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Dedicated
to professor **Dr Svetozar Milić**
on the ocasion of his 65th birthday

RECTANGULAR LOOPS

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ABSTRACT. Rectangular groups, i.e. direct products of rectangular bands and groups play a significant role in the semilattice decomposition theory of semigroups. In our attempt to generalize this theory to groupoids, we start by investigating *rectangular loops*, i.e. direct products of rectangular bands and loops.

The standard method of R. A. Knoebel gives us an axiom system for rectangular loops consisting of 21 identities in an extended language. We give a simpler and more intuitive equivalent system of only 12 identities.

Other important properties of rectangular loops are derived.

1. Introduction

One of the prominent results in the theory of semilattice decomposition of semigroups is the following theorem (see for example Petrich [**3**]):

THEOREM 1.1. *Every completely regular semigroup in which idempotents form a subsemigroup is a semilattice of rectangular groups.*

A component structure from this theorem, a rectangular group, is a particularly simple semigroup, being the direct product of a rectangular band and a group.

Approaching the structure theory of groupoids from the easy end, we attempt to generalize the notion of the rectangular group. One possible way to do this is to relax the group property and allow a loop instead. The resulting groupoid, a rectangular loop, has a subgroupoid of idempotents which is a rectangular band. Even though regularity properties (in particular inverse elements) are lost, retaining a band of idempotents gives us a good enough tool to analyze this groupoid.

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2. Rectangular loops

Of the several possible ways to define rectangular loops we choose the following:

DEFINITION 2.1. Groupoid S is a *rectangular loop* iff it is isomorphic to the direct product of a left zero semigroup, a loop and a right zero semigroup.

We are now faced with the problem of the axiomatization of the class of all rectangular loops. For that we have at our disposal the standard method of Knoebel [**2**]. Adjusting the types of the left/right zero semigroups to that of (equational) loops, we define $x\backslash y = x/y = xy$ in both of them. Different definition of $\backslash$ and $/$ would affect the form of the axioms for rectangular loops. As it is, the resulting axiom system (K) consists of the following 21 identities:

$$(xy\backslash xy)\cdot t(t/t) = (x\backslash x)\cdot t(t/t) \tag{K1}$$

$$((x\backslash y)\backslash(x\backslash y))\cdot t(t/t) = (x\backslash x)\cdot t(t/t) \tag{K2}$$

$$((x/y)\backslash(x/y))\cdot t(t/t) = (x\backslash x)\cdot t(t/t) \tag{K3}$$

$$(t\backslash t)\cdot(x\backslash xy)(t/t) = (t\backslash t)\cdot y(t/t) \tag{K4}$$

$$(t\backslash t)\cdot(x(x\backslash y))(t/t) = (t\backslash t)\cdot y(t/t) \tag{K5}$$

$$(t\backslash t)\cdot(xy/y)(t/t) = (t\backslash t)\cdot x(t/t) \tag{K6}$$

$$(t\backslash t)\cdot((x/y)y)(t/t) = (t\backslash t)\cdot x(t/t) \tag{K7}$$

$$(t\backslash t)\cdot(x\backslash x)(t/t) = (t\backslash t)\cdot(y/y)(t/t) \tag{K8}$$

$$(t\backslash t)\cdot t(xy/xy) = (t\backslash t)\cdot t(y/y) \tag{K9}$$

$$(t\backslash t)\cdot t((x\backslash y)/(x\backslash y)) = (t\backslash t)\cdot t(y/y) \tag{K10}$$

$$(t\backslash t)\cdot t((x/y)/(x/y)) = (t\backslash t)\cdot t(y/y) \tag{K11}$$

$$\begin{aligned}&(((x\backslash x)\cdot t(t/t))\backslash((x\backslash x)\cdot t(t/t)))\\&\quad\cdot((t\backslash t)\cdot x(t/t))(((t\backslash t)\cdot t(x/x))/((t\backslash t)\cdot t(x/x))) = x\end{aligned} \tag{K12}$$

$$(((x\backslash x)\cdot t(t/t))\cdot(y\backslash y)\cdot t(t/t)))\cdot t(t/t) = (xy\backslash xy)\cdot t(t/t) \tag{K13}$$

$$(((x\backslash x)\cdot t(t/t))\backslash(y\backslash y)\cdot t(t/t)))\cdot t(t/t) = ((x\backslash y)\backslash(x\backslash y))\cdot t(t/t) \tag{K14}$$

$$(((x\backslash x)\cdot t(t/t))/(y\backslash y)\cdot t(t/t)))\cdot t(t/t) = ((x/y)\backslash(x/y))\cdot t(t/t) \tag{K15}$$

$$((t\backslash t)\cdot(((t\backslash t)\cdot x(t/t))\cdot((t\backslash t)\cdot y(t/t)))(t/t) = (t\backslash t)(xy\cdot(t/t)) \tag{K16}$$

$$((t\backslash t)\cdot(((t\backslash t)\cdot x(t/t))\backslash((t\backslash t)\cdot y(t/t)))(t/t) = (t\backslash t)\cdot(x\backslash y)(t/t) \tag{K17}$$

$$((t\backslash t)\cdot(((t\backslash t)\cdot x(t/t))/((t\backslash t)\cdot y(t/t)))(t/t) = (t\backslash t)\cdot(x/y)(t/t) \tag{K18}$$

$$(t\backslash t)\cdot t(((t\backslash t)\cdot t(x/x))\cdot((t\backslash t)\cdot t(y/y))) = (t\backslash t)\cdot t(xy/xy) \tag{K19}$$

$$(t\backslash t)\cdot t(((t\backslash t)\cdot t(x/x))\backslash((t\backslash t)\cdot t(y/y))) = (t\backslash t)\cdot t((x\backslash y)/(x\backslash y)) \tag{K20}$$

$$(t\backslash t)\cdot t(((t\backslash t)\cdot t(x/x))/((t\backslash t)\cdot t(y/y))) = (t\backslash t)\cdot t((x/y)/(x/y)) \tag{K21}$$

Obviously, the axiom system (K) is far from being elegant. Therefore we propose another one, called $(\Box\Lambda)$, which consists of the following 12 identities:

(Q1) $$x\backslash xy = (x\backslash x)y$$

(Q2) $$x(x\backslash y) = (x\backslash x)y$$

(Q3) $$xy/y = x(y/y)$$

(Q4) $$(x/y)y = x(y/y)$$

(E1) $$(x/x)x = x$$

(E2) $$x\backslash x = x/x$$

(A1) $$(x/x)\cdot yz = (x/x)y\cdot z$$

(A2) $$x\cdot(y/y)z = xz$$

(A3) $$xy\cdot(z/z) = x\cdot y(z/z)$$

(U1) $$xy/xy = (x/x)(y/y)$$

(U2) $$(x\backslash y)/(x\backslash y) = (x/x)(y/y)$$

(U3) $$(x/y)/(x/y) = (x/x)(y/y)$$

In the theorem 2.5 we shall prove that this system is equivalent to (K). For the proof we need a series of interim results collected in theorems 2.1–2.4.

For the rest of this section (except in the theorem 2.5) we assume to work in the system $(\square\Lambda)$.

Definition 2.2. For any $a \in S$ we define $e_a = a/a$, $Q_a = \{x \in S | e_x = e_a\}$ and $a^2 = aa$. Also, $E = \{x \in S | x^2 = x\}$.

Theorem 2.1. *Let $*$ be any of the operations $\cdot$, $\backslash$, $/$. Then:*

(a) $x\backslash x = x/x = e_x$	(f) $e_x x = xe_x = x\backslash x^2 = x^2/x = x$
(b) $x\backslash xy = e_x y$	(g) $e_x * e_x = e_{e_x} = e_{x^2} = e_x$
(c) $x(x\backslash y) = e_x y$	(h) $e_x\backslash x = x/e_x = x$
(d) $xy/y = xe_y$	(i) $e_{x*y} = e_x e_y$
(e) $(x/y)y = xe_y$	(j) *x is an idempotent iff $x = e_x$.*

Proof. (a) Trivial; (b)–(e) are axioms (Q1)–(Q4) rewritten.

$(f1)$ $e_x x = (x/x)x = x$
$(f2)$ $xe_x = x(x/x) = x(x\backslash x) = e_x x = x$
$(f3)$ $x\backslash x^2 = x\backslash xx = e_x x = x$
$(f4)$ $x^2/x = xx/x = xe_x = x$
$(g1)$ $e_x e_x = (e_x x)/x = x/x = e_x$
$(g2)$ $e_x\backslash e_x = e_x\backslash e_x e_x = e_x$
$(g3)$ $e_x/e_x = e_x e_x/e_x = e_x$
$(g4)$ $e_{e_x} = e_x/e_x = e_x$
$(g5)$ $e_{x^2} = x^2/x^2 = (x/x)(x/x) = e_x e_x = e_x$
$(h1)$ $e_x\backslash x = e_x\backslash e_x x = (e_x\backslash e_x)x = e_x x = x$
$(h2)$ $x/e_x = xe_x/e_x = x(e_x/e_x) = xe_x = x$
(i) These are axioms (U1)–(U3) rewritten and combined together.

$(j1)$ If $x^2 = x$ then $e_x = x/x = x^2/x = x$
$(j2)$ If $e_x = x$ then $x^2 = xx = e_x x = x$. □

THEOREM 2.2. *The following implications are true:*

(a) $xz = yz, e_x = e_y \Rightarrow x = y$
(b) $zx = zy, e_x = e_y \Rightarrow x = y$
(c) $z\backslash x = z\backslash y, e_x = e_y \Rightarrow x = y$
(d) $x/z = y/z, e_x = e_y \Rightarrow x = y$
(e) $x \neq y, e_x = e_y \Rightarrow xz \neq yz$
(f) $x \neq y, e_x = e_y \Rightarrow zx \neq zy$
(g) $x \neq y, e_x = e_y \Rightarrow z\backslash x \neq z\backslash y$
(h) $x \neq y, e_x = e_y \Rightarrow x/z \neq y/z$.

PROOF. (a) Assume $xz = yz$ and $e_x = e_y$. Then $x = xe_x = x \cdot (z/z)e_x = x(z/z) \cdot e_x = (xz/z)e_x = (yz/z)e_y = y(z/z) \cdot e_y = y \cdot (z/z)e_y = ye_y = y$.

The items $(b), (c)$ and (d) can be proved analogously, while (e), (f), (g) and (h) are just contrapositives of (a), (b), (c) and (d) respectively. □

THEOREM 2.3. *In all rectangular loops we have:*

(a) $e_x \cdot yz = e_x y \cdot z$
(b) $xe_y \cdot z = xz$
(c) $x \cdot e_y z = xz$
(d) $xy \cdot e_z = x \cdot ye_z$
(e) $e_x\backslash y = e_x y$
(f) $x/e_y = xe_y$
(g) $e_x\backslash e_y = e_x/e_y = e_x e_y$
(h) $e_x y\backslash z = e_x(y\backslash z)$
(i) $xe_y\backslash z = x\backslash z$
(j) $x\backslash e_y z = x\backslash z$
(k) $x\backslash ye_z = (x\backslash y)e_z$
(l) $e_x y/z = e_x(y/z)$
(m) $xe_y/z = x/z$
(n) $x/e_y z = x/z$
(o) $x/ye_z = (x/y)e_z$.

PROOF. (a), (c) and (d) are just (A1), (A2) and (A3) respectively, rewritten using axiom (E2) and definition 2.1.

(b) Assume statement not true. Then there are elements a, b, c such that $ae_b \cdot c \neq ac$. Also $e_{ae_b \cdot c} = e_{ae_b}e_c = e_a e_{e_b} \cdot e_c = e_a \cdot e_{e_b}e_c = e_a e_c$. By the theorem 2.1$(h)$ we get $(ae_b{\cdot}c)/c \neq ac/c$ and $ae_c = a(c/c) \neq (ae_b{\cdot}c)/c = ae_b{\cdot}e_c = a{\cdot}e_b e_c = ae_c$ which is a contradiction. Therefore statement must be true.

(e) Assume statement not true. Then there are elements a, b such that $e_a\backslash b \neq e_a b$. Also $e_{e_a\backslash b} = e_{e_a}e_b = e_{e_a b}$. By the theorem 2.1$(f)$ we get $e_a(e_a\backslash b) \neq e_a \cdot e_a b$ and $e_a b = e_{e_a} b = e_a(e_a\backslash b) \neq e_a \cdot e_a b = e_a b$ which is a contradiction. Therefore statement must be true.

(f) Analogously to (e).

(g) Trivially from (e) and (f).

The rest of the statements (h)–(o) are proved similarly to the case (b). As an example we prove

(l) Assume statement not true. Then there are elements a, b, c such that $e_a b/c \neq e_a(b/c)$. Also $e_{e_a b/c} = e_{e_a b}e_c = e_{e_a}e_b \cdot e_c = e_{e_a} \cdot e_b e_c = e_{e_a}e_{b/c} = e_{e_a(b/c)}$. By the theorem 2.1$(e)$ we get $(e_a b/c)c \neq e_a(b/c){\cdot}c$ and $e_a{\cdot}be_c = e_a b{\cdot}e_c = (e_a b/c)c \neq e_a(b/c) \cdot c = e_a \cdot (b/c)c = e_a \cdot be_c$ which is a contradiction. Therefore statement must be true. □

THEOREM 2.4. *Let $a_1, \ldots, a_n$ be a sequence of elements of the rectangular loop S, such that at most two of them are nonidempotents. Then all products of $a_1, \ldots, a_n$, in that order, are equal to the following product of at most four of them: a_1, nonidempotents of $a_2, \ldots, a_{n-1}$ if any (the one with the smaller index first), a_n if $n > 1$.*

PROOF. Let $\bar{a}$ be the sequence $a_1, \ldots, a_n$ $(n > 1)$ of elements from S, with at most two nonidempotents. Let us define the *core* of such a sequence. If all elements in $\bar{a}$ are idempotents, let $\operatorname{core}(\bar{a}) = a_1$. If there is only one nonidempotent a_j $(1 \le j \le n)$ in $\bar{a}$, then let $\operatorname{core}(\bar{a}) = a_j$. Finally, if there are two nonidempotents a_j, a_k $(1 \le j < k \le n)$, then let $\operatorname{core}(\bar{a}) = a_j a_k$. We shall prove that any product p of elements $a_1, \ldots, a_n$, in that order, is equal to $e_{a_1} \operatorname{core}(\bar{a}) e_{a_n}$.

(a) First, we prove that the product $e_{a_1} \operatorname{core}(\bar{a}) e_{a_n}$ is unambiguous.

($a1$) If $\operatorname{core}(\bar{a}) = a_j$ for some j $(1 \le j \le n)$, then there are two possible products of elements e_{a_1}, a_j, e_{a_n} in that order: $p_1 = e_{a_1} \cdot a_j e_{a_n}$ and $p_2 = e_{a_1} a_j \cdot e_{a_n}$. As e_{a_1} is idempotent, $p_1 = p_2$ by (A1).

($a2$) If $\operatorname{core}(\bar{a}) = a_j a_k$ for some j, k $(1 \le j < k \le n)$, then there are five possible products of elements e_{a_1}, a_j, a_k, e_{a_n} in that order: $q_1 = e_{a_1}(a_j \cdot a_k e_{a_n})$, $q_2 = e_{a_1}(a_j a_k \cdot e_{a_n})$, $q_3 = e_{a_1} a_j \cdot a_k e_{a_n}$, $q_4 = (e_{a_1} \cdot a_j a_k) e_{a_n}$ and $q_5 = (e_{a_1} a_j \cdot a_k) e_{a_n}$. Then $q_1 = q_2$ by (A3), $q_4 = q_5$ by (A1), $q_1 = q_3$ by (A1) and $q_3 = q_5$ by (A3). Therefore all products q_i $(i = 1, \ldots, 5)$ are mutually equal.

(b) Next, we prove $p = e_{a_1} \operatorname{core}(\bar{a}) e_{a_n}$, by induction on the length n of the sequence $\bar{a}$.

($b1$) For $n = 1$ $\operatorname{core}(\bar{a}) = a_1$ and $e_{a_1} = e_{a_n}$. Therefore $p = a_1 = e_{a_1} a_1 e_{a_n} = e_{a_1} \operatorname{core}(\bar{a}) e_{a_n}$.

($b2$) Assume statement (b) true for all products with $< m$ $(m > 1)$ elements and prove it for the product of m elements.

Any product p of elements $a_1, \ldots, a_m$, in that order, is equal to qr where q is some product of $a_1, \ldots, a_s$ $(1 \le s < m)$, in that order, and r is some product of elements $a_{s+1}, \ldots, a_m$ in that order. Denote sequences $a_1, \ldots, a_s$ and $a_{s+1}, \ldots, a_m$ by $\bar{b}$ and $\bar{c}$ respectively. Then, by the induction hypothesis, $q = e_{a_1} \operatorname{core}(\bar{b}) e_{a_s}$ and $r = e_{a_{s+1}} \operatorname{core}(\bar{c}) e_{a_m}$. Therefore $p = qr = e_{a_1} \operatorname{core}(\bar{b}) e_{a_s} \cdot e_{a_{s+1}} \operatorname{core}(\bar{c}) e_{a_m} = e_{a_1} \operatorname{core}(\bar{b}) e_{a_s} \cdot (e_{a_{s+1}} \cdot \operatorname{core}(\bar{c}) e_{a_m}) = (e_{a_1} \operatorname{core}(\bar{b}) \cdot e_{a_s}) \cdot \operatorname{core}(\bar{c}) e_{a_m} = e_{a_1} \operatorname{core}(\bar{b}) \cdot \operatorname{core}(\bar{c}) e_{a_m} = (e_{a_1} \operatorname{core}(\bar{b}) \cdot \operatorname{core}(\bar{c})) e_{a_m} = e_{a_1}(\operatorname{core}(\bar{b}) \operatorname{core}(\bar{c})) e_{a_m}$.
Three cases are possible.

($b2.1$) Assume either $\bar{a}$ has no nonidempotents or they all belong to $\bar{b}$. Then $\operatorname{core}(\bar{a}) = \operatorname{core}(\bar{b}), \operatorname{core}(\bar{c}) = a_{s+1}$ is idempotent and $p = e_{a_1}(\operatorname{core}(\bar{b}) \operatorname{core}(\bar{c})) \cdot e_{a_m} = (e_{a_1} \operatorname{core}(\bar{b}) \cdot a_{s+1}) e_{a_m} = e_{a_1} \operatorname{core}(\bar{a}) e_{a_m}$.

($b2.2$) If the two nonidempotents of $\bar{a}$ belong, one to $\bar{b}$ and the other to $\bar{c}$, then $\operatorname{core}(\bar{a}) = \operatorname{core}(\bar{b}) \operatorname{core}(\bar{c})$ and $p = e_{a_1}(\operatorname{core}(\bar{b}) \operatorname{core}(\bar{c})) e_{a_m} = e_{a_1} \operatorname{core}(\bar{a}) e_{a_m}$.

($b2.3$) If all nonidempotents of $\bar{a}$ belong to $\bar{c}$ then $\operatorname{core}(\bar{b}) = a_1$ is idempotent, $\operatorname{core}(\bar{a}) = \operatorname{core}(\bar{c})$ and $p = e_{a_1}(\operatorname{core}(\bar{b}) \operatorname{core}(\bar{c})) \cdot e_{a_m} = (e_{a_1} a_1 \cdot \operatorname{core}(\bar{a})) e_{a_m} = a_1 \operatorname{core}(\bar{a}) e_{a_m} = e_{a_1} \operatorname{core}(\bar{a}) e_{a_m}$.

By induction, $p = e_{a_1} \operatorname{core}(\bar{a}) e_{a_n}$ for all n.

(c) Finally, we prove that $e_{a_1}\operatorname{core}(\bar a)e_{a_n}$ is equal to the product of at most four elements of the sequence $\bar a$, namely: a_1, nonidempotents of $a_2,\dots,a_{n-1}$ if any (the one with the smaller index first), a_n if $n>1$.

($c1$) If $n=1$ then $\operatorname{core}(\bar a)=a_1$ and $p=e_{a_1}a_1e_{a_n}=a_1e_{a_1}=a_1$.

($c2$) If both a_1 and a_n are idempotents then $p=e_{a_1}\operatorname{core}(\bar a)e_{a_n}=a_1\operatorname{core}(\bar a)a_n$. As $\operatorname{core}(\bar a)$ is by definition the product of at most two elements from $\bar a$, p is the product of at most four elements from $\bar a$. If $\operatorname{core}(\bar a)=a_1$ is idempotent then $p=a_1a_n$ by (A2). In all cases it has the prescribed form.

($c3$) If a_1 is idempotent but a_n is not, then either $\operatorname{core}(\bar a)=a_n$ or $\operatorname{core}(\bar a)=a_ja_n$ $(1<j<n)$ depending on whether there is one or two nonidempotents in $\bar a$. In the first case $p=e_{a_1}\operatorname{core}(\bar a)e_{a_n}=a_1a_ne_{a_n}=a_1a_n$. Otherwise $p=e_{a_1}\operatorname{core}(\bar a)e_{a_n}=a_1a_ja_ne_{a_n}=a_1a_ja_n$.

($c4$) The case where a_1 is nonidempotent and a_n is idempotent is proved analogously to ($c3$).

($c5$) If both a_1 and a_n are nonidempotents then $p=e_{a_1}\operatorname{core}(\bar a)e_{a_n}=e_{a_1}a_1a_ne_{a_n}=a_1a_n$. □

We use the following notation: $\mathcal{L}$ for the class of all left zero semigroups, $\mathcal{R}$ for the class of all right zero semigroups, $\mathcal{B}$ for the class of all rectangular bands, $\mathcal{Q}_1$ for the class of all loops and $\mathcal{H}$ for the class of all rectangular loops.

THEOREM 2.5. *The following conditions for the groupoid S are equivalent:*

(a) S is a rectangular loop

(b) $S\simeq L\times Q\times R$, $L\in\mathcal{L}$, $Q\in\mathcal{Q}_1$, $R\in\mathcal{R}$

(c) $S\simeq B\times Q$, $B\in\mathcal{B}, Q\in\mathcal{Q}_1$

(d) S satisfies (K)

(e) S satisfies $(\Box\Lambda)$.

PROOF. (a) and (b) are equivalent by the definition of rectangular loop. Every rectangular band is isomorphic to the direct product of a left and a right zero semigroup and therefore (b) and (c) are equivalent. As previously noted the equivalence of (b) and (d) follows from the Theorem of Knoebel [**2**].

(b) $\Rightarrow$ (e) requires just tedious checking. As an example we prove $(A2)$.

Let $x,y,z\in S$. Then there are $a,b,c\in L\in\mathcal{L}$, $u,v,w\in Q\in\mathcal{Q}_1$ and $p,q,r\in R\in\mathcal{R}$ such that $x=(a,u,p)$, $y=(b,v,q)$, $z=(c,w,r)$. Therefore $x\cdot(y/y)z=x(((b,v,q)/(b,v,q))\cdot z)=x((b,v/v,q)z)=x((b,1,q)(c,w,r))=(a,u,p)(b,w,r)=(a,uw,r)=(a,u,p)(c,w,r)=xz$.

(e) $\Rightarrow$ (d) is also straightforward. Again we prove just one of the more intimidating formulas – (K12). However, it should be noted that most of the results of theorems 2.1–2.4 are needed to prove (K).

$(((x\backslash x)\cdot t(t/t))\backslash((x\backslash x)\cdot t(t/t)))\cdot((t\backslash t)\cdot x(t/t))(((t\backslash t)\cdot t(x/x))/((t\backslash t)\cdot t(x/x)))=((e_x\cdot te_t)\backslash(e_x\cdot te_t))\cdot(e_t\cdot xe_t)((e_t\cdot te_x)/(e_t\cdot te_x))=e_{e_x\cdot te_t}\cdot x\cdot e_{e_t\cdot te_x}=e_{e_x}e_{te_t}\cdot x\cdot e_{e_t}e_{te_x})=e_x\cdot x\cdot e_te_{e_x}=xe_x=x$. □

DEFINITION 2.3. $\operatorname{head}(t)$ $(\operatorname{tail}(t))$ is the first (last) variable of the term t.

THEOREM 2.6. *$u=v$ is true in all rectangular loops iff* $\operatorname{head}(u)=\operatorname{head}(v)$, $\operatorname{tail}(u)=\operatorname{tail}(v)$ *and $u=v$ is true in all loops.*

Using the result of Evans [**1**] we get:

COROLLARY 2.1. *The word problem for rectangular loops is solvable.*

Also:

COROLLARY 2.2. *If* $u = v$ *is true in all loops then* $e_x u e_y = e_x v e_y$ *is true in all rectangular loops.*

COROLLARY 2.3. *If* $u = v$ *is true in all loops then* $x/(u\backslash x) = (x/v)\backslash x$ *is true in all rectangular loops.*

COROLLARY 2.4. *Let* $*$ *and* $\circ$ *be any of the operations* $\cdot, \backslash, /$. *If* $u = v$ *is true in all loops, then* $x * (u \circ y) = x * (v \circ y)$ *and* $(x * u) \circ y = (x * v) \circ y$ *are true in all rectangular loops.*

THEOREM 2.7. *Let* i *be an idempotent of* S. *Then:*

(a) E *is the greatest subband of* S. *Therefore it is unique and happen to be rectangular.*

(b) There are $|iE|$ *maximal left zero subsemigroups of* S. *They are all isomorphic and of the form* Ee $(e \in E)$.

(c) There are $|Ei|$ *maximal right zero subsemigroups of* S. *They are all isomorphic and of the form* eE $(e \in E)$.

(d) There are $|E|$ *maximal subloops of* S. *They are all isomorphic and for all of them* $Q_e = \{x \in S | e_x = e\} = eS \cap Se = eSe$ $(e \in E)$.

(e) $S \simeq Ei \times iSi \times iE$ *and the isomorphism is given by* $f(x) = (e_x i, ixi, ie_x)$.

DEFINITION 2.4. This is just a reminder of the standard notation.
$\mathrm{Sub}(S)$ is the lattice of subalgebras of S.
$\mathrm{Sub}^0(S)$ is the lattice of subalgebras of S with the empty set added as the smallest element (used when two subalgebras have an empty intersection).
$\mathrm{Con}(S)$ – the lattice of congruences of S.
$\mathrm{Eq}(S)$ – the lattice of equivalences of S.
$\mathrm{Hom}(S, T)$ – the set of homomorphisms from S to T.
$\mathrm{End}(S)$ – the monoid of endomorphisms of S.
$\mathrm{Aut}(S)$ – the group of automorphisms of S.
$\mathrm{Var}(\mathcal{K})$ is the lattice of varieties of a class $\mathcal{K}$ of algebras.

COROLLARY 2.5. *For all* $L, M \in \mathcal{L}$; $Q, Q' \in \mathcal{Q}_1$ *and* $R, N \in \mathcal{R}$:

(a) $\mathrm{Sub}^0(L \times Q \times R) \simeq \mathbf{2}^L \times \mathrm{Sub}(Q) \times \mathbf{2}^R$.

(b) $\mathrm{Con}(L \times Q \times R) \simeq \mathrm{Eq}(L) \times \mathrm{Con}(Q) \times \mathrm{Eq}(R)$.

(c) $\mathrm{Hom}(L \times Q \times R,\, M \times Q' \times N) = M^L \times \mathrm{Hom}(Q, Q') \times N^R$.

(d) $\mathrm{End}(L \times Q \times R) \simeq L^L \times \mathrm{End}(Q) \times R^R$.

(e) $\mathrm{Aut}(L \times Q \times R) \simeq S_{|L|} \times \mathrm{Aut}(Q) \times S_{|R|}$.

COROLLARY 2.6. $\mathrm{FreeRectangularLoop}(n) \simeq L_n \times \mathrm{FreeLoop}(n) \times R_n$.

FreeRectangularLoop(n) and FreeLoop(n) are just what the names indicate: appropriate free structures with n generators. $L_n(R_n)$ is the unique n-element left (right) zero semigroup – which also happen to be free.

COROLLARY 2.7. $\mathrm{Var}(\mathcal{H}) \simeq \mathbf{2} \times \mathrm{Var}(\mathcal{Q}_1) \times \mathbf{2}$.

THEOREM 2.8. *Equation* $ax = b$ *has a solution iff* $e_a b = b$ *and then it has exactly* $|Ee_a|$ *solutions. All are of the form* $e(a\backslash b)$ *for some* $e \in E$ $(e \in Ee_a)$.

Of course, the dual theorem is also true.

3. Independence of axioms

The author proved the independence of several of the axioms of the system $(\Box\Lambda)$. The results were verified by FINDER, a finite model generator program, developed by Slaney [**4**]. However, it is author's belief that the axiom system $(\Box\Lambda)$ is not independent. Therefore:

PROBLEM 1. Give an independent axiom system for the class of all rectangular loops.

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