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FINITE DIFFERENCE SCHEMES ON NONUNIFORM MESHES FOR PARABOLIC PROBLEMS WITH GENERALIZED SOLUTIONS

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Abstract. We investigate the convergence of finite difference schemes for one dimensional heat conduction equation on nonuniform rectangular meshes. For schemes with averaged right hand sides convergence rate estimates consistent with the smoothness of the solution in discrete L_2 norm are obtained. Possible extensions of obtained results are noted.

Introduction

Nonuniform meshes are often used for approximation of problems with generalized solutions. In this case, the order of local error is usually reduced.

In many papers it is shown that the accuracy of the method can be increased using approximation of the considered differential equation in some non–mesh points (see e.g. [**1**], [**5**], [**6**]). In [**15**] for one dimensional heat conduction equation on a nonuniform in space variable rectangular mesh finite difference schemes (FDSs) of second order accuracy on x are constructed. The convergence of these schemes in discrete C–norm is proved under some restrictions on the step sizes of the mesh. Analogous results are obtained for hyperbolic problems. In [**16**] similar results for the Poisson equation are obtained.

Clearly, the employment of strong norms (e.g. C^k) in the proofs of convergence involves supplementary restrictions on the smoothness of solutions of boundary value problems, restricting the class of admissible problems. Notice that the first results on the convergence of discrete methods for the problems with solutions from

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Sobolev spaces were obtained in finite element method theory framework [**3**]. However, the ways of construction of difference schemes and methods for establishing the convergence rate estimates for the finite difference methods are essentially different from those for finite element methods (see [**12**], [**13**]). In particular, so called convergence rate estimates consistent with the smoothness of data are of the major interest [**10**]. In the case of uniform meshes such estimates are obtained for a large class of boundary value problems (see bibliographies in [**9**] and [**14**]). Estimates of this type are usually based on the application of the Bramble–Hilbert lemma [**2**], [**4**].

In present paper the convergence of FDSs defined in [**15**] is proved in discrete L_2–norm assuming that the generalized solution of the considered initial boundary value problem (IBVP) belongs to the corresponding Sobolev space. Obtained convergence rate estimates are consistent with the smoothness of data. These estimates are obtained by direct estimation of the truncation error, using the interpolation theory of function spaces. In such a way, application of the Bramble-Hilbert lemma, which usually involves unnecessary restriction on the mesh step sizes is avoided. Contrary to FDSs considered in [**15**], where the right hand sides of equations are taken in some intermediate non–mesh points, we replace the right hand sides with some averaged values. This is necessary because in the problems with generalized solutions, the right hand sides of equations may be discontinuous functions.

Preliminaries and notations.

Let us consider the first initial–boundary value problem for the heat conduction equation

$$\begin{gathered}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,\,t)\,, \qquad (x,\,t)\in Q=(0,\,l)\times(0,\,T)\\ u(x,\,0)=u_0(x)\,, \qquad 0<x<l\,,\\ u(0,\,t)=u(l,\,t)=0\,, \qquad 0<t<T\,.\end{gathered} \tag{1}$$

In the domain $\bar{Q}$ we define the mesh $\bar{Q}_{h\tau}=\hat{\bar{\omega}}\times\bar{\omega}_\tau$, where

$$\hat{\bar{\omega}}=\big\{x=x_i=x_{i-1}+h_i\,,\ \ i=1,\,2,\,\dots\,,\,n-1\,,\ \ x_0=0\,,\ x_n=l\big\}\,, \qquad \sum_{i=1}^{n}h_i=l\,,$$

is nonuniform mesh on $[0,\,l]$ and $\bar{\omega}_\tau$ – uniform mesh on $[0,\,T]$ with the step size $\tau=T/m$. We assume that

$$\frac{1}{c_1}\le\frac{h_{i+1}}{h_i}\le c_1\,, \tag{2}$$

where c_1 is a positive constant.

Denote

$$\hat{\omega}=\hat{\bar{\omega}}\cap(0,\,l),\ \ \hat{\omega}^+=\hat{\bar{\omega}}\cap(0,\,l],\ \ \omega_\tau^+=\bar{\omega}_\tau\cap(0,\,T],\ \ Q_{h\tau}=\hat{\omega}\times\omega_\tau^+,\ \ Q_{h\tau}^0=\hat{\omega}\times\bar{\omega}_\tau.$$

In the sequel we shall use the notation

$$x = x_i\,,\quad x_\pm = x_{i\pm 1}\,,\quad t = t_j = j\,\tau\,,\quad h = h_i\,,\quad h_\pm = h_{i\pm 1}\,,\quad \hbar = (h + h_+)/2\,,$$
$$v = v(x,\,t)\,,\quad v_\pm = v(x_\pm,\,t)\,,\quad \hat v = v(x,\,t+\tau)\,,\quad \check v = v(x,\,t-\tau)\,.$$

We introduce finite differences in a standard way (see [**12**])

$$v_x = \big(v_+ - v\big)/h_+\,,\qquad v_{\bar x} = \big(v - v_-\big)/h\,,\qquad v_{\hat x} = \big(v_+ - v\big)/\hbar\,,$$
$$v_t = (\hat v - v)/\tau\,,\qquad v_{\bar t} = (v - \check v)/\tau\,,$$

Let us introduce the discrete inner products

$$(u,\,v)_* = \sum_{x\in\hat\omega} u\,v\,\hbar\,,\qquad (u,\,v] = \sum_{x\in\hat\omega^+} u\,v\,h\,,$$

and the norms

$$\|v\|^2_{L_2(\hat\omega)} = \|v\|^2_* = (v,\,v)_*\,,\qquad \|v]|^2 = (v,\,v]\,,\qquad \|v\|^2_{L_2(Q_{h\tau})} = \tau \sum_{t\in\omega_\tau^+} \|v(\,\cdot\,,\,t)\|^2_*\,.$$

The following assertion holds true.

Lemma 1. *[**13**] For arbitrary mesh functions $v(x)$ and $w(x)$, defined on $\hat{\bar\omega}$ which vanish for $x = 0,\,l$, the following relations hold*

$$(v_{\hat x},\,w)_* = -(v,\,w_{\bar x}]\,,$$
$$(v_{\bar x\hat x},\,w)_* = -(v_x,\,w_{\bar x}]\,.$$

Let us define the Steklov smoothing operator on variable t

$$T_{\bar t} f(x,\,t) = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,\,t')\,dt'\,,$$

and asymmetric Steklov smoothing operators on x (for $x\in\hat\omega$):

$$T_1 f(x,\,t) = \frac{1}{\hbar}\int_{x-h/2}^{x+h_+/2} f(x',\,t)\,dx'\,,\qquad T_2 f(x,\,t) = \frac{1}{\hbar}\int_{x_-}^{x_+} K(x')\,f(x',\,t)\,dx'\,,$$

where

$$K(x') = \begin{cases} 1 + \frac{x'-x}{h}\,, & x'\in(x_-,\,x) \\ 1 - \frac{x'-x}{h_+}\,, & x'\in(x,\,x_+)\,. \end{cases}$$

Notice that in the mesh nodes: $T_{\bar t}\dfrac{\partial u}{\partial t} = u_{\bar t}$ and $T_2\dfrac{\partial^2 u}{\partial x^2} = u_{\bar x\hat x}\,.$

As usual, let $W_2^s(\Omega)$ be the Sobolev space in Ω. Following [**11**], we also define spaces $W_2^s\big((0,\,T);\,W_2^r(\Omega)\big)$ and anisotropic Sobolev spaces in Q: $W_2^{s,\,r}(Q) = W_2^0\big((0,\,T);\,W_2^s(0,\,l)\big)\cap W_2^r\big((0,\,T);\,W_2^0(0,\,l)\big)$.

The following assertion holds true (comp. [**11**]).

LEMMA 2. *If* $a \in W_2^\sigma(0,\,1)$, $0.5 < \sigma < 1$ *and* $a(0) = 0$, *then*

$$\Big(\int_0^1 \xi^{-2\sigma}\, a^2(\xi)\, d\xi\Big)^{1/2} \le \frac{2\,\sigma+1}{2\,\sigma-1}\, |a|_{W_2^\sigma(0,\,1)}\,.$$

In the sequel we need some results of interpolation theory of function spaces. Let X and Y be two Hilbert spaces, X dense in Y, and continuously imbedded. Following [**11**], let us introduce interpolation space $[X,\,Y]_\theta$. The following assertions hold true (see [**11**]).

LEMMA 3. *For* $u \in X$ *and* $0 < \theta < 1$ *the following inequality holds*

$$\|u\|_{[X,\,Y]_\theta} \le C_\theta\, \|u\|_X^{1-\theta}\, \|u\|_Y^\theta\,.$$

LEMMA 4. *If* $s_1,\, s_2,\, r_1,\, r_2 \ge 0$ *and* $0 < \theta < 1$ *then*

$$\big[W_2^{s_1}\big((0,T);\, W_2^{r_1}(0,l)\big),\, W_2^{s_2}\big((0,T);\, W_2^{r_2}(0,l)\big)\big]_\theta = W_2^{(1-\theta)\,s_1+\theta\,s_2}\big((0,T);\, W_2^{(1-\theta)\,r_1+\theta\,r_2}(0,l)\big)\,.$$

In the following, by C and C_i we shall denote positive generic constants independent of mesh step sizes.

Divergent scheme.

We approximate the problem (1) by implicit FDS

$$(3)\qquad \begin{gathered} v_{\bar t} + \Big(\frac{h^2}{6}\, v_{\bar t \bar x}\Big)_{\hat x} = v_{\bar x \hat x} + T_2 T_{\bar t} f \quad \text{in} \quad Q_{h\tau}\,; \\ v = T_2 u_0 - \Big(\frac{h^2}{6}\, u_{0,\,\bar x}\Big)_{\hat x} \quad \text{for} \quad t = 0\,; \qquad v = 0 \quad \text{for} \quad x = 0,\, l\,. \end{gathered}$$

The FDS (3) is proposed in [**15**] for problems with smooth solutions. The value of the right hand side is taken in an intermediate point $(\bar x,\, t)$, where $\bar x = x + (h_+ - h)/3$. In our case the right hand side may be discontinuous and we must take its averaged value. Notice that FDS (3) is exactly the scheme produced by the finite element method for linear elements (on variable x).

The error $z = u - v$ satisfies the following conditions

$$(4)\qquad \begin{gathered} z_{\bar t} + \Big(\frac{h^2}{6}\, z_{\bar t \bar x}\Big)_{\hat x} = z_{\bar x \hat x} + \varphi_{\bar x \hat x} + \psi_{\bar t} \ \text{ in } \ Q_{h\tau}\,; \\ z = \psi(\,\cdot\,,\,0) \ \text{ for } \ t = 0\,; \qquad z = 0 \ \text{ for } \ x = 0,\, l\,, \end{gathered}$$

where

$$\varphi = T_{\bar t} u - u \qquad \text{and} \qquad \psi = u - T_2 u + \Big(\frac{h^2}{6}\, u_{\bar x}\Big)_{\hat x}\,.$$

To derive the a priori estimate in discrete L_2–norm, let us set $z = z^{(1)} + z^{(2)}$ where $z^{(1)}$ and $z^{(2)}$ are the solutions of the following FDSs

$$
\begin{gathered}
z^{(1)}_{\bar t} + \Big(\frac{h^2}{6}\, z^{(1)}_{\bar t\bar x}\Big)_{\hat x} = z^{(1)}_{\bar x\hat x} + \varphi_{\bar x\hat x} \quad \text{in} \quad Q_{h\tau}\,; \\
z^{(1)} = 0 \quad \text{for} \quad t = 0\,; \qquad z^{(1)} = 0 \quad \text{for} \quad x = 0,\, l\,,
\end{gathered} \tag{5}
$$

and

$$
\begin{gathered}
z^{(2)}_{\bar t} + \Big(\frac{h^2}{6}\, z^{(2)}_{\bar t\bar x}\Big)_{\hat x} = z^{(2)}_{\bar x\hat x} + \psi_{\bar t} \ \ \text{in} \ \ Q_{h\tau}\,; \\
z^{(2)} = \psi(\,\cdot\,,0) \ \ \text{for} \ \ t = 0\,; \qquad z^{(2)} = 0 \ \ \text{for} \ \ x = 0,\, l\,.
\end{gathered} \tag{6}
$$

Let ζ be a mesh function satisfying

$$
-\zeta_{\bar x\hat x} = z^{(1)} + \Big(\frac{h^2}{6}\, z^{(1)}_{\bar x}\Big)_{\hat x} \quad \text{in} \quad Q^0_{h\tau}\,; \qquad \zeta = 0 \quad \text{for} \quad x = 0,\, l\,.
$$

Multiplying (5) by ζ and summing over the mesh $\hat\omega$, we immediately obtain

$$
\begin{gathered}
\frac{\tau}{2}\,\|\zeta_{\bar x\bar t}]|^2 + \frac{1}{2\,\tau}\,\big(\|\zeta_{\bar x}]|^2 - \|\check\zeta_{\bar x}]|^2\big) + \|z^{(1)}\|^2_* \\
= \frac{1}{6}\,\|h\, z^{(1)}_{\bar x}]|^2 + \frac{1}{6}\,(h\, z^{(1)}_{\bar x},\, h\,\varphi_{\bar x}] - (\varphi,\, z^{(1)})_*\,.
\end{gathered} \tag{7}
$$

Using the inequality

$$
\|h\, z_{\bar x}]|^2 = \sum_{x\in\hat\omega^+} (z - z_-)^2\, h \le 4\,\|z\|^2_*
$$

and ε–inequality, from (7) we obtain

$$
\begin{aligned}
&\frac{\tau}{2}\,\|\zeta_{\bar x\bar t}]|^2 + \frac{1}{2\,\tau}\,\big(\|\zeta_{\bar x}]|^2 - \|\check\zeta_{\bar x}]|^2\big) + \|z^{(1)}\|^2_* \\
&\qquad \le \frac{2}{3}\,\|z^{(1)}\|^2_* + \frac{4}{12}\,\Big(\varepsilon\,\|z^{(1)}\|^2_* + \frac{1}{\varepsilon}\,\|\varphi\|^2_*\Big) + \frac{1}{2}\,\Big(\varepsilon\,\|z^{(1)}\|^2_* + \frac{1}{\varepsilon}\,\|\varphi\|^2_*\Big)\,.
\end{aligned} \tag{8}
$$

From (8), summing over the mesh ω^+_τ, we obtain

$$
\Big(\frac{1}{3} - \frac{5\,\varepsilon}{6}\Big)\,\|z^{(1)}\|^2_{L_2(Q_{h\tau})} \le \frac{5}{6\,\varepsilon}\,\|\varphi\|^2_{L_2(Q_{h\tau})}\,,
$$

wherefrom, for $0 < \varepsilon < 2/5$, we obtain the estimate

$$
\|z^{(1)}\|_{L_2(Q_{h\tau})} \le C\,\|\varphi\|_{L_2(Q_{h\tau})}\,. \tag{9}
$$

Let us now estimate $z^{(2)}$. We multiply (6) in scalar way by η, where

$$-\eta_t = z^{(2)} \quad \text{in} \quad Q^0_{h\tau}\,; \qquad \eta = 0 \quad \text{for} \quad t = T+\tau\,.$$

Applying partial summing on t, we obtain

$$-\tau\sum_{t=\tau}^{T}(z^{(2)},\eta_t)_* - \tau\sum_{t=\tau}^{T}\Big(\Big(\frac{h^2}{6}\,z^{(2)}_{\bar x}\Big)_{\hat x},\eta_t\Big)_* - \Big(\Big(\frac{h^2}{6}\,\psi_{\bar x}(\,\cdot\,,0)\Big)_{\hat x},\eta(\,\cdot\,,\tau)\Big)_*$$
$$= \tau\sum_{t=\tau}^{T}(z^{(2)}_{\bar x\hat x},\,\eta)_* - \tau\sum_{t=\tau}^{T}(\psi,\,\eta_t)_*\,.$$

Further

$$(z^{(2)}_{\bar x\hat x},\,\eta)_* = -(z^{(2)}_{\bar x},\eta_{\bar x}] = (\eta_{\bar x t},\eta_{\bar x}] = \frac{1}{2\,\tau}\,(\|\hat\eta_{\bar x}]|^2 - \|\eta_{\bar x}]|^2) - \frac{\tau}{2}\,\|\eta_{\bar x t}]|^2$$
$$= \frac{1}{2\,\tau}\,(\|\hat\eta_{\bar x}]|^2 - \|\eta_{\bar x}]|^2) - \frac{\tau}{2}\,\|z^{(2)}_{\bar x}]|^2\,,$$

$$\Big(\Big(\frac{h^2}{6}\,z^{(2)}_{\bar x}\Big)_{\hat x},\eta_t\Big)_* = -\Big(\Big(\frac{h^2}{6}\,z^{(2)}_{\bar x}\Big)_{\hat x},z^{(2)}\Big)_* = \frac{1}{6}\,\|h\,z^{(2)}_{\bar x}]|^2 \le \frac{2}{3}\,\|z^{(2)}\|^2_*\,,$$

and

$$\Big(\Big(\frac{h^2}{6}\,\psi_{\bar x}(\,\cdot\,,0)\Big)_{\hat x},\eta(\,\cdot\,,\tau)\Big)_* = -\Big(\frac{h^2}{6}\,\psi_{\bar x}(\,\cdot\,,0),\eta_{\bar x}(\,\cdot\,,\tau)\Big]$$
$$\le \frac{1}{2}\,\|\eta_{\bar x}(\,\cdot\,,\tau)]|^2 + \frac{1}{2}\,\Big\|\frac{h^2}{6}\,\psi_{\bar x}(\,\cdot\,,0)\Big]\Big|^2 \le \frac{1}{2}\,\|\eta_{\bar x}(\,\cdot\,,\tau)]|^2 + \frac{h^2_{\max}}{72}\,\|h\,\psi_{\bar x}(\,\cdot\,,0)]|^2$$
$$\le \frac{1}{2}\,\|\eta_{\bar x}(\,\cdot\,,\tau)]|^2 + \frac{h^2_{\max}}{18}\,\|\psi(\,\cdot\,,0)\|^2_*\,.$$

It follows

$$\tau\sum_{t=\tau}^{T}\|z^{(2)}\|^2_* + \frac{1}{2}\,\|\eta_{\bar x}(\,\cdot\,,\tau)]|^2 + \frac{\tau^2}{2}\sum_{t=\tau}^{T}\|z^{(2)}_{\bar x}]|^2$$
$$(10)\qquad = \tau\sum_{t=\tau}^{T}(\psi,\,z^{(2)})_* + \frac{\tau}{6}\sum_{t=\tau}^{T}\|h\,z^{(2)}_{\bar x}]|^2 - \Big(\frac{h^2}{6}\,\psi_{\bar x}(\,\cdot\,,0),\eta_{\bar x}(\,\cdot\,,\tau)\Big]$$
$$\le \Big(\varepsilon+\frac{2}{3}\Big)\,\tau\sum_{t=0}^{T}\|z^{(2)}\|^2_* + \frac{1}{4\,\varepsilon}\,\tau\sum_{t=0}^{T}\|\psi\|^2_* + \frac{1}{2}\,\|\eta_{\bar x}(\,\cdot\,,\tau)]|^2 + \frac{h^2_{\max}}{18}\,\|\psi(\,\cdot\,,0)\|^2_*\,.$$

From (10) for $0<\varepsilon<1/3$ it follows

$$(11)\qquad \|z^{(2)}\|_{L_2(Q_{h\tau})} \le C\,\big[\|\psi\|_{L_2(Q_{h\tau})} + h_{\max}\,\|\psi(\,\cdot\,,0)\|_{L_2(\hat\omega)}\big]\,.$$

From (9) and (11) we obtain the desired a priori estimate

$$\|z\|_{L_2(Q_{h\tau})} \le C\left[\|\varphi\|_{L_2(Q_{h\tau})} + \|\psi\|_{L_2(Q_{h\tau})} + h_{\max}\,\|\psi(\,\cdot\,,0)\|_{L_2(\hat\omega)}\right]. \tag{12}$$

In such a manner, to obtain the convergence rate estimate of FDS (3), we need to estimate the right hand side terms in (12). Let us set

$$\varphi = \varphi_1 + \varphi_2 = T_1(T_{\bar t}u - u) + \left[(T_{\bar t}u - u) - T_1(T_{\bar t}u - u)\right].$$

Further

$$\varphi_1(x,\,t) = \frac{1}{\hbar}\int_{x-h/2}^{x+h_+/2}\left[\frac{1}{\tau}\int_{t-\tau}^{t} u(x',\,t')\,dt' - u(x',\,t)\right]dx'$$

$$= \frac{1}{\hbar\tau}\int\limits_{x-h/2}^{x+h_+/2}\int\limits_{t-\tau}^{t}\left[u(x',\,t') - u(x',\,t)\right]dt'\,dx' = \frac{1}{\hbar\tau}\int\limits_{x-h/2}^{x+h_+/2}\int\limits_{t-\tau}^{t}\int\limits_{t}^{t'}\frac{\partial u(x',\,t'')}{\partial t}\,dt''\,dt'\,dx'$$

Estimating integrals in the right hand side by Cauchy–Schwartz inequality, we obtain

$$|\varphi_1(x,\,t)| \le \frac{\tau}{\sqrt{\hbar\tau}}\left\|\frac{\partial u}{\partial t}\right\|_{L_2(e)}, \qquad \text{where} \qquad e = (x - h/2,\, x + h_+/2)\times(t-\tau,\,t).$$

Summing over the mesh $Q_{h\tau}$, we obtain:

$$\|\varphi_1\|_{L_2(Q_{h\tau})} \le \tau\left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q)}. \tag{13}$$

To estimate φ_2 let us set $\alpha(t) = T_1u - u$. For $1/2 < \sigma < 1$, from lemma 2 follows

$$|\varphi_2(x,\,t)| = |\alpha - T_{\bar t}\alpha| = \left|\frac{1}{\tau}\int_{t-\tau}^{t}\left[\alpha(t) - \alpha(t')\right]dt'\right| \le$$

$$\le \frac{1}{\tau}\left\{\int_{t-\tau}^{t}(t-t')^{-2\sigma}\left[\alpha(t) - \alpha(t')\right]^2 dt'\right\}^{1/2}\left\{\int_{t-\tau}^{t}(t-t')^{2\sigma}\,dt'\right\}^{1/2}$$

$$\le \frac{\sqrt{2\sigma+1}}{2\sigma-1}\,\tau^{\sigma-1/2}\left(\int_{t-\tau}^{t}\int_{t-\tau}^{t}\frac{|\alpha(t') - \alpha(t'')|^2}{|t'-t''|^{1+2\sigma}}\,dt'\,dt''\right)^{1/2}.$$

Further

$$|\alpha(t') - \alpha(t'')| = \left|\frac{1}{\hbar}\int_{x-h/2}^{x+h_+/2}\left[u(x',\,t') - u(x,\,t') - u(x',\,t'') + u(x,\,t'')\right]dx'\right|$$

$$\le \frac{1}{\hbar}\int_{x-h/2}^{x+h_+/2}|\beta(x') - \beta(x)|\,dx',$$

where

$$\beta(x) = u(x,\, t') - u(x,\, t'')\,.$$

Estimating $\beta(x) - \beta(x')$ in an analogous manner, we finally obtain

$$|\varphi_2(x,\, t)| \le \frac{C\,\hbar^{\varrho-1/2}\,\tau^{\sigma-1/2}}{(\varrho-1/2)\,(\sigma-1/2)}\,|u|_{(\varrho,\,\sigma);\,e}\,,$$

where $|u|^2_{(\varrho,\,\sigma);\,e} =$

$$= \int\limits_{x-h/2}^{x+h_+/2} \int\limits_{x-h/2}^{x+h_+/2} \int\limits_{t-\tau}^{t} \int\limits_{t-\tau}^{t} \frac{|u(x',t') - u(x'',t') - u(x',t'') + u(x'',t'')|^2}{|x'-x''|^{1+2\varrho}\,|t'-t''|^{1+2\sigma}}\,dt'\,dt''\,dx'\,dx''$$

and $1/2 < \varrho,\, \sigma < 1$. Summing over the mesh we obtain

$$\|\varphi_2\|_{L_2(Q_{h\tau})} \le C\,h^{\varrho}_{\max}\,\tau^{\sigma}\,|u|^2_{(\varrho,\,\sigma);\,Q}\,. \tag{14}$$

Let us choose $\sigma \in (1/2,\, 3/4)$ and set $\varrho = 2\,(1-\sigma)$. It follows that $\varrho + 2\,\sigma = 2$ and $h^{\varrho}_{\max}\,\tau^{\sigma} \le h^2_{\max} + \tau$. Using lemma 4, we obtain

$$\big[W_2^0\big((0,T);\, W_2^2(0,l)\big),\, W_2^1\big((0,T);\, W_2^0(0,l)\big)\big]_{\sigma} = W_2^{\sigma}\big((0,T);\, W_2^{2-2\,\sigma}(0,l)\big)\,.$$

Finally, accordingly to lemma 3

$$\begin{aligned} |u|_{(2-2\,\sigma,\,\sigma);\,Q} &\le \|u\|_{W_2^{\sigma}\left((0,T);\, W_2^{2-2\,\sigma}(0,l)\right)} \\ &\le C\,\|u\|^{1-\sigma}_{W_2^0\left((0,T);\, W_2^2(0,l)\right)}\,\|u\|^{\sigma}_{W_2^1\left((0,T);\, W_2^0(0,l)\right)} \\ &\le C\left(\|u\|^2_{W_2^0\left((0,T);\, W_2^2(0,l)\right)} + \|u\|^2_{W_2^1\left((0,T);\, W_2^0(0,l)\right)}\right)^{1/2} = C\,\|u\|_{W_2^{2,1}(Q)}\,. \end{aligned} \tag{15}$$

In such a way, from (13–15) follows

$$\|\varphi\|_{L_2(Q_{h\tau})} \le C\,(h^2_{\max} + \tau)\,\|u\|_{W_2^{2,1}(Q)}\,. \tag{16}$$

To estimate the term ψ, let us set

$$\begin{aligned} \psi = \psi_1 + \psi_2 &= T_{\bar t}\Big[u - T_2 u + \Big(\frac{h^2}{6}\,u_{\bar x}\Big)_{\hat x}\Big] \\ &+ \Big\{\Big[u - T_2 u + \Big(\frac{h^2}{6}\,u_{\bar x}\Big)_{\hat x}\Big] - T_{\bar t}\Big[u - T_2 u + \Big(\frac{h^2}{6}\,u_{\bar x}\Big)_{\hat x}\Big]\Big\}\,. \end{aligned}$$

Further

$$\psi_2 = -(T_{\bar t}u - u) + T_2(T_{\bar t}u - u) - \Big[\frac{h^2}{6}\,(T_{\bar t}u - u)_{\bar x}\Big]_{\hat x} = -\varphi + T_2\varphi - \Big(\frac{h^2}{6}\,\varphi_{\bar x}\Big)_{\hat x}\,,$$

and, using estimate (16), we immediately obtain

$$\|\psi_2\|_{L_2(Q_{h\tau})} \le C\,(h_{\max}^2+\tau)\,\|u\|_{W_2^{2,1}(Q)}\,. \tag{17}$$

The term ψ_1 can be presented in the form

$$\psi_1(x,\,t) = \frac{1}{\hbar\tau}\int_{x_-}^{x_+}\int_{x'}^{x}\int_{x}^{x''}\int_{t-\tau}^{t} K(x')\,\frac{\partial^2 u(x''',\,t')}{\partial x^2}\,dt'\,dx'''\,dx''\,dx'$$

$$+\frac{h}{6\,\hbar\tau}\int_{x_-}^{x}\int_{x'}^{x}\int_{t-\tau}^{t}\frac{\partial^2 u(x'',\,t')}{\partial x^2}\,dt'\,dx''\,dx' + \frac{h_+}{6\,\hbar\tau}\int_{x}^{x_+}\int_{x}^{x'}\int_{t-\tau}^{t}\frac{\partial^2 u(x'',\,t')}{\partial x^2}\,dt'\,dx''\,dx'\,.$$

Estimating integrals at the right hand side by the Cauchy–Schwartz inequality, one obtains

$$|\psi_1(x,\,t)| \le \frac{C\,h_{\max}^2}{\sqrt{\hbar\tau}}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(e')}, \qquad \text{where} \qquad e' = (x_-,\,x_+)\times(t-\tau,\,t)\,.$$

Summing over the mesh $Q_{h\tau}$, we obtain:

$$\|\psi_1\|_{L_2(Q_{h\tau})} \le C\,h_{\max}^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(Q)}. \tag{18}$$

The term $\psi(x,\,0)$ can be presented in the form

$$\psi(x,\,0) = \frac{1}{\hbar}\int_{x_-}^{x_+}\int_{x'}^{x} K(x')\,u_0{}'(x'')\,dx''\,dx' + \frac{h_+}{6\,\hbar}\int_{x}^{x_+} u_0{}'(x')\,dx' - \frac{h}{6\,\hbar}\int_{x_-}^{x} u_0{}'(x')\,dx'\,,$$

wherefrom follows

$$|\psi(x,\,0)| \le C\sqrt{\hbar}\,\|u_0{}'\|_{L_2(i)}\,, \qquad i = (x_-,\,x_+)$$

and by summing over the mesh $\hat\omega$

$$\|\psi(\,\cdot\,,0)\|_{L_2(\hat\omega)} \le C\,h_{\max}\,\|u_0\|_{W_2^1(0,l)} \le C\,h_{\max}\,\|u\|_{W_2^{2,1}(Q)}\,. \tag{19}$$

From (12) and (16–19) we obtain the desired convergence rate estimate for FDS (3)

$$\|u-v\|_{L_2(Q_{h\tau})} \le C\,(h_{\max}^2+\tau)\,\|u\|_{W_2^{2,1}(Q)}\,. \tag{20}$$

In such a manner, the following assertion is proved.

Theorem 1. *FDS (3) converges in the discrete L_2 norm and the inequality (20) holds.*

Notice, that the convergence rate estimate (20) is consistent with the smoothness of the solution of IBVP (1) (see [**10**]). In the case of uniform mesh estimate of the form (20) is obtained in [**7**] for $\tau \asymp h^2$. Equation with variable coefficients is considered in [**8**]. For the less smooth solutions ($u \in W_2^{2s,s}$, $0 < s < 1$) analogous lower order convergence rate estimate can be obtained. Analogous result (in continuous L_2–norm) follows from the existing finite element theory.

Monotonous scheme.

Let us approximate now the IBVP (1) with the following implicite monotonous FDS (see [**15**])

$$
\begin{gathered}
v_{\bar t} + \frac{h_+ - h}{3}\, v_{\bar t \bar x} = v_{\bar x \hat x} + T_2 T_{\bar t} f \quad \text{in} \quad Q_{h\tau}\,; \\
v + \frac{h_+ - h}{3}\, v_{\bar x} = T_2 u_0 \quad \text{for} \quad t = 0\,; \qquad v = 0 \quad \text{for} \quad x = 0,\, l\,.
\end{gathered} \tag{21}
$$

For the sake of simplicity we assume that the mesh $\hat\omega$ satisfies the condition

$$
h < h_+ < c_1\, h\,, \tag{22}
$$

The error $z = u - v$ satisfies the conditions

$$
\begin{gathered}
z_{\bar t} + \frac{h_+ - h}{3}\, z_{\bar t \bar x} = z_{\bar x \hat x} + \varphi_{\bar x \hat x} + \psi_{\bar t} \quad \text{in} \quad Q_{h\tau}\,; \\
z + \frac{h_+ - h}{3}\, z_{\bar x} = \psi(x,\, 0) \quad \text{for} \quad t = 0\,; \qquad z = 0 \quad \text{for} \quad x = 0,\, l\,,
\end{gathered} \tag{23}
$$

where

$$
\varphi = T_{\bar t} u - u \qquad \text{and} \qquad \psi = u - T_2 u + \frac{h_+ - h}{3}\, u_{\bar x}\,.
$$

To obtain the a priori estimate in discrete L_2–norm, let us set $z = z^{(1)} + z^{(2)}$, where $z^{(1)}$ and $z^{(2)}$ are the solutions of the following FDSs

$$
\begin{gathered}
z^{(1)}_{\bar t} + \frac{h_+ - h}{3}\, z^{(1)}_{\bar t \bar x} = z^{(1)}_{\bar x \hat x} + \varphi_{\bar x \hat x} \quad \text{in} \quad Q_{h\tau}\,; \\
z^{(1)} = 0 \quad \text{for} \quad t = 0\,; \qquad z^{(1)} = 0 \quad \text{for} \quad x = 0,\, l\,,
\end{gathered} \tag{24}
$$

and

$$
\begin{gathered}
z^{(2)}_{\bar t} + \frac{h_+ - h}{3}\, z^{(2)}_{\bar t \bar x} = z^{(2)}_{\bar x \hat x} + \psi_{\bar t} \quad \text{in} \quad Q_{h\tau}\,; \\
z^{(2)} + \frac{h_+ - h}{3}\, z^{(2)}_{\bar x} = \psi(x,\, 0) \quad \text{for} \quad t = 0\,; \qquad z^{(2)} = 0 \quad \text{for} \quad x = 0,\, l\,.
\end{gathered} \tag{25}
$$

Let ζ be a mesh–function satisfying

$$
-\zeta_{\bar x \hat x} = z^{(1)} + \frac{h_+ - h}{3}\, z^{(1)}_{\bar x} \quad \text{in} \quad Q^0_{h\tau}\,; \qquad \zeta = 0 \quad \text{for} \quad x = 0,\, l\,. \tag{26}
$$

Multiplying (24) by ζ, after some simple transformations, we obtain

$$
(\zeta_{\bar x \bar t},\, \zeta_{\bar x}] - (z^{(1)},\, \zeta_{\bar x \hat x})_* = (\varphi,\, \zeta_{\bar x \hat x})_*\,. \tag{27}
$$

From (27) and (26), using the identity

$$(\zeta_{\bar{x}\bar{t}},\, \zeta_{\bar{x}}] = \frac{\tau}{2}\, \|\zeta_{\bar{x}\bar{t}}]|^2 + \frac{1}{2\,\tau}\left(\|\zeta_{\bar{x}}]|^2 - \|\check{\zeta}_{\bar{x}}]|^2\right),$$

we obtain

$$\frac{1}{2\,\tau}\left(\|\zeta_{\bar{x}}]|^2 - \|\check{\zeta}_{\bar{x}}]|^2\right) + \frac{\tau}{2}\, \|\zeta_{\bar{x}\bar{t}}]|^2 + \|z^{(1)}\|_*^2$$
$$= -(\varphi,\, z^{(1)})_* - \left(\varphi,\, \frac{h_+ - h}{3}\, z_{\bar{x}}^{(1)}\right)_* - \left(z^{(1)},\, \frac{h_+ - h}{3}\, z_{\bar{x}}^{(1)}\right)_*.$$

Further

$$\left\|\frac{h_+ - h}{3}\, z_{\bar{x}}\right\|_*^2 = \sum_{x\in\hat{\omega}} \left(\frac{h_+ - h}{3}\cdot\frac{z - z_-}{h}\right)^2 \hbar \le \frac{2\,(c_1 - 1)^2}{9} \sum_{x\in\hat{\omega}} (z^2 + z_-^2)\,\hbar$$
$$= \frac{2\,(c_1 - 1)^2}{9}\left(\sum_{x\in\hat{\omega}} z^2\,\hbar + \sum_{x\in\hat{\omega}\setminus\{x_{n-1}\}} z^2\,\hbar_+\right) \le \frac{2}{9}\,(c_1 - 1)^2\,(c_1 + 1)\,\|z\|_*^2,$$

wherefrom follows

$$\left|\left(z,\, \frac{h_+ - h}{3}\, z_{\bar{x}}\right)_*\right| \le \frac{c_1 - 1}{3}\,\sqrt{2\,(c_1 + 1)}\,\|z\|_*^2,$$
$$\left|\left(\varphi,\, \frac{h_+ - h}{3}\, z_{\bar{x}}\right)_*\right| \le \frac{c_1 - 1}{3}\,\sqrt{2\,(c_1 + 1)}\,\|\varphi\|_*\,\|z\|_*$$
$$\le \varepsilon\,\|z\|_*^2 + \frac{2\,(c_1 + 1)\,(c_1 - 1)^2}{36\,\varepsilon}\,\|\varphi\|_*^2$$

and

$$\left|(\varphi,\, z)_*\right| \le \varepsilon\,\|z\|_*^2 + \frac{1}{4\,\varepsilon}\,\|\varphi\|_*^2.$$

From here follows

$$\frac{1}{2\,\tau}\left(\|\zeta_{\bar{x}}]|^2 - \|\check{\zeta}_{\bar{x}}]|^2\right) + \frac{\tau}{2}\,\|\zeta_{\bar{x}\bar{t}}]|^2 + \left[1 - \frac{c_1 - 1}{3}\,\sqrt{2\,(c_1 + 1)} - 2\,\varepsilon\right]\|z^{(1)}\|_*^2$$
$$\le \frac{1}{4\,\varepsilon}\left[\frac{2\,(c_1 + 1)\,(c_1 - 1)^2}{9} + 1\right]\|\varphi\|_*^2.$$

If

$$1 - \frac{c_1 - 1}{3}\,\sqrt{2\,(c_1 + 1)} > 0\,, \tag{28}$$

then, for sufficiently small $\varepsilon > 0$, one obtains

$$\frac{1}{2\,\tau}\left(\|\zeta_{\bar{x}}]|^2 - \|\check{\zeta}_{\bar{x}}]|^2\right) + \frac{\tau}{2}\,\|\zeta_{\bar{x}\bar{t}}]|^2 + C_0\,\|z^{(1)}\|_*^2 \le C_1\,\|\varphi\|_*^2.$$

From here, summing over the mesh ω_τ we obtain

$$\frac{1}{2}\,\|\zeta_{\bar{x}}(\,\cdot\,,T)]|^2+\frac{\tau^2}{2}\sum_{t=\tau}^{T}\|\zeta_{\bar{x}\bar{t}}]|^2+C_0\,\tau\sum_{t=\tau}^{T}\|z^{(1)}\|_*^2\le C_1\,\tau\sum_{t=\tau}^{T}\|\varphi\|_*^2\,,$$

and finally

(29) $$\|z^{(1)}\|_{L_2(Q_{h\tau})}\le C_2\,\|\varphi\|_{L_2(Q_{h\tau})}\,.$$

Notice that the condition (28) is satisfied for

$$(c_1-1)\,(c_1+1)^2<4.5$$

wherefrom follows

(30) $$c_1<2.188\,.$$

Let us estimate $z^{(2)}$. Multiplying (25) in a scalar way by η, where

$$-\eta_t=z^{(2)}\quad\text{in}\quad Q^0_{h\tau}\,;\qquad\quad \eta=0\quad\text{for}\quad t=T+\tau\,,$$

and using partial summing on t, we obtain

$$-\tau\sum_{t=\tau}^{T}(z^{(2)},\eta_t)_*-\tau\sum_{t=\tau}^{T}\Big(\frac{h_+-h}{3}\,z^{(2)}_{\bar{x}},\eta_t\Big)_*=\tau\sum_{t=\tau}^{T}(z^{(2)}_{\bar{x}\hat{x}},\,\eta)_*-\tau\sum_{t=\tau}^{T}(\psi,\,\eta_t)_*\,.$$

Further

$$(z^{(2)}_{\bar{x}\hat{x}},\,\eta)_*=-(z^{(2)}_{\bar{x}},\eta_{\bar{x}}]=(\eta_{\bar{x}t},\eta_{\bar{x}}]$$
$$=\frac{1}{2\,\tau}\left(\|\hat{\eta}_{\bar{x}}]|^2-\|\eta_{\bar{x}}]|^2\right)-\frac{\tau}{2}\,\|\eta_{\bar{x}t}]|^2=\frac{1}{2\,\tau}\left(\|\hat{\eta}_{\bar{x}}]|^2-\|\eta_{\bar{x}}]|^2\right)-\frac{\tau}{2}\,\|z^{(2)}_{\bar{x}}]|^2$$

wherefrom follows

(31) $$\begin{aligned}&\tau\sum_{t=\tau}^{T}\|z^{(2)}\|_*^2+\frac{1}{2}\,\|\eta_{\bar{x}}(\,\cdot\,,\tau)]|^2+\frac{\tau^2}{2}\sum_{t=\tau}^{T}\|z^{(2)}_{\bar{x}}]|^2\\&=\tau\sum_{t=\tau}^{T}(\psi,\,z^{(2)})_*-\tau\sum_{t=\tau}^{T}\Big(\frac{h_+-h}{3}\,z^{(2)}_{\bar{x}},\,z^{(2)}\Big)_*\,.\end{aligned}$$

From (31) follows

$$\begin{aligned}\tau\sum_{t=\tau}^{T}\|z^{(2)}\|_*^2&\le-\tau\sum_{t=\tau}^{T}\Big(\frac{h_+-h}{3}\,z^{(2)}_{\bar{x}},\,z^{(2)}\Big)_*+\tau\sum_{t=\tau}^{T}(\psi,\,z^{(2)})_*\\&\le\frac{c_1-1}{3}\,\sqrt{2\,(c_1+1)}\,\tau\sum_{t=\tau}^{T}\|z^{(2)}\|_*^2+\varepsilon\,\tau\sum_{t=\tau}^{T}\|z^{(2)}\|_*^2+\frac{1}{4\,\varepsilon}\,\tau\sum_{t=\tau}^{T}\|\psi\|_*^2\,.\end{aligned}$$

From here, under the condition (30), for sufficiently small $\varepsilon > 0$, we obtain

$$\|z^{(2)}\|_{L_2(Q_{h\tau})} \le C_3\,\|\psi\|_{L_2(Q_{h\tau})}\,. \tag{32}$$

From (29) and (32) we obtain the desired a priori estimate

$$\|z\|_{L_2(Q_{h\tau})} \le C\left(\|\varphi\|_{L_2(Q_{h\tau})} + \|\psi\|_{L_2(Q_{h\tau})}\right). \tag{33}$$

Notice that contrary to the case of divergent scheme (3), this a priori estimate is obtained under restriction to the ratio h_+/h.

The terms φ and ψ in the right hand side of (33) can be estimated in an analogous way as in the case of divergent FDS: φ is the same as in the previous case, while ψ may be represented in the form

$$\psi = \psi_1 + \psi_2 = T_{\bar t}\Big(u - T_2 u + \frac{h_+ - h}{3}\,u_{\bar x}\Big)$$
$$+\Big[\Big(u - T_2 u + \frac{h_+ - h}{3}\,u_{\bar x}\Big) - T_{\bar t}\Big(u - T_2 u + \frac{h_+ - h}{3}\,u_{\bar x}\Big)\Big].$$

Further

$$\psi_2 = -(T_{\bar t}u - u) + T_2(T_{\bar t}u - u) - \Big[\frac{h_+ - h}{3}\,\big(T_{\bar t}u - u\big)_{\bar x}\Big] = -\varphi + T_2\varphi - \frac{h_+ - h}{3}\,\varphi_{\bar x}\,,$$

and

$$\psi_1(x,\,t) = \frac{1}{\hbar\tau}\int_{x_-}^{x_+}\int_{x'}^{x}\int_{x}^{x''}\int_{t-\tau}^{t} K(x')\,\frac{\partial^2 u(x''',\,t')}{\partial x^2}\,dt'\,dx'''\,dx''\,dx'$$
$$+\frac{h_+ - h}{3\,\hbar\tau}\int_{x_-}^{x}\int_{x}^{x'}\int_{t-\tau}^{t}\frac{\partial^2 u(x'',\,t')}{\partial x^2}\,dt'\,dx''\,dx'\,.$$

In such a manner, estimates (16–18) and the convergence rate estimate (20) are satisfied.

Theorem 2. *FDS (21) converges in discrete norm L_2, under assumptions (22) and (30), and the convergence rate estimate (20) holds.*

Remark. Analogous results can be obtained for FDS approximating other boundary value problems, including problems with variable coefficients and hyperbolic problems.

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