(s)}\, ds - \int_T^t H(t,s)\varrho^\alpha(s)\frac{p(s)}{a(s)}\frac{x'(s)}{f(x(s))}\, ds. \end{aligned}$$

Following the procedure of the proof of Theorem 1, we get

$$\begin{aligned} \int_T^t H(t,s)\varrho^\alpha(s)\frac{q(s)}{a(s)}\, ds &\le \Big(w'(T) + \varrho^\alpha(T)\, \frac{p(T)}{a(T)} F_1[x(T)] \Big)\, H(t,T) \\ &\qquad - \frac{\partial H}{\partial s}(t,T) w(T) + \frac{1}{M}\int_T^t H(t,s)\, \varrho^\alpha(s)\, \frac{|r(s)|}{a(s)}\, ds, \end{aligned}$$

which implies

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\varrho^\alpha(s)\frac{q(s)}{a(s)}\, ds &\le w'(T) + \varrho^\alpha(T)\, \frac{p(T)}{a(T)} F_1[x(T)] \\ - w(T) \liminf_{t\to\infty} \frac{\frac{\partial H}{\partial s}(t,T)}{H(t,T)} &+ \frac{1}{M} \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\, \varrho^\alpha(s)\, \frac{|r(s)|}{a(s)}\, ds. \end{aligned}$$

In view od conditions (C_2), (H_4) and (C_5), we obtain the desired contradiction. $\square$

Corollary 2. *Let conditions* (C_2) *and* (C_5) *in Theorem* 4 *be replaced by*

$$(C_6) \qquad \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s) \frac{\varrho^\alpha(s)}{a(s)} \big(q(s) - P\,|r(s)|\big)\, ds = \infty,$$

for any $T \geq t_0$ *and* $P > 0$; *then the conclusion of Theorem* 5 *holds.*

By a similar argument, from Theorems 2 and 3 we can derive the following results.

Theorem 5. *Every solution* $x(t)$ *of* (E_1) *satisfies* $\liminf_{t\to\infty} |x(t)| = 0$ *if the hypotheses of Theorem* 2 *hold and the condition* (C_5) *is satisfied.*

Corollary 3. *Let conditions* (C_2) *and* (C_5) *in Theorem* 5 *be replaced by* (C_6); *then the conclusion of Theorem* 5 *holds.*

Theorem 6. *Every solution* $x(t)$ *of* (E_1) *satisfies* $\liminf_{t\to\infty} |x(t)| = 0$ *if the hypotheses of Theorem* 3 *hold and the condition* (C_5) *is satisfied.*

Corollary 4. *Let conditions* (C_2) *and* (C_5) *in Theorem* 6 *be replaced by* (C_6); *then the conclusion of Theorem* 6 *holds.*

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(Received 10 12 1998)
(Revised 08 05 2000)