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NOTE ON A QUESTION OF REINHOLD BAER ON PREGROUPS

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ABSTRACT. Reinhold Baer asked the relationship between certain properties in a pregroup. The paper gives a partial answer to his question.

1. Introduction

Let P be a nonempty set with a partial operation (called an "add" by Baer [11]). Note that a partial operation on P is a mapping $m : D \to P$, where $D \subseteq P \times P$. If $(a, b) \in D$, we will denote $m(a, b)$ by ab and say that ab is *defined* or that *ab exists*.

The *universal group* $G(P)$ of an add P is the group with the following presentation:

$$G(P) = \mathrm{gp}(P; \text{operation } m).$$

That is, P is the set of generators, and the defining relations are of the form $ab = c$, where $m(a, b) = c$. P is said to be *group-embeddable* or, simply, *embeddable*, if P is embedded in $G(P)$.

Next follows two classical examples of embeddable adds.

EXAMPLE 1.1. Let K and H be groups which intersect in a subgroup A. Then the amalgam $P = H \cup_A K$ is group-embeddable where $G(P) = H *_A K$, the free product of H and K with A amalgamated.

EXAMPLE 1.2. Let $T = (H_i, A_{st})$ be a tree graph of groups with vertex groups H_i and with edge groups A_{st}. (Here A_{st} is a subgroup of the vertex groups H_s and H_t.) Let $P = \bigcup_i(H_i; A_{st})$, the amalgam of the groups in T. Then P is an add which is embeddable in $G(P)$, the tree product of the vertex groups H_i with the edge groups A_{st} amalgamated.

Baer [1] and Stallings [11] gave, independently, sets of axioms which guarantee that an add P is embeddable in $G(P)$. In particular, Stallings invented the name "pregroup" for an add P which satisfies the following four axioms:

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- [P_1] (Identity) There exists an element $1 \in P$ such that, for every $a \in P$, $1a$ and $a1$ are defined and $1a = a1 = a$.
- [P_2] (Inverses) For every $a \in P$, there exists $a^{-1} \in P$ such that aa^{-1} and $a^{-1}a$ are defined and $aa^{-1} = a^{-1}a = 1$.
- [A] (Weak Associative Law) If ab and bc are defined, then $a(bc)$ is defined if and only if $(ab)c$ is defined, and, in such a case $a(bc) = (ab)c$. [We then say that the *triple* $abc = (ab)c = a(bc)$ is *defined*.]
- [B] Suppose ab, bc, cd are defined. Then $(ab)c$ or $(bc)d$ is defined.

We state the above result formally.

THEOREM 1.1 (Baer 1950, and Stallings 1971). *A pregroup P is group-embeddable.*

We note that a pregroup is a generalization of the add P in Example 1.1, but not of the add P in Example 1.2.

Axiom [B] was actually incorporated in Baer's Postulate XI [1, page 648] which we state below:

POSTULATE XI. (Consists of three parts):

- (a) If ab, bc, cd exist in P, then $a(bc)$ or $(bc)d$ exist in P.
- (b) If bc, cd and $a(bc)$ exist in P, then ab or $(bc)d$ exists in P.
- (c) If ab, bc and $(bc)d$ exist in P, then $a(bc)$ or cd exists in P.

Baer then states

"In certain instances it is possible to deduce properties (b), (c) from (a);
but whether or not this is true in general, the author does not know."

This paper gives a partial answer to Baer's question. Specifically, consider the following four axioms:

- [B_1] If ab, $(ab)c$, $((ab)c)d$ are defined, then bc or cd is defined.
- [B_2] If cd, $b(cd)$, $a(b(cd))$ are defined, then ab or bc is defined
- [B_3] If bc, cd, $a(bc)$ are defined, then ab or $(bc)d$ is defined.
- [B_4] If ab, bc, $(bc)d$ are defined, then $a(bc)$ or cd is defined.

Observe that, assuming axiom [A], Postulate XI(a) is axiom [B], and that (b) and (c) are, respectively, axioms [B_3] and [B_4].

Our main result follows.

THEOREM 1.2. *Suppose P is an add which satisfies axioms* [P_1], [P_2] *and* [A]. *Then the axioms* [B], [B_1], [B_2], [B_3], [B_4] *are equivalent.*

COROLLARY 1.1. *Suppose P satisfies axioms* [P_1], [P_2], [A] *and one of the axioms* [B_1], [B_2], [B_3], [B_4]. *Then P is embeddable in $G(P)$.*

2. Prees

Suppose an add P satisfies axioms [P_1], [P_2], and [A]. Then P will be called a *pree*. We emphasize that a pree P need not be embeddable in $G(P)$. However, a pree P does not satisfy the following properties (which we prove here for completeness):

- P(i) Suppose ab is defined. Then $a^{-1}(ab)$ is defined and $a^{-1}(ab) = b$. Dually, $(ab)b^{-1}$ is defined and $(ab)b^{-1} = a$

P(ii) Suppose ab is defined. Then $b^{-1}a^{-1}$ is defined and $(ab)^{-1} = b^{-1}a^{-1}$.

Proof of P(i). We have $1b = (a^{-1}a)b$ is defined. Hence $a^{-1}(ab)$ is defined and $a^{-1}(ab) = (a^{-1}a)b = 1b = b$. Dually $a1 = a(bb^{-1})$ is defined. Hence $(ab)b^{-1}$ is defined and $(ab)b^{-1} = a(bb^{-1}) = a1 = a$. □

Proof of P(ii). Let $r = (ab)^{-1}$, $s = (ab)b^{-1} = a$, $t = a^{-1}$. Then $rs = b^{-1}$ and $st = 1$ are defined. Also, $r(st) = r1$ is defined. Hence $(rs)t = b^{-1}a^{-1}$ is defined. Furthermore

$$(ab)^{-1} = r = r(st) = (rs)t = b^{-1}a^{-1}.$$

Thus P(ii) is proved. □

3. Proof of Theorem 1.2

Here we assume that our add P is a pree, that is, that P satisfies axioms [P$_1$], [P$_2$], and [A]. For notational convenience we restate axiom [B] using the letters r, s, t, u instead of a, b, c, d.

[B] If rs, st, tu are defined, then rst or stu is defined.

Lemma 3.1. *[B] and [B$_1$] are equivalent.*

Proof. Suppose [B] holds, and suppose ab, $(ab)c$, $((ab)c)d$ are defined. Let:

$$r = b, \quad s = (ab)^{-1} = b^{-1}a^{-1}, \quad t = (ab)c, \quad u = d.$$

Then rs, st, tu are defined. By axiom [B], $rst = bc$ or $stu = cd$ is defined. Thus [B] implies [B$_1$].

Conversely, suppose [B$_1$] holds, and suppose rs, st, tu are defined. Let:

$$a = (rs)^{-1} = s^{-1}r^{-1}, \quad b = r, \quad c = st, \quad d = u.$$

Then ab, $(ab)c$, $((ab)c)d$ are defined, so the hypothesis of [B$_1$] is satisfied. By axiom [B$_1$], $bc = rst$ or $cd = stu$ is defined. Thus [B$_1$] implies [B]. □

Lemma 3.2. *[B] and [B$_2$] are equivalent.*

Proof. Suppose [B] holds, and suppose cd, $b(cd)$, $a(b(cd))$ are defined. Let:

$$r = a, \quad s = b(cd), \quad t = (cd)^{-1} = d^{-1}c^{-1}, \quad u = c.$$

Then rs, st, tu are defined. By axiom [B], $rst = ab$ or $stu = bc$ is defined. Thus [B] implies [B$_2$].

Conversely, suppose [B$_2$] holds, and suppose rs, st, tu are defined. Let:

$$a = r, \quad b = st, \quad c = u, \quad d = (tu)^{-1} = u^{-1}t^{-1}.$$

Then cd, $b(cd)$, $a(b(cd))$ are defined, so the hypothesis of [B$_2$] is satisfied. By axiom [B$_2$], $ab = rst$ or $bc = stu$ is defined. Thus [B$_2$] implies [B]. □

Lemma 3.3. *[B] and [B$_3$] are equivalent.*

PROOF. Suppose [B] holds, and suppose bc, cd, $a(bc)$ are defined. Let:

$$r = a, \quad s = bc, \quad t = c^{-1}, \quad u = cd.$$

Then rs, st, tu are defined. By axiom [B], $rst = ab$ or $stu = bcd$ is defined. Thus [B] implies [B_3].

Conversely, suppose [B_3] holds, and suppose rs, st, tu are defined. Let:

$$a = r, \quad b = st, \quad c = t^{-1}, \quad d = tu.$$

Then bc, cd, $a(bc)$ are defined, so the hypothesis of [B_3] is satisfied. By axiom [B_3], $ab = rst$ or $(bc)d = stu$ is defined. Thus [B_3] implies [B]. □

LEMMA 3.4. *[B] and [B_4] are equivalent.*

PROOF. Suppose [B] holds, and suppose ab, bc, $(bc)d$ are defined. Let:

$$r = ab, \quad s = b^{-1}, \quad t = bc, \quad u = d.$$

Then rs, st, tu are defined. By axiom [B], $rst = abc$ or $stu = cd$ is defined. Thus [B] implies [B_4].

Conversely, suppose [B_4] holds, and suppose rs, st, tu are defined. Let:

$$a = rs, \quad b = s^{-1}, \quad c = st, \quad d = u.$$

Then ab, bc, $(bc)d$ are defined, so the hypothesis of [B_4] is satisfied. By axiom [B_4], $a(bc) = rst$ or $cd = stu$ is defined. Thus [B_4] implies [B]. □

We have shown that [B], [B_1], [B_2], [B_3], [B_4] are equivalent in a pree P. That is, we have proved Theorem 1.2.

4. Generalizations and Questions

Many authors (e.g. [1]–[8]) have generalized the Baer–Stallings pregroup by giving a weaker sets of axioms which also guarantee that a pre P is embeddable in $G(P)$. Specifically, the following axioms were considered by various authors:

- [S_n, $n \geq 4$] (Baer 1953) Suppose $a_1a_2^{-1}, a_2a_3^{-1}, \ldots, a_na_1^{-1}$, are defined. Then, for some i, $a_ia_{i+2}^{-1}$ (mod n) is defined.
- [T_n] (Kushner and Lipschutz 1988) Suppose a_1a_2, $a_2a_3, \ldots, a_{n+2}a_{n+3}$, are defined. Then, for some i, $(a_ia_{i+1})a_{i+2}$ is defined.
- [K] (Kushner 1988) Suppose ab, bc, cd, and $(ab)(cd)$ are defined. Then $(ab)c$ or $(bc)d$ is defined.
- [L] (Lipschutz 1994) Suppose ab, bc, cd are defined but $(ab)(cd)$, $(ab)c$, $a(bc)$ are undefined. If $(ab)z$ and $z^{-1}(cd)$ are defined, then bz and $z^{-1}c$ are defined.
- [M] (Lipschutz 1994) Suppose X and Y are fully reduced words and $X =_G Y$. Then X and Y have the same length.

Note that [T_1] = [B] and [T_n] implies [T_{n+1}]. We also note that axiom [M] is analogous to the following axiom of Baer [1, page 684]:

"Similar irreducible vectors have the same length."

Moreover, Lipschutz [10] showed that [K] is equivalent to each of the following two axioms:

[K′] Suppose $[x, y]$ is reduced and $y = ab = cd$, where xa and xc are defined. Then $a^{-1}c$ is defined.

[K″] Suppose $W = [x, y, z]$ is reduced. Then W is not reducible to a word of length one.

Furthermore, recently, Hoare [4] showed that [S_4] and [S_5] are equivalent to [K] and [L].

Let Z be a set of axioms. Following Stallings, who invented the names pregroup and S-pregroup, an add P will be called a *Z-pregroup* if P satisfies [P_1], [P_2], [A] and also satisfies the axioms Z.

Besides Theorem 1.1, that a pregroup is embeddable, we have the following results.

THEOREM 4.1. *Each of the following is embeddable:*
(1) *S-pregroup* ([1], Baer 1953);
(2) *KT_2-pregroup* ([6], Kushner and Lipschutz 1988);
(3) *T_2-pregroup* ([5], Kushner 1978 and [3], Hoare 1992);
(4) *KT_3-pregroup* ([7], Kushner and Lipschutz 1993);
(5) *KLM-pregroup* ([8], Lipschutz 1993);
(6) *KL-pregroup* = *S_4S_5-pregroup* ([2], Gilman 1998 and [4], Hoare 1998).

We note that Gilman and Hoare proved (6) independently. In fact, Gilman [2] proved (6) using small cancellation theory, and Hoare [4] proved (6) by showing that [M] follows from [K] and [L]. Lipschutz [9] showed that the KL-pregroups include the S-pregroups which include the adds in Example 1.2.

Observe that the axioms [T_n] are a direct generalization of axiom [B]. Accordingly, the following axioms [T^*] and [T^{**}] are a direct generalization, respectively, of [B_1[and [B_2]:

[T_n^*] If $a_1a_2, (a_1a_2)a_3, \ldots, (\ldots((a_1a_2)a_3)\ldots)a_{n+3}$, are defined, then, for some $i \geq 2$, a_ia_{i+1} is defined.

[T_n^{**}] If $a_{n+2}a_{n+3},\ a_{n+1}(a_{n+2}a_{n+3}), \ldots,\ a_1(\ldots(a_{n+1}(a_{n+2}a_{n+3}))\ldots)$ are defined, then, for some $i \leq n+1$, a_ia_{i+1} is defined.

QUESTION 1. What role, if any, do the axioms [T_n^*] and [T_n^{**}] play in the embedding of a pree P in its universal group?

QUESTION 2. How would one generalize axioms [B_3] and [B_4]. and what role would they play in the embedding of a pree P in its universal group?

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