PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 68(82) (2000), 37–45

GENERALIZED LINE GRAPHS WITH THE SECOND LARGEST EIGENVALUE AT MOST 1

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Communicated by Žarko Mijajlović

Abstract. All connected generalized line graphs whose second largest eigenvalue does not exceed 1 are characterized. Besides, all minimal generalized line graphs with second largest eigenvalue greater than 1 are determined.

1. Introduction

In this paper we consider simple graphs with (0,1) adjacency matrix. The eigenvalues of a graph are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The second largest eigenvalue $\lambda_2(G)$ of a graph G has attracted much attention in literature (see, for example, [3] and [6]).

Graphs with $\lambda_2(G) \leq 1$ have been studied in 1982 by Cvetković [2]. It turned out that some of these graphs are the complements of the graphs whose least eigenvalue is greater than or equal to -2, while, on the other hand, the complement of a graph whose least eigenvalue is not less than -2 always has $\lambda_2 \leq 1$. A representation of graphs with $\lambda_2(G) = 1$ in the Lorentz space is given in 1983 by Neumaier and Seidel [8]. Bipartite graphs with $\lambda_2(G) \leq 1$ have been characterized in 1991 by Petrović [9]. In particular, trees with second largest eigenvalue less than 1 were treated by Neumaier [7]. Line graphs whose second largest eigenvalue does not exceed 1 have been studied in 1998 by Petrović and Milekić [10].

The exact characterization of graphs with second largest eigenvalue around 1 still remains an interesting open question in the spectral theory of graphs.

In this paper we explicitely characterize all connected generalized line graphs with property $\lambda_2(G) \leq 1$. We prove that a connected generalized line graph G has this property if and only if G is an induced subgraph of any of 11 graphs displayed in Fig. 2. We note that one of the mentioned 11 graphs represents in fact a class of graphs.

1991 *Mathematics Subject Classification.* Primary 05C50.
Key words and phrases. Generalized line graphs, minimal generalized line graphs, second largest eigenvalue of graph.

In this paper we also determine all minimal generalized line graphs with the property $\lambda_2(G) > 1$. There are exactly 21 such graphs (see Fig. 3).

Throughout this paper $H \subset G$ will denote that H is an induced subgraph of a graph G.

We now recall to some known classes of graphs.

DEFINITION 1. The cocktail party graph on $2n$ vertices denoted by $CP(n)$, is the regular graph on $2n$ vertices of degree $2n-2$.

DEFINITION 2. A generalized line graph, denoted by $L(H; a_1, \dots, a_n)$, is constructed from a graph H with n vertices $v_1, \dots, v_n$ and nonnegative integers $a_1, \dots,$ a_n in the following way: it consists of disjoint copies of $L(H)$ and $CP(a_i)$ $(i = 1, \dots, n)$, with additional lines joining a vertex in $L(H)$ with a vertex in $CP(a_i)$ if the vertex in $L(H)$ corresponds to a line in H that has v_i as an end point.

Special cases include an ordinary line graph $(a_1 = \cdots = a_n = 0)$ and the cocktail party graph $CP(m)$ $(n = 1$ and $a_1 = m)$.

DEFINITION 3. A generalized cocktail party graph (GCP) is a graph obtained by deletion of independent edges from the complete graph K_n. Any vertex of degree $n-1$ is said to be of l-type, while the other are said to be of a-type.

D. Cvetković, M. Doob and S. Simić characterized generalized line graphs by showing that there are exactly 31 minimal nongeneralized line graphs.

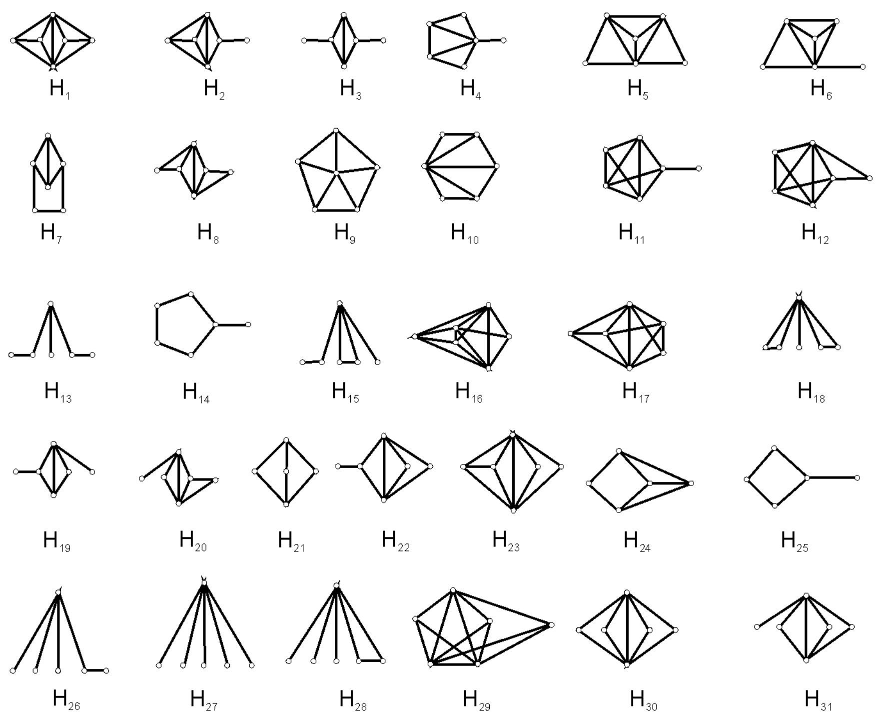

Fig. 1

PROPOSITION 1 ([4] AND [5]). *A graph G is a generalized line graph if and only if it does not contain any of the 31 graphs in Fig. 1 as an induced subgraph.*

2. Main results

Let $F_1, \dots, F_{11}$ denote the generalized line graphs displayed in Fig. 2. Here the line between $\boxed{CP(r)}$ and $\boxed{K_{s+2t}}$ denotes the join of graphs $CP(r)$ and K_{s+2t}, i.e. all posible edges between the graphs $CP(r)$ and K_{s+2t} are present. The graph F_{11} is a graph with $2r + 3s + 3t$ vertices which contains the generalized cocktail party graph (GCP) with $2r$ a-type vertices and $(s + 2t)$ l-type vertices as an induced subgraph ($r \ge 1$, $s \ge 1$, $t \ge 1$).

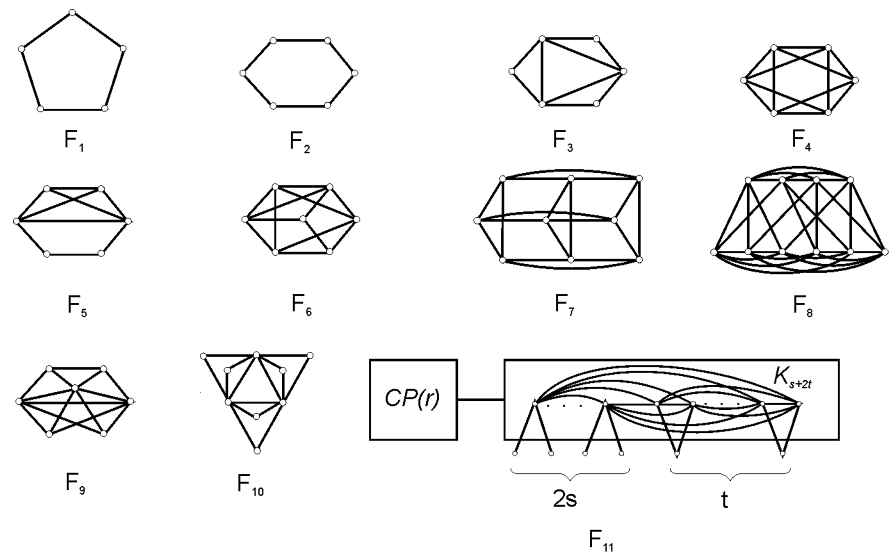

Fig. 2

THEOREM 1. *Graphs F_1–F_{11} from Fig. 2 have the property $\lambda_2(F_i) \le 1$ ($i = 1, \dots, 11$).*

PROOF. We easily get by computer that $\lambda_2(F_i) \le 1$ ($i = 1, \dots, 10$).

Let A be the adjacency matrix of the graph F_{11}, let λ be an eigenvalue of F_{11} distinct from $\pm 1, -2$ and 0, and let $\mathbf{x}$ be an eigenvector of F_{11} belonging to the eigenvalue λ. From equality

$$A\mathbf{x} = \lambda\mathbf{x} \ ,$$

we get that $\mathbf{x} = (\underbrace{x, \dots, x}_{2r}, \underbrace{y, \dots, y}_{s+2t}, \underbrace{z, \dots, z}_{2s}, \underbrace{2z, \dots, 2z}_{t})$ and all eigenvalues of the graph F_{11} distinct from $\pm 1, -2$ and 0 are determined by equation

$$P(\lambda) = C_0\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0 \, , \tag{1}$$

where

$$C_0 = 1\,,$$
$$C_1 = -(2r + s + 2t - 3)\,,$$
$$C_2 = -2(r + s + 2t)\,,$$
$$C_3 = 4(r - 1)\,.$$

For $r = 1$, in the sequence (C_0, C_1, C_2, C_3) there is exactly one sign change, and for $r > 1$ there are exactly two sign changes in this sequence. Since $P(0) = 4(r-1) \geq 0$ and $P(1) = -3(s+2t) < 0$ we conclude that equation (1) has exactly one root greater than 1. It follows that $\lambda_2(F_{11}) \leq 1$. $\square$

In the sequel, we shall determine all connected generalized line graphs G with the property

(2) $$\lambda_2(G) \leq 1\,.$$

The property (2) is hereditary because, whenever G satisfies (2) and $H \subset G$, it follows that H also satisfies (2). The hereditary property (2) implies that there are minimal generalized line graphs that do not satisfy (2); such graphs are called *forbidden subgraphs.*

In the set of all generalized line graphs with at most 7 vertices, there are exactly 21 forbidden subgraphs (18 connected and 3 disconnected); see Fig. 3. Exactly 5 of these graphs are not line graphs: G_2, G_4, G_{10}, G_{13} and G_{18}. We use them in the proofs of Lemmas 2 and 3. The remaining graphs from the set $\{G_1, \ldots, G_{21}\}$ are taken from the results in [10].

Now, let $\mathcal{L}$ denote the set of all connected generalized line graphs G such that G contains as an induced subgraph neither of the graphs G_1–G_{21} in Fig. 3. Clearly, since the complement of a generalized cocktail party graph G is a graph with the least eigenvalue -1 (in fact, it is a line graph), its second largest eigenvalue is less than 1 and it belongs to $\mathcal{L}$. Denote by $\mathcal{L}_0$ the set of all other members of $\mathcal{L}$ distinct from generalized cocktail party graphs.

Let $G = L(H; a_1, \cdots, a_n)$, where $a_1 = \cdots = a_n = 0$. Then G is a line graph and the following lemma holds.

LEMMA 1 [10]. *If $G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0$, $a_1 = \cdots = a_n = 0$, then G is an induced subgraph of some of the graphs F_1–F_8 and F_{11} displayed in Fig. 2.*

Now, let $G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0$, where $V(H) = \{v_1, \ldots, v_n\}$, $a_1 \geq \cdots \geq a_n$ and $a_1 > 0$.

Denote by G_0 generalized cocktail party graph induced by vertices of the graph $CP(a_1)$ and vertices of the graph $L(H)$ which correspond to lines in H that have v_1 as an end point. Let $\{x_1, \ldots, x_m\}$ be the set of all l-type vertices of the graph G_0. Then $m \geq 1$ (in the opposite case we would have that G is disconnected graph, what is a contradiction).

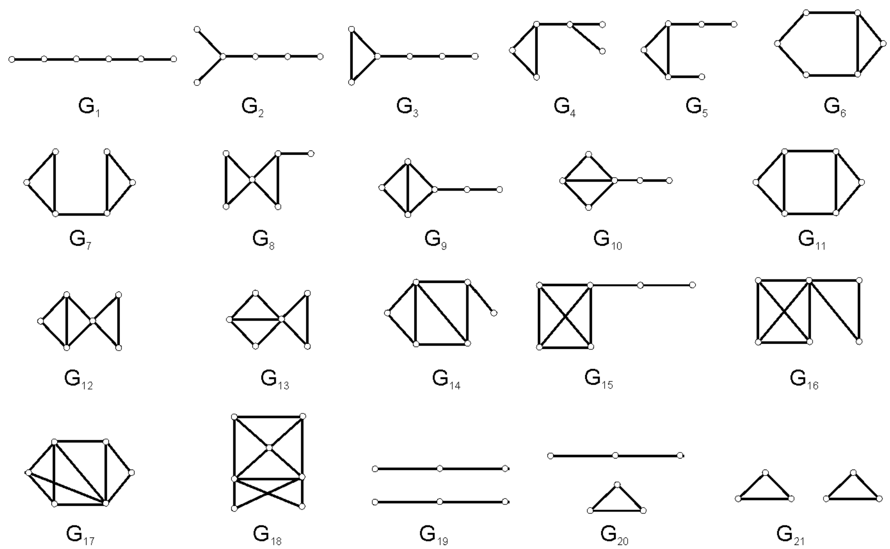

Fig. 3

Denote by T the set $V(G) \setminus V(G_0)$. By Definition 2 we have that the vertices from T are not adjacent to a-type vertices of G_0. Also, they can be adjacent to at most two vertices from the set $\{x_1, \dots, x_m\}$ (in the opposite case we would have $H_{29} \subset G$, what is a contradiction*). Hence we have

$$T = T_0 \cup T_1 \cup T_2 ,$$

where T_0 is the set of vertices which are not adjacent to vertices from $\{x_1, \dots, x_m\}$, T_1 is the set of vertices which are adjacent to exactly one vertex from $\{x_1, \dots, x_m\}$ and T_2 is the set of vertices which are adjacent to exactly two vertices from $\{x_1, \dots, x_m\}$. Also, we have

$$T_1 = T_{x_1} \cup \dots \cup T_{x_m}$$

and

$$T_2 = T_{x_1 x_2} \cup \dots \cup T_{x_{m-1} x_m} ,$$

where T_{x_i} is the set of vertices from T_1 which are adjacent to a vertex x_i, and $T_{x_i x_j}$ is the set of vertices from T_2 which are adjacent to vertices x_i and x_j of the set $\{x_1, \dots, x_m\}$.

*To be short, we shall often reduce the mentioned sentence symply by "$H_{29} \subset G$"

Lemma 2. *If $G = L(H; a_1, \dots, a_n) \in \mathcal{L}_0$, $a_1 \geq \dots \geq a_n$ and $a_1 = 1$, then G is an induced subgraph of some of the graphs F_9,F_{10} and F_{11} displayed in Fig. 2.*

Proof. In the proof we distinguish the following four cases:

$$(A) \quad m = 1\,; \quad (B) \quad m = 2\,; \quad (C) \quad m = 3\,; \quad (D) \quad m \geq 4\,.$$

Case A. In this case we have

$$T = T_0 \cup T_{x_1}\,.$$

Each vertex from the set T_{x_1} can be nonadjacent to at most one vertex from this set ($H_{27} \subset G \vee H_{28} \subset G$). It follows that the graph induced by vertices of the set T_{x_1} is generalized cocktail party graph. Denote by $\{y_1, \dots, y_p\}$ the set of all l-type vertices of this graph.

The vertices from T_0 are not adjacent to a-type vertices of generalized cocktail party graph induced by vertices of T_{x_1} ($H_{25} \subset G \vee H_{26} \subset G$), and they can be adjacent to at most two vertices from the set $\{y_1, \dots, y_p\}$ ($H_{11} \subset G$). Hence we have

$$T_0 = T_0^0 \cup T_1^0 \cup T_2^0\,,$$

where T_0^0 is the set of vertices which are not adjacent to vertices from $\{y_1, \dots, y_p\}$, T_1^0 is the set of vertices which are adjacent to exactly one vertex from $\{y_1, \dots, y_p\}$ and T_2^0 is the set of vertices which are adjacent to exactly two vertices from $\{y_1, \dots, y_p\}$. Also, we have

$$T_1^0 = T_{y_1}^0 \cup \dots \cup T_{y_p}^0$$

and

$$T_2^0 = T_{y_1 y_2}^0 \cup \dots \cup T_{y_{p-1} y_p}^0\,,$$

where $T_{y_i}^0$ is the set of vertices from T_1^0 which are adjacent to a vertex y_i, and $T_{y_i y_j}^0$ is the set of vertices from T_2^0 which are adjacent to vertices y_i and y_j of the set $\{y_1, \dots, y_p\}$.

The vertices of the set T_0 have the following properties:

(1) $T_0^0 = \emptyset$ ($G_2 \subset G$);

(2) The set $T_{y_i}^0$ does not contain adjacent vertices ($G_4 \subset G$) and $|T_{y_i}^0| \leq 2$ ($H_{26} \subset G$);

(3) $|T_{y_i y_j}^0| \leq 1$ ($H_{22} \subset G \vee G_4 \subset G$);

(4) The sets $T_{y_i}^0$ and $T_{y_i y_j}^0$ are not coexistent ($H_{19} \subset G \vee G_4 \subset G$). The sets $T_{y_i y_j}^0$ and $T_{y_i y_k}^0$ are not coexistent, too ($H_{19} \subset G \vee G_4 \subset G$);

(5) The graph which is induced by vertices of the set T_0 is the graph without edges ($G_2 \subset G$).

By properties (1)–(5) we conclude that the graph G is an induced subgraph of the graph F_{11} in Fig. 2.

Case B. In this case we have

$$T = T_0 \cup T_{x_1} \cup T_{x_2} \cup T_{x_1 x_2} .$$

The vertices of the set T have the following properties:

(1) $T_0 = \emptyset$ $(G_{10} \subset G \vee H_{22} \subset G)$;

(2) The set T_{x_i} does not contain adjacent vertices $(G_{13} \subset G)$ and $|T_{x_i}| \leq 2$ $(H_{27} \subset G)$;

(3) $|T_{x_1 x_2}| \leq 1$ $(H_{23} \subset G \vee H_{30} \subset G)$;

(4) If $T_{x_i} \neq \emptyset$ and $T_{x_1 x_2} \neq \emptyset$, then a vertex from the set T_{x_i} is adjacent to a vertex from the set $T_{x_1 x_2}$ $(H_{31} \subset G)$;

(5) If $T_{x_1} \neq \emptyset$ and $T_{x_2} \neq \emptyset$ and if vertices $x \in T_{x_1}$ and $y \in T_{x_2}$ are adjacent, then $|T_{x_1}| = |T_{x_2}| = 1$ $(H_{21} \subset G \vee H_{25} \subset G)$ and $T_{x_1 x_2} = \emptyset$ $(G_{18} \subset G)$.

In view of properties (1)–(5) we have that the graph G is an induced subgraph of the graphs F_9 or F_{10} from Fig. 2.

Case C. The vertices of the set T have the following properties:

(1) $T_0 = \emptyset$ $(G_{10} \subset G \vee H_{22} \subset G)$;

(2) The set T_{x_i} does not contain adjacent vertices $(G_{13} \subset G)$ and $|T_{x_i}| \leq 2$ $(H_{27} \subset G)$;

(3) $|T_{x_i x_j}| \leq 1$ $(H_{23} \subset G \vee H_{30} \subset G)$;

(4) The sets T_{x_i} and $T_{x_i x_j}$ are not coexistent $(G_{10} \subset G \vee H_{31} \subset G)$;

(5) If $|T_2| \leq 1$, then the graph which is induced by vertices of the set T is the graph without edges $(G_{10} \subset G \vee H_{24} \subset G)$;

(6) If $|T_2| > 1$, then the graph which is induced by vertices of the set $T = T_2$ is the complete graph $(H_{31} \subset G)$, and $|T_2| = 2$ $(G_{18} \subset G)$.

In view of properties (1)–(6) we have that the graph G is an induced subgraph of the graphs F_9 or F_{11} from Fig. 2.

Case D. The vertices of the set T have the properties (1)–(4) from *Case C.* The following properties also hold:

(5) The sets $T_{x_i x_j}$ and $T_{x_i x_k}$ are not coexistent $(H_{31} \subset G \vee G_{10} \subset G)$;

(6) The graph which is induced by vertices of the set T is the graph without edges $(G_{10} \subset G \vee H_{22} \subset G \vee H_{24} \subset G)$.

Now using the mentioned properties we find that the graph G is an induced subgraph of the graph F_{11} displayed in Fig. 2. $\square$

LEMMA 3. *If $G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0$, $a_1 \geq \cdots \geq a_n$ and $a_1 > 1$, then G is an induced subgraph of the graph F_{11} displayed in Fig. 2.*

PROOF. The vertices of the set T have the following properties:

(1) $T_0 = \emptyset$ $(G_{10} \subset G)$;

(2) The set T_{x_i} does not contain adjacent vertices $(G_{13} \subset G)$ and $|T_{x_i}| \leq 2$ $(H_{27} \subset G)$;

(3) $|T_{x_i x_j}| \leq 1$ $(H_{23} \subset G \vee H_{30} \subset G)$;

(4) The sets T_{x_i} and $T_{x_i x_j}$ are not coexistent $(H_{31} \subset G \vee G_{10} \subset G)$. The sets $T_{x_i x_j}$ and $T_{x_i x_k}$ are not coexistent, too $(H_{31} \subset G \vee G_{10} \subset G)$;

(5) The graph which is induced by vertices of the set T is the graph without edges ($G_{10} \subset G \vee G_{13} \subset G$).

By properties (1)–(5) we conclude that the graph G is an induced subgraph of the graph F_{11} in Fig. 2. $\Box$

Thus, collecting the former conclusions from Lemmas 1–3, we arrive to the following theorem. We note that a generalized cocktail party graph is an induced subgraph of the graph F_{11}.

THEOREM 2. *If a connected generalized line graph contains as an induced subgraph neither of the graphs G_1–G_{21} in Fig. 3, then G is an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2.*

THEOREM 3. *A connected generalized line graph G has the property $\lambda_2(G) \leq 1$ if and only if G is an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2.*

PROOF. Assume that G is a connected generalized line graph with the property $\lambda_2(G) \leq 1$. Then by the Interlacing theorem (cf. [3, p. 19]) we conclude that G does not contain any of graphs G_1–G_{21} in Fig. 3 as an induced subgraph. In view of Theorem 2, G must be an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2.

Conversely, let a connected generalized line graph G is an induced subgraph of any of the graphs F_1–F_{11} in Fig. 2. Because the property (2) is hereditary and the Theorem 1 holds, we have that $\lambda_2(G) \leq 1$. $\Box$

In the sequel, we shall determine all minimal generalized line graphs with the property $\lambda_2(G) > 1$.

THEOREM 4. *There are exactly 21 minimal generalized line graphs with the property $\lambda_2(G) > 1$. These are the graphs G_1–G_{21} in Fig. 3.*

PROOF. By a straightforward verification one can easily prove that graphs G_1–G_{21} in Fig. 3 are minimal with respect to the property $\lambda_2(G) > 1$. We shall prove that they are only generalized line graphs which are minimal with respect to this property.

Let G be an arbitrary connected generalized line graph which is minimal with respect to the property $\lambda_2(G) > 1$ and which is distinct from the graphs G_1–G_{18}. Then G does not contain any of graphs G_1–G_{21} as an induced subgraph. By Theorem 2 we get that G is an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2. But Theorem 1 and the Interlacing theorem also give $\lambda_2(G) \leq 1$, which is a contradiction. Thus, G_1–G_{18} are the only minimal connected generalized line graphs with the property $\lambda_2(G) > 1$.

Now assume that G is an arbitrary disconnected generalized line graph which is minimal with respect to the property $\lambda_2(G) > 1$. Then G has no isolated vertices and it has exactly two connected components E_1 and E_2, where $\lambda_1(E_1) > 1$ and $\lambda_1(E_2) > 1$. Hence, we have that graphs E_1 and E_2 contain the graph P_3 or the graph K_3 as an induced subgraph and $P_3 \cup P_3 \subset G$ or $P_3 \cup K_3 \subset G$ or $K_3 \cup K_3 \subset G$. So we get that G_{19}–G_{21} are the only minimal disconnected generalized line graphs with the property $\lambda_2(G) > 1$. $\Box$

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(Received 02 02 1999)
(Revised 02 02 2000)