PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 68(82) (2000), 77–82

A LITTLEWOOD–PALEY THEOREM FOR SUBHARMONIC FUNCTIONS

Miroslav Pavlović

Communicated by Miroljub Jevtić

ABSTRACT. If $u(z) > 0$ $(|z| < 1)$ is a subharmonic function of class C^2 such that Δu is subharmonic and if $\int u(re^{it})\,dt$ $(q > 1)$ is bounded when $0 < r < 1$, then

$$\iint (1-|z|)^{2q-1}\bigl(\Delta u(z)\bigr)^q\,dx\,dy < \infty.$$

In the case $u = h^2$ and $q = p/2 < 1$, where h is harmonic, this reduces to the Littlewood–Paley theorem. In the case $0 < q < 1$ we prove a theorem in the oposite direction.

1. Introduction

Let $\mathbf{D}$ denote the open unit disk in the complex plane. For a function u defined on $\mathbf{D}$ we write

$$I(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})\,dt$$

provided the integral is defined for all $r < 1$, and

$$I(u) = \sup_{0<r<1} I(r,u),$$

where the value ∞ is permitted. In this paper we prove the following theorem.

THEOREM 1.1. *Let $u \ge 0$ be a subharmonic function of class $C^2(\mathbf{D})$ such that its Laplacian, Δu, is subharmonic as well. If $q \ge 1$ and $I(u^q) < \infty$, then*

$$\int_{\mathbf{D}} (1-|z|)^{2q-1}\bigl(\Delta u(z)\bigr)^q\,dm(z) \le C_q\bigl(I(u^q) - u(0)^q\bigr), \tag{1.1}$$

where C_q is a constant depending only on q.

1991 *Mathematics Subject Classification.* Primary 30D55.
Key words and phrases. Riesz' measure of a subharmonic function.

Here dm denotes the area measure in the plane.

An important special case of (1.1) is the Littlewood–Paley inequality [**3**]; namely, if $p \geq 2$ and $I(|h|^p) < \infty$, where h is a real-valued function harmonic in $\mathbf{D}$, then

$$\int_{\mathbf{D}} (1-|z|)^{p-1}\, |\nabla u|^p \, dm < C_p\big(I(|h|^p - |h(0)|^p)\big). \tag{1.2}$$

To obtain (1.2) from (1.1) we take $u = h^2$ and $q = p/2$. The function u satisfies the hypotheses of Theorem 1.1 because $\Delta u = 2|\nabla h|^2$.

Inequality (1.2) is usually stated in the weaker form

$$\int_{\mathbf{D}} (1-|z|)^{p-1}\, |\nabla h|^p \, dm \leq C_p \, I(|h|^p) \quad (p > 2). \tag{1.3}$$

The usual method of proving (1.3) is to use the Riesz–Thorin theorem. A quick elementary proof is given in [**6**]; it is based on the Hardy–Stein identity and the inequality $|\nabla h(z)| \leq 2h(z)/(1-|z|)$ which holds when $h > 0$. An earlier proof based on the Hardy–Stein inequality and some local estimates is due to Luecking [**5**]. Our proof of Theorem 1.1 is similar to Luecking's proof of (1.3) (see Lemma 2.2 and 3.1). However, some simplifications are made so that we can treat the case $q < 1$ as well (see Theorem 4.1). This provides, in particular, a new proof of the reverse Littlewood–Paley inequality which holds for harmonic functions when $1 < p < 2$ and for analytic functions when $0 < p < 2$. Moreover, a special case of Theorems 1.1 and 4.1 is the Littlewood–Paley inequality for vector valued functions. More precisely, inequality (1.3) remains true for $p \geq 2$ if we assume that h is a harmonic function with values in ℓ^2, $|h(z)|^2 = \sum h_n(z)^2$ and $|\nabla h(z)|^2 = \sum |\nabla h_n(z)|^2$. The reverse inequality holds for $1 < p < 2$.

2. Local estimates for Riesz' measure

From now on we shall assume that u is an arbitrary nonnegative subharmonic function defined on $\mathbf{D}$. Then there exists a positive measure $d\mu$ on $\mathbf{D}$, called the Riesz measure of u, such that $\Delta u = d\mu$ in the sense of distribution theory. (If u is of class C^2, then $d\mu(z) = \Delta u(z)\, dm(z)$.) There holds the formula

$$I(r,u) - u(0) = \frac{1}{2\pi} \int_{r\mathbf{D}} \log \frac{r}{|z|}\, d\mu(z) \quad (0 < r < 1), \tag{2.1}$$

which can be deduced, for example, from the Riesz representation formula (see [**4**], Theorem 3.3.6.)

LEMMA 2.1. *We have*

$$I(u) - u(0) = \frac{1}{2\pi} \int_{\mathbf{D}} \log \frac{1}{|z|}\, d\mu(z).$$

PROOF. Write (2.1) in the form

$$I(r,u) - u(0) = \frac{1}{2\pi} \int_{\mathbf{D}} K_r(z) \log \frac{r}{|z|}\, d\mu(z),$$

where $K_r(z)$ is the characteristic function of the disk $r\mathbf{D}$. Since $K_r(z)\log(r/|z|)$ increases with r we have

$$\lim_{r\to 1}(r,u) - u(0) = \frac{1}{2\pi}\int_{\mathbf{D}} \lim_{r\to 1} K_r(z)\log\frac{r}{|z|}\,d\mu(z).$$

And since $I(r,u)$ increases with r we have $I(u) = \lim_{r\to 1} I(r,u)$. The result follows. □

LEMMA 2.2. *Let $q \geq 1$ and let μ and μ_q be the Riesz measures of u and u^q respectively. Then*

$$\mu(E)^q \leq C_q\,\mu_q(5E) \tag{2.2}$$

for any disk E such that $6E \subset \mathbf{D}$. The constant C_q depends only on q.

If E is a disk of radius R, then rE denotes the *concetric* disk of radius Rr.

PROOF. By translation the proof is reduced to the case where E is centered at 0. Then since $\mu(E) = \nu((1/r)E)$, where ν is the Riesz measure of the function $u(rz)$, we can assume that the radius of E is fixed. e.g., $E = \varepsilon\mathbf{D}$ with $\varepsilon = 1/6$. Assuming this we use the simple inequalities

$$\big(I(r,u) - u(0)\big)^q \leq \big(I(r,u)\big)^q - u(0)^q$$

and $\big(I(r,u)\big)^q \leq I(r,u^q)$, which hold because $q > 1$, to deduce from (2.1) (applied to u and u^q) that

$$\left(\frac{1}{2\pi}\int_{r\mathbf{D}} \log\frac{r}{|z|}\,d\mu(z)\right)^q \leq \frac{1}{2\pi}\int_{r\mathbf{D}} \log\frac{r}{|z|}\,d\mu_q(z). \tag{2.3}$$

Putting $r = 4\varepsilon$ we get

$$\mu(2\varepsilon\mathbf{D})^q \leq C\int_{4\varepsilon\mathbf{D}} |z|^{-1}\,d\mu_q(z), \tag{2.4}$$

where we have used the estimates $\log(4\varepsilon/|z|) \geq \log 2$ for $|z| < 2\varepsilon$ and $\log(4\varepsilon/|z|) \leq 1/|z|$. Thus to prove (2.2) we have to eliminate $|z|^{-1}$ in the integral. To do this we change the 'center' of (2.4) and we get

$$\mu(2\varepsilon D_a)^q \leq C\int_{4\varepsilon D_a} |z-a|^{-1}\,d\mu_q(z)$$

for $a \in \varepsilon\mathbf{D}$, where $D_a = \{z : |z-a| < 1\}$. Since $\varepsilon\mathbf{D} \subset 2\varepsilon D_a$ and $4\varepsilon D_a \subset 5\varepsilon\mathbf{D}$ we have

$$\mu(\varepsilon\mathbf{D})^q \leq C\int_{4\varepsilon D_a} |z-a|^{-1}\,d\mu_q(z).$$

Now we integrate this inequality over $\varepsilon\mathbf{D}$ with respect to $dm(a)$ and use Fubini's theorem. This concludes the proof because

$$\sup_{z\in\mathbf{D}}\int_{\varepsilon\mathbf{D}} |z-a|^{-1}\,dm(a) < \infty.$$

□

3. Proof of Theorem 1.1

Theorem 1.1 is a consequence of the following.

THEOREM 3.1. *Let $u \geq 0$ be a subharmonic function in $\mathbf{D}$ and let μ be the Riesz measure of u. If $q \geq 1$ and $I(u^q) < \infty$, then there holds the inequality*

$$\int_{\mathbf{D}} (1-|z|)^{-1} \big(\mu(E_\varepsilon(z))\big)^q \, dm \leq C_q \big(I(u^q) - u(0)^q\big), \tag{3.1}$$

where $\varepsilon = 1/6$ and

$$E_\varepsilon(z) = \{w : |w-z| < \varepsilon(1-|z|)\}.$$

If in addition u is C^2 and Δu is subharmonic, then

$$\mu(E_\varepsilon(z)) = \int_{E_\varepsilon(z)} \Delta u \, dm \geq \pi\varepsilon^2(1-|z|)^2 \Delta u(z)$$

because of the sub-mean-value property of Δu, and this shows that (3.1) implies (1.2).

PROOF. It follows from (2.2) that

$$\int_{\mathbf{D}} (1-|z|)^{-1} \big(\mu(E_\varepsilon(z))\big)^q \, dm C \int_{\mathbf{D}} (1-|z|)^{-1} \, \mu_q(E_{5\varepsilon}(z)) \, dm(z). \tag{3.2}$$

Next we write

$$\mu_q(E_{5\varepsilon}(z)) = \int_{E_{5\varepsilon}(z)} d\mu_q(w)$$

and use Fubini's theorem to conclude that the right hand side of (3.2) is equal to

$$\int_{\mathbf{D}} d\mu_q(w) \int_{G(w)} (1-|z|)^{-1} \, dm(z),$$

where $G(w) = \{z : |z-w| < 5\varepsilon(1-|z|)\}$. Since $z \in G(w)$ implies $|z| - |w| < 5\varepsilon(1-|z|)$, whence $1_{|}z| < (1+5\varepsilon)(1-|z|)$, we have

$$\int_{G(w)} (1-|z|)^{-1} \, dm(z) \leq (1+5\varepsilon) \, m(G(w)) \, (1-|w|)^{-1}.$$

And since $(1+5\varepsilon)(1-|z|) < 1-|w|$ for $z \in G(w)$, we have $m(G(w)) \leq C'(1-|w|)^2$, where $C' = \pi(5\varepsilon/(1-5\varepsilon))^2$. Combining the previous results we see that

$$\int_{\mathbf{D}} (1-|z|)^{-1} \big(\mu(E_\varepsilon(z))\big)^q \, dm \leq C_q \int_{\mathbf{D}} (1-|w|) \, d\mu_q(w).$$

This finishes the proof of (3.1) because of Lemma 2.1 and the inequality $1-|w| \leq \log(1/|w|)$. □

4. The case $q < 1$

THEOREM 4.1. *Let $0 < q < 1$ and let $u \geq 0$ be a C^2−function such that u^q and Δu are subharmonic. If $\int_b D(1-|z|)^{2q-1}(\Delta u)^q\,dm < \infty$, then $I(u^q) < \infty$ and there holds the inequality*

$$I(u^q) - u(0)^q \leq C_q \int_{\mathbf{D}} (1-|z|)^{2q-1}(\Delta u)^q\,dm. \tag{4.1}$$

Observe that, in contrast to the case $q > 1$, the function u^q need not be smooth.

PROOF. Fix $\varepsilon < 1/6$. Applying Lemma 2.2 to the pair u^q, $(u^q)^{1/q}$ we get, because $1/q > 1$,

$$\mu_q(E_\varepsilon(z)) \leq C_q\big(\mu(E_{5\varepsilon}(z))\big)^q,$$

where μ_q and μ are the Riesz measure of u^q and u. On the other hand

$$\begin{aligned}\big(\mu(E_{5\varepsilon}(z))\big)^q &= \left(\int_{E_{5\varepsilon}(z)} \Delta u\,dm\right)^q \\ &\leq C'(1-|z|)^{2q}\sup\{(\Delta u(w))^q : w \in E_{5\varepsilon}(z)\}.\end{aligned} \tag{4.2}$$

The function $(\Delta u)^q$ need not be subharmonic. Nevertheless, by a result of Hardy and Littlewood [**2**] and Fefferman and Stein [**1**], it possesses a weak form of the sub-mean-value property, namely

$$(\Delta u(z))^q \leq \frac{C}{m(E)}\int_E (\Delta u)^q\,dm, \tag{4.3}$$

where $E \subset \mathbf{D}$ is any disk centered at z, and C depends only on q. Using (4.3) one shows that

$$\sup_{E_{5\varepsilon}(z)} (\Delta u)^q \leq C"(1-|z|)^{-2}\int_{E_{6\varepsilon}(z)} (\Delta u)^q\,dm.$$

It follows that

$$\int_{\mathbf{D}} (1-|z|)^{-1}\mu_q(E_\varepsilon(z))\,dm(z) \leq C\int_{\mathbf{D}} (1-|z|)^{2q-3}\,dm(z)\int_{E_{6\varepsilon}(z)} (\Delta u)^q\,dm,$$

where C depends only on q. Hence, as in the proof of Theorem 3.1,

$$\int_{\mathbf{D}} (1-|z|)\,d\mu_q(z) \leq C_q\int_{\mathbf{D}} (1-|z|)^{2q-1}(\Delta u)^q\,dm. \tag{4.4}$$

This implies that $I(u^q) < \infty$ because of Lemma 2.1 applied to u^q.

In order to prove (4.1) additional work is needed. We rewrite (2.3) as

$$\left(\frac{1}{2\pi}\int_{r\mathbf{D}} \log\frac{r}{|z|}\,d\mu_q(z)\right)^q \leq \frac{1}{2\pi}\int_{r\mathbf{D}} \log\frac{r}{|z|}\,d\mu(z).$$

Hence

$$\int_{\varepsilon\mathbf{D}} \log\varepsilon|z|\,d\mu_q(z) \leq C\sup_{\varepsilon\mathbf{D}}(\Delta u)^q \leq C'\int_{2\varepsilon\mathbf{D}} (\Delta u)^q\,dm,$$

where we have used (4.3). Now it is easy to show that (4.4) remains true if we replace the left integral by

$$\frac{1}{2\pi}\int_{\mathbf{D}} \log\frac{1}{|z|}\, d\mu_q(z) = I(u^q) - u(0)^q.$$

□

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Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia
pavlovic@matf.bg.ac.yu

(Received 25 02 1999)