

NEW SPECIAL GEODESIC MAPPINGS OF GENERALIZED RIEMANNIAN SPACES

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ABSTRACT. We define R_θ -projective geodesic mappings ($\theta = 1, \dots, 5$) of two generalized Riemannian spaces and obtain some invariant geometric objects of these mappings, generalizing Weyl's tensor. Also, we define R_θ -projectively flat generalized Riemannian spaces $G\overline{R}_N$ and find necessary conditions for the space GR_N to be R_θ -projectively flat.

0. Introduction

DEFINITION 0.1. A differentiable N -dimensional manifold GR_N is a generalized Riemannian space in the sense of Eisenhart's definition [2] if it is equipped with a nonsymmetric basic tensor g_{ij} .

Consider two N -dimensional generalized Riemannian spaces GR_N and $G\overline{R}_N$. The connection coefficients of these spaces are generalized Christoffel's symbols of second kind, Γ_{jk}^i and $\overline{\Gamma}_{jk}^i$ respectively. In the general case $\Gamma_{jk}^i \neq \Gamma_{kj}^i$.

DEFINITION 0.2. The mapping $f : GR_N \rightarrow G\overline{R}_N$ is geodesic [6], if the geodesics of the space GR_N are mapped into to the geodesics of the space $G\overline{R}_N$.

We can consider these spaces in the common by this mapping system of local coordinates, i.e., if $f : M(x) \equiv M(x^1, \dots, x^N) \mapsto \overline{M}$, then we have $\overline{M}(x) \equiv \overline{M}(x^1, \dots, x^N)$. For Christoffel's symbols of second kind of the spaces GR_N and $G\overline{R}_N$ in the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$(0.1) \quad \overline{\Gamma}_{jk}^i(x) = \Gamma_{jk}^i(x) + P_{jk}^i(x) \quad (i, j, k = 1, \dots, N),$$

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where $P_{jk}^i(x)$ is the **deformation tensor** of the connection Γ of GR_N according to the mapping $f : GR_N \rightarrow G\overline{R}_N$.

A necessary and sufficient condition [6] for the mapping $f : GR_N \rightarrow G\overline{R}_N$ to be geodesic is that the deformation tensor P_{jk}^i from (0.1) at the mapping f has the form

$$(0.2) \quad P_{jk}^i(x) = \delta_j^i \psi_k(x) + \delta_k^i \psi_j(x) + \xi_{jk}^i(x),$$

where

$$(0.3) \quad \psi_i(x) = \frac{1}{N+1}(\overline{\Gamma}_{i\alpha}^\alpha(x) - \Gamma_{i\alpha}^\alpha(x)), \quad \xi_{jk}^i(x) = P_{jk}^i = \frac{1}{2}(P_{jk}^i - P_{kj}^i).$$

In a generalized Riemannian space we can define four kinds of covariant derivative [3,4]. Denote by $\underset{\theta}{|}, \underset{\theta}{\rfloor}$ a covariant derivative of the kind θ in GR_N and $G\overline{R}_N$ respectively.

For the torsion tensor of the space GR_N it holds (eq. (2.10) in [6])

$$(0.4) \quad \Gamma_{i\alpha}^\alpha = 0.$$

In the space GR_N we have five independent curvature tensors [5]:

$$(0.5) \quad R_1^i{}_{jmn} = \Gamma_{jm,n}^i - \Gamma_{jn,m}^i + \Gamma_{jm}^\alpha \Gamma_{\alpha n}^i - \Gamma_{jn}^\alpha \Gamma_{\alpha m}^i,$$

$$(0.6) \quad R_2^i{}_{jmn} = \Gamma_{mj,n}^i - \Gamma_{nj,m}^i + \Gamma_{mj}^\alpha \Gamma_{n\alpha}^i - \Gamma_{nj}^\alpha \Gamma_{m\alpha}^i,$$

$$(0.7) \quad R_3^i{}_{jmn} = \Gamma_{jm,n}^i - \Gamma_{nj,m}^i + \Gamma_{jm}^\alpha \Gamma_{n\alpha}^i - \Gamma_{nj}^\alpha \Gamma_{\alpha m}^i + \Gamma_{nm}^\alpha (\Gamma_{\alpha j}^i - \Gamma_{j\alpha}^i),$$

$$(0.8) \quad R_4^i{}_{jmn} = \Gamma_{jm,n}^i - \Gamma_{nj,m}^i + \Gamma_{jm}^\alpha \Gamma_{n\alpha}^i - \Gamma_{nj}^\alpha \Gamma_{\alpha m}^i + \Gamma_{mn}^\alpha (\Gamma_{\alpha j}^i - \Gamma_{j\alpha}^i),$$

$$(0.9) \quad R_5^i{}_{jmn} = \frac{1}{2}(\Gamma_{jm,n}^i + \Gamma_{mj,n}^i - \Gamma_{jn,m}^i - \Gamma_{nj,m}^i + \Gamma_{jm}^\alpha \Gamma_{\alpha n}^i + \Gamma_{mj}^\alpha \Gamma_{n\alpha}^i - \Gamma_{jn}^\alpha \Gamma_{m\alpha}^i - \Gamma_{nj}^\alpha \Gamma_{\alpha m}^i).$$

Denote by $\overline{R}_{\theta}^i{}_{jmn}$ ($\theta = 1, \dots, 5$) the corresponding curvature tensors of the space $G\overline{R}_N$.

In the case of a geodesic mapping $f : R_N \rightarrow \overline{R}_N$ of Riemannian spaces R_N and \overline{R}_N we have an invariant geometric object ([8, p. 80] or [1, eq. (37.5)]):

$$(0.10) \quad W^i{}_{jmn} = R^i{}_{jmn} + \frac{1}{N-1}(\delta_m^i R_{jn} - \delta_n^i R_{jm}).$$

where $R^i{}_{jmn}$ is Riemann-Christoffel's curvature tensor of the space R_N , and R_{jm} is Ricci's tensor. The object $W^i{}_{jmn}$ is called *Weyl's tensor*, or the tensor of projective curvature [1], [8]. Having a geodesic mapping of two generalized Riemannian spaces, we can not find a generalization of Weyl's tensor as an invariant of geodesic mapping in the general case. For that reason in [7] we defined a special geodesic mapping $f : GR_N \rightarrow G\overline{R}_N$, that we called equitortion geodesic mapping. Here we define and study new kinds of geodesic mappings of generalized Riemannian spaces.

1. R -projective mappings

DEFINITION 1.1. The geodesic mapping $f : GR_N \rightarrow G\overline{R}_N$ is R -projective if the following condition is satisfied

$$(1.1) \quad \begin{aligned} & \delta_j^i (\mathcal{D}_{mn} - \mathcal{D}_{nm}) + \delta_m^i \mathcal{D}_{jn} - \delta_n^i \mathcal{D}_{jm} + (\delta_n^i \xi_{jm}^\alpha - \delta_m^i \xi_{jn}^\alpha) \psi_\alpha \\ & + 2\psi_j \xi_{mn}^i + \xi_{jm}^\alpha \xi_{\alpha n}^i - \xi_{jn}^\alpha \xi_{\alpha m}^i + 2\Gamma_{m\nu}^i \psi_j + 2\Gamma_{m\nu}^\alpha \psi_\alpha \delta_j^i \\ & + 2\Gamma_{m\nu}^\alpha \xi_{j\alpha}^i = 0, \end{aligned}$$

where

$$(1.1') \quad \begin{aligned} \mathcal{D}_{jm}^i &= \frac{N-1}{N+1} \xi_{jm}^\alpha \psi_\alpha - \frac{1}{N^2-1} (4\Gamma_{m\nu}^\alpha \psi_\alpha + 2\Gamma_{m\nu}^\alpha \xi_{j\alpha}^\beta - 2\Gamma_{j\nu}^\alpha \xi_{m\alpha}^\beta) \\ & - \frac{1}{N-1} (\xi_{j\beta}^\alpha \xi_{\alpha m}^\beta - 2\Gamma_{m\nu}^\alpha \psi_\alpha - 2\Gamma_{m\nu}^\alpha \xi_{j\alpha}^\beta), \end{aligned}$$

and mj , as in (0.3), denotes an antisymmetrisation with respect to indices m and j .

DEFINITION 1.2. The space GR_N is R -projectively flat if there exists an R -projective mapping of the space GR_N to a flat space (i.e., to a space, whose connection coefficients in special coordinates are zero).

Using (0.1), we get the relation between the first kind curvature tensors (0.5) of the spaces GR_N and $G\overline{R}_N$

$$(1.2) \quad \overline{R}_{jmn}^i = R_{jmn}^i + P_{jm|n}^i - P_{jn|m}^i + P_{jm}^\alpha P_{\alpha n}^i - P_{jn}^\alpha P_{\alpha m}^i + 2\Gamma_{m\nu}^\alpha P_{j\alpha}^i.$$

Denoting

$$(1.3) \quad \psi_{mn} = \psi_{m|n} - \psi_m \psi_n \quad (\theta = 1, 2)$$

and substituting P with respect to (0.2) in (1.2), we obtain

$$(1.4) \quad \begin{aligned} \overline{R}_{jmn}^i &= R_{jmn}^i + \delta_j^i (\psi_{mn} - \psi_{nm}) + \delta_m^i \psi_{jn} - \delta_n^i \psi_{jm} - \delta_m^i \xi_{jn}^\alpha \psi_\alpha \\ & + \delta_n^i \xi_{jm}^\alpha \psi_\alpha + \xi_{jm|n}^i - \xi_{jn|m}^i + 2\psi_j \xi_{mn}^i + \xi_{jm}^\alpha \xi_{\alpha n}^i - \xi_{jn}^\alpha \xi_{\alpha m}^i \\ & + 2\Gamma_{m\nu}^i \psi_j + 2\Gamma_{m\nu}^\alpha \psi_\alpha \delta_j^i + 2\Gamma_{m\nu}^\alpha \xi_{j\alpha}^i. \end{aligned}$$

Contracting (1.4) with respect to i and n and using (0.4) we get

$$(1.5) \quad \begin{aligned} \overline{R}_{jm} &= R_{jm} - 2\psi_{jm} + (1-N)\psi_{jm} + (N-1)\xi_{jm}^\alpha \psi_\alpha + \xi_{jm| \beta}^\beta \\ & - \xi_{j\beta}^\alpha \xi_{\alpha m}^\beta + 2\Gamma_{m\nu}^\beta \psi_j + 2\Gamma_{m\nu}^\alpha \psi_\alpha + 2\Gamma_{m\nu}^\alpha \xi_{j\alpha}^\beta. \end{aligned}$$

Here \overline{R}_{1jm} and R_{1jm} denote the first kind Ricci tensors of the space $G\overline{R}_N$ and GR_N respectively.

From (1.5) we obtain

$$(1.6) \quad \begin{aligned} (1+N)\psi_{1j\check{\nu}} = R_{1j\check{\nu}} - \overline{R}_{1j\check{\nu}} + (N-1)\xi_{jm}^\alpha \psi_\alpha + \xi_{jm|\beta}^\beta + 2\Gamma_{m\check{\nu}}^\alpha \psi_\alpha \\ + \Gamma_{m\beta}^\alpha \xi_{j\beta}^\beta - \Gamma_{j\check{\nu}}^\alpha \xi_{m\alpha}^\beta. \end{aligned}$$

Substituting (0.3) in (1.6) we get

$$\begin{aligned} \overline{R}_{1jm} = R_{1jm} - \frac{1}{N+1} [2R_{1j\check{\nu}} - 2\overline{R}_{1j\check{\nu}} + 2(N-1)\xi_{jm}^\alpha \psi_\alpha + 2\xi_{jm|\beta}^\beta \\ + 4\Gamma_{m\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{m\beta}^\alpha \xi_{j\alpha}^\beta - 2\Gamma_{j\check{\nu}}^\alpha \xi_{m\alpha}^\beta] + (1-N)\psi_{1jm} + (N-1)\xi_{jm}^\alpha \psi_\alpha \\ + \xi_{jm|\beta}^\beta - \xi_{j\beta}^\alpha \xi_{\alpha m}^\beta + 2\Gamma_{m\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{m\beta}^\alpha \xi_{j\alpha}^\beta, \end{aligned}$$

i.e.

$$(1.7) \quad \begin{aligned} (N-1)\psi_{1jm} = R_{1jm} - \overline{R}_{1jm} - \frac{1}{N+1} [2R_{1j\check{\nu}} - 2\overline{R}_{1j\check{\nu}} + 2(N-1)\xi_{jm}^\alpha \psi_\alpha \\ + 2\xi_{jm|\beta}^\beta + 4\Gamma_{m\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{m\beta}^\alpha \xi_{j\alpha}^\beta - 2\Gamma_{j\check{\nu}}^\alpha \xi_{m\alpha}^\beta] + (N-1)\xi_{jm}^\alpha \psi_\alpha + \xi_{jm|\beta}^\beta \\ - \xi_{j\beta}^\alpha \xi_{\alpha m}^\beta + 2\Gamma_{m\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{m\beta}^\alpha \xi_{j\alpha}^\beta. \end{aligned}$$

Replacing (1.7) in (1.4) we get

$$\begin{aligned} \overline{R}_{1jmn}^i = R_{1jmn}^i + \frac{1}{N-1} \delta_j^i \{ R_{mn} - \overline{R}_{mn} - \frac{1}{N+1} [2R_{m\check{\nu}} - 2\overline{R}_{m\check{\nu}} \\ + 2(N-1)\xi_{mn}^\alpha \psi_\alpha + 2\xi_{mn|\beta}^\beta + 4\Gamma_{n\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{n\beta}^\alpha \xi_{m\alpha}^\beta - 2\Gamma_{m\check{\nu}}^\alpha \xi_{n\alpha}^\beta] \\ + (N-1)\xi_{mn}^\alpha \psi_\alpha + \xi_{mn|\beta}^\beta - \xi_{m\beta}^\alpha \xi_{\alpha n}^\beta + 2\Gamma_{n\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{n\beta}^\alpha \xi_{m\alpha}^\beta - R_{nm} + \overline{R}_{nm} \\ + \frac{1}{N+1} [2R_{n\check{\nu}} - 2\overline{R}_{n\check{\nu}} + 2(N-1)\xi_{nm}^\alpha \psi_\alpha + 2\xi_{nm|\beta}^\beta + 4\Gamma_{m\check{\nu}}^\alpha \psi_\alpha \\ + 2\Gamma_{m\beta}^\alpha \xi_{n\alpha}^\beta - 2\Gamma_{n\beta}^\alpha \xi_{m\alpha}^\beta] - (N-1)\xi_{nm}^\alpha \psi_\alpha - \xi_{nm|\beta}^\beta + \xi_{n\beta}^\alpha \xi_{\alpha m}^\beta - 2\Gamma_{m\check{\nu}}^\alpha \psi_\alpha \\ - 2\Gamma_{m\beta}^\alpha \xi_{n\alpha}^\beta \} + \frac{1}{N-1} \delta_m^i \{ R_{jn} - \overline{R}_{jn} - \frac{1}{N+1} [2R_{j\check{\nu}} - 2\overline{R}_{j\check{\nu}} + 2(n-1)\xi_{jn}^\alpha \psi_\alpha \\ + 2\xi_{jn|\beta}^\beta + 4\Gamma_{n\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{n\beta}^\alpha \xi_{j\alpha}^\beta - 2\Gamma_{j\check{\nu}}^\alpha \xi_{n\alpha}^\beta] + (N-1)\xi_{jn}^\alpha \psi_\alpha + \xi_{jn|\beta}^\beta - \xi_{j\beta}^\alpha \xi_{\alpha n}^\beta \\ + 2\Gamma_{n\check{\nu}}^\alpha \psi_\alpha + 2\Gamma_{n\beta}^\alpha \xi_{j\alpha}^\beta \} - \frac{1}{N-1} \delta_n^i \{ R_{jm} - \overline{R}_{jm} - \frac{1}{N+1} [2R_{1j\check{\nu}} - 2\overline{R}_{1j\check{\nu}} \end{aligned}$$

$$\begin{aligned}
& + 2(N-1)\xi_{jm}^\alpha \psi_\alpha + 2\xi_{jm|\beta}^\beta + 4\Gamma_{m\check{v}}^\alpha \psi_\alpha + 2\Gamma_{m\check{v}}^\alpha \xi_{j\alpha}^\beta - 2\Gamma_{j\check{v}}^\alpha \xi_{m\alpha}^\beta + (N-1)\xi_{jm}^\alpha \psi_\alpha \\
& + \xi_{jm|\beta}^\beta - \xi_{j\beta}^\alpha \xi_{\alpha m}^\beta + 2\Gamma_{m\check{v}}^\alpha \psi_\alpha + 2\Gamma_{m\check{v}}^\alpha \xi_{j\alpha}^\beta \} + (\delta_n^i \xi_{jm}^\alpha - \delta_m^i \xi_{jn}^\alpha) \psi_\alpha + \xi_{jm|n}^i \\
& - \xi_{jn|m}^i + 2\psi_j \xi_{mn}^i + \xi_{jm}^\alpha \xi_{\alpha n}^i - \xi_{jn}^\alpha \xi_{\alpha m}^i + 2\Gamma_{m\check{v}}^i \psi_j + 2\Gamma_{m\check{v}}^\alpha \psi_\alpha \delta_j^i + 2\Gamma_{m\check{v}}^\alpha \xi_{j\alpha}^i.
\end{aligned}$$

Taking into account (1.1) and (1.1'), we can write the last equation in the form

$$(1.8) \quad \overline{W}(\overline{R})^i{}_{jmn} = W(R)^i{}_{jmn},$$

where

$$\begin{aligned}
(1.9) \quad W(R)^i{}_{jmn} & = R_1^i{}_{jmn} + \frac{2}{N+1} \delta_j^i R_{1\check{v}}^i{}_{m\check{v}} + \frac{1}{N^2-1} [(NR_1^{jn} - R_{1n}^j) \delta_m^i \\
& - (NR_1^{jm} - R_{1m}^j) \delta_n^i] - \frac{2}{N+1} \delta_j^i \Gamma_{m\check{v}|1}^\beta - \frac{1}{N+1} \delta_m^i \Gamma_{j\check{v}|1}^\beta \\
& + \frac{1}{N+1} \delta_n^i \Gamma_{j\check{v}|1}^\beta - \Gamma_{j\check{v}|1}^i{}_{m\check{v}} - \Gamma_{j\check{v}|1}^i{}_{m\check{v}}.
\end{aligned}$$

Therefore, we proved the next theorem

THEOREM 1.1. *The tensor (1.9) is an invariant of an R_1 -projective mapping.*

THEOREM 1.2. *If GR_N is R_1 -projectively flat, then*

$$(1.10) \quad W(R)^i{}_{jmn} = 0.$$

Proof. $G\overline{R}_N$ is a flat space. Then by (1.9) we get $\overline{W}(\overline{R})^i{}_{jmn} = 0$, and using (1.8) we can see that (1.10) holds. \square

2. R_2 -projective mappings

DEFINITION 2.1. The geodesic mapping $f : GR_N \rightarrow G\overline{R}_N$ is R_2 -projective if the following condition is satisfied

$$\begin{aligned}
& \delta_j^i (\mathcal{D}_{mn} - \mathcal{D}_{nm}) + \delta_m^i \mathcal{D}_{jn} - \delta_n^i \mathcal{D}_{jm} + (\delta_n^i \xi_{mj}^\alpha - \delta_m^i \xi_{nj}^\alpha) \psi_\alpha \\
& + 2\psi_j \xi_{nm}^i + \xi_{mj}^\alpha \xi_{n\alpha}^i - \xi_{nj}^\alpha \xi_{m\alpha}^i + 2\Gamma_{n\check{v}}^i \psi_j + 2\Gamma_{n\check{v}}^\alpha \psi_\alpha \delta_j^i \\
& + 2\Gamma_{n\check{v}}^\alpha \xi_{\alpha j}^i = 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}_{jm} & = \frac{N-1}{N+1} \xi_{mj}^\alpha \psi_\alpha - \frac{1}{N^2-1} (4\Gamma_{m\check{v}}^\alpha \psi_\alpha + 2\Gamma_{\beta\check{v}}^\alpha \xi_{\alpha j}^\beta - 2\Gamma_{\beta\check{v}}^\alpha \xi_{\alpha m}^\beta) \\
& - \frac{1}{N-1} (\xi_{\beta j}^\alpha \xi_{m\alpha}^\beta - 2\Gamma_{j\check{v}}^\alpha \psi_\alpha - 2\Gamma_{\beta\check{v}}^\alpha \xi_{\alpha j}^\beta).
\end{aligned}$$

DEFINITION 2.2. The space GR_N is R_2 -projectively flat if there exists an R_2 -projective mapping of the space GR_N into a flat space.

For curvature tensors of the second kind (0.6) of the spaces GR_N and $G\overline{R}_N$ we get the relation

$$\overline{R}_{2jmn}^i = R_{2jmn}^i + P_{mj|n}^i - P_{nj|m}^i + P_{mj}^\alpha P_{n\alpha}^i - P_{nj}^\alpha P_{m\alpha}^i + 2\Gamma_{nm}^\alpha P_{\alpha j}^i,$$

i.e., using (0.2) and (1.3),

$$\begin{aligned} \overline{R}_{2jmn}^i &= R_{2jmn}^i + \delta_j^i (\psi_{mn} - \psi_{nm}) + \delta_m^i \psi_{jn} - \delta_n^i \psi_{jm} \\ &\quad - \delta_m^i \xi_{nj}^\alpha \psi_\alpha + \delta_n^i \xi_{mj}^\alpha \psi_\alpha + \xi_{mj|n}^i - \xi_{nj|m}^i + 2\psi_j \xi_{nm}^i + \xi_{mj}^\alpha \xi_{n\alpha}^i \\ &\quad - \xi_{nj}^\alpha \xi_{m\alpha}^i + 2\Gamma_{nm}^i \delta_j^i + 2\Gamma_{nm}^\alpha \psi_\alpha \delta_j^i + 2\Gamma_{nm}^\alpha \xi_{\alpha j}^i. \end{aligned}$$

Analogously to the previous case, we get an invariant of an R_2 -projective mapping $f : GR_N \rightarrow G\overline{R}_N$

$$\begin{aligned} (2.1) \quad W(R)_2^i{}_{jmn} &= R_{2jmn}^i + \frac{2}{N+1} \delta_j^i R_{m\nu}^\beta + \frac{1}{N^2-1} [(NR_{2jn} - R_{2nj})\delta_m^i \\ &\quad - (NR_{2jm} - R_{2mj})\delta_n^i] - \frac{2}{N+1} \delta_j^i \Gamma_{nm|\beta}^\beta - \frac{1}{N+1} \delta_m^i \Gamma_{nj|\beta}^\beta \\ &\quad + \frac{1}{N+1} \delta_n^i \Gamma_{mj|\beta}^\beta - \Gamma_{mj|n}^i - \Gamma_{nj|m}^i. \end{aligned}$$

Consequently, the next theorems are valid

THEOREM 2.1. *The tensor (2.1) is an invariant of an R_2 -projective mapping.*

THEOREM 2.2. *If GR_N is R_2 -projectively flat then we have*

$$W(R)_2^i{}_{jmn} = 0.$$

3. R_3 -projective mappings

DEFINITION 3.1. The geodesic mapping $f : GR_N \rightarrow G\overline{R}_N$ is R_3 -projective if the following condition holds

$$\begin{aligned} (3.1) \quad &\delta_j^i (\mathcal{D}_{3mn} - \mathcal{D}_{3nm}) + \delta_m^i \mathcal{D}_{3jn} - \delta_n^i \mathcal{D}_{3jm} + 2(\delta_j^i \Gamma_{m\nu}^\alpha + \delta_m^i \Gamma_{j\nu}^\alpha) \psi_\alpha \\ &+ (\delta_n^i \xi_{jm}^\alpha - \delta_m^i \xi_{nj}^\alpha) \psi_\alpha + \xi_{jm}^\alpha \xi_{n\alpha}^i - \xi_{jn}^\alpha \xi_{\alpha m}^i + 2\psi_n (\Gamma_{mj}^i + \xi_{mj}^i) \\ &+ 2\psi_m (\Gamma_{nj}^i + \xi_{nj}^i) + 2\xi_{nm}^\alpha (\Gamma_{\alpha j}^i + \xi_{\alpha j}^i) = 0, \end{aligned}$$

where

$$(3.1') \quad \begin{aligned} \mathcal{D}_3^{j m} &= \xi_{j m}^\alpha \psi_\alpha - \frac{1}{N^2 - 1} [4\psi_\beta (\Gamma_{m \check{\nu}}^\beta + \xi_{m j}^\beta) + 2\xi_{\beta m}^\alpha \Gamma_{\check{\nu}}^\beta - 2\xi_{\beta j}^\alpha \Gamma_{\check{\nu}}^\beta] \\ &+ \frac{1}{N - 1} [\xi_{j \beta}^\alpha \xi_{\alpha m}^\beta + 2\psi_\beta (\Gamma_{m \check{\nu}}^\beta + \xi_{m j}^\beta) + 2\xi_{\beta m}^\alpha (\Gamma_{\check{\nu}}^\beta + \xi_{\alpha j}^\beta)]. \end{aligned}$$

DEFINITION 3.2. The space GR_N is R -projectively flat if there exists an R_3 -projective mapping of the space GR_N into a flat space.

In the case of curvature tensors of the third kind (0.7) of the spaces GR_N and $G\overline{R}_N$ we get the relation

$$\begin{aligned} \overline{R}_3^i{}_{j m n} &= R_3^i{}_{j m n} + P_{j m}^i|_n - P_{n j}^i|_m + P_{j m}^\alpha P_{n \alpha}^i - P_{n j}^\alpha P_{\alpha m}^i \\ &+ 2P_{n m}^\alpha \Gamma_{\check{\nu}}^i + 2P_{n m}^\alpha P_{\check{\nu}}^i \end{aligned}$$

i.e., in virtue of (0.2) and (1.3)

$$(3.2) \quad \begin{aligned} \overline{R}_3^i{}_{j m n} &= R_3^i{}_{j m n} + \delta_j^i (\psi_{m n} - \psi_{n m}) + \delta_m^i \psi_{j n} - \delta_n^i \psi_{j m} \\ &+ \psi_\alpha (\delta_n^i \xi_{j m}^\alpha - \delta_m^i \xi_{n j}^\alpha) + \xi_{j m}^i|_n - \xi_{n j}^i|_m + \xi_{j m}^\alpha \xi_{n \alpha}^i - \xi_{j n}^\alpha \xi_{\alpha m}^i \\ &+ 2\psi_n (\Gamma_{m \check{\nu}}^i + \xi_{m j}^i) + 2\psi_m (\Gamma_{n \check{\nu}}^i + \xi_{n j}^i) + 2\xi_{n m}^\alpha (\Gamma_{\check{\nu}}^i + \xi_{\alpha j}^i). \end{aligned}$$

Also, it holds

$$(3.3) \quad \psi_{m n} = \psi_{m n} + 2\Gamma_{m \check{\nu}}^\alpha \psi_\alpha.$$

From (3.2) and (3.3) we get

$$(3.4) \quad \begin{aligned} \overline{R}_3^i{}_{j m n} &= R_3^i{}_{j m n} + \delta_j^i (\psi_{m n} - \psi_{n m}) + \delta_m^i \psi_{j n} - \delta_n^i \psi_{j m} \\ &+ 2(\delta_j^i \Gamma_{m \check{\nu}}^\alpha + \delta_n^i \Gamma_{j \check{\nu}}^\alpha) \psi_\alpha + (\delta_n^i \xi_{j m}^\alpha - \delta_m^i \xi_{n j}^\alpha) \psi_\alpha \\ &+ \xi_{j m}^i|_n - \xi_{n j}^i|_m + \xi_{j m}^\alpha \xi_{n \alpha}^i - \xi_{j n}^\alpha \xi_{\alpha m}^i \\ &+ 2\psi_n (\Gamma_{m \check{\nu}}^i + \xi_{m j}^i) + 2\psi_m (\Gamma_{n \check{\nu}}^i + \xi_{n j}^i) + 2\xi_{n m}^\alpha (\Gamma_{\check{\nu}}^i + \xi_{\alpha j}^i). \end{aligned}$$

Contracting (3.4) with respect to i and n we get

$$(3.5) \quad \begin{aligned} \overline{R}_3^{j m} &= R_3^{j m} - 2\psi_{j m} + (1 - N)\psi_{j m} + (N + 1)\xi_{j m}^\alpha \psi_\alpha + \xi_{j m}^\beta|_2 \\ &- \xi_{j \beta}^\alpha \xi_{\alpha m}^\beta + 2\psi_\beta (\Gamma_{m \check{\nu}}^\beta + \xi_{m j}^\beta) + 2\xi_{\beta m}^\alpha (\Gamma_{\check{\nu}}^\beta + \xi_{\alpha j}^\beta), \end{aligned}$$

hence

$$(3.6) \quad \begin{aligned} (1+N)\psi_{j\underset{1}{\vee}m} &= R_{j\underset{3}{\vee}m} - \overline{R}_{j\underset{3}{\vee}m} + (N+1)\xi_{jm}^\alpha \psi_\alpha + \xi_{jm|\beta}^\beta \\ &+ 2\psi_\beta(\Gamma_{m\underset{3}{\vee}j}^\beta + \xi_{mj}^\beta) + \xi_{\beta m}^\alpha \Gamma_{\alpha j}^\beta - \xi_{\beta j}^\alpha \Gamma_{\alpha m}^\beta. \end{aligned}$$

Substituting (3.6) in (3.5) we get

$$(3.7) \quad \begin{aligned} (N-1)\psi_{jm} &= R_{jm} - \overline{R}_{jm} - \frac{1}{N+1} [2R_{j\underset{3}{\vee}m} - 2\overline{R}_{j\underset{3}{\vee}m} + 2(N+1)\xi_{jm}^\alpha \psi_\alpha + 2\xi_{jm|\beta}^\beta \\ &+ 4\psi_\beta(\Gamma_{m\underset{3}{\vee}j}^\beta + \xi_{mj}^\beta) + 2\xi_{\beta m}^\alpha \Gamma_{\alpha j}^\beta - 2\xi_{\beta j}^\alpha \Gamma_{\alpha m}^\beta] + (N+1)\xi_{jm}^\alpha \psi_\alpha \\ &+ \xi_{jm|\beta}^\beta - \xi_{j\beta}^\alpha \xi_{\alpha m}^\beta + 2\psi_\beta(\Gamma_{m\underset{3}{\vee}j}^\beta + \xi_{mj}^\beta) + 2\xi_{\beta m}^\alpha (\Gamma_{\alpha j}^\beta + \xi_{\alpha j}^\beta). \end{aligned}$$

Now, from (3.4,6,7) by conditions (3.1,1') we get

$$\overline{W}(\overline{R})^i_{jmn} = W(R)^i_{jmn},$$

where

$$(3.8) \quad \begin{aligned} W(R)^i_{jmn} &= R^i_{jmn} + \frac{2}{N+1} \delta_j^i R_{mn} + \frac{1}{N^2-1} [(NR_{jn} - R_{nj})\delta_m^i \\ &- (NR_{jm} - R_{mj})\delta_n^i] - \frac{2}{N+1} \delta_j^i \Gamma_{m\underset{2}{\vee}n|\beta}^\beta \\ &- \frac{1}{N+1} \delta_m^i \Gamma_{j\underset{2}{\vee}n|\beta}^\beta + \frac{1}{N+1} \delta_n^i \Gamma_{j\underset{2}{\vee}m|\beta}^\beta - \Gamma_{j\underset{2}{\vee}m|n}^i + \Gamma_{n\underset{2}{\vee}j|m}^i. \end{aligned}$$

Consequently, the next theorems hold:

THEOREM 3.1. *The tensor (3.8) is an invariant of an R_3 -projective mapping.*

THEOREM 3.3. *If GR_N is R_3 -projectively flat then*

$$W(R)^i_{jmn} = 0.$$

4. R_4 -projective mappings

DEFINITION 4.1. A geodesic mapping $f : GR_N \rightarrow G\overline{R}_N$ is R_4 -projective if the following condition is satisfied

$$\begin{aligned} &\delta_j^i (\mathcal{D}_{4mn} - \mathcal{D}_{4nm}) + \delta_m^i \mathcal{D}_{4jn} - \delta_n^i \mathcal{D}_{4jm} + 2(\delta_j^i \Gamma_{mn}^\alpha + \delta_m^i \Gamma_{jn}^\alpha) \psi_\alpha \\ &+ (\delta_n^i \xi_{jm}^\alpha - \delta_m^i \xi_{nj}^\alpha) \psi_\alpha + \xi_{jm}^\alpha \xi_{n\alpha}^i - \xi_{jn}^\alpha \xi_{\alpha m}^i + 2\psi_n (\Gamma_{mj}^i + \xi_{mj}^i) \\ &+ 2\psi_m (\Gamma_{nj}^i + \xi_{nj}^i) + 2\xi_{mn}^\alpha (\Gamma_{\alpha j}^i + \xi_{\alpha j}^i) = 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_4^{jm} &= \xi_{jm}^\alpha \psi_\alpha - \frac{1}{N^2 - 1} [4\psi_\beta (\Gamma_{m\check{v}}^\beta + \xi_{mj}^\beta) + 2\xi_{m\beta}^\alpha \Gamma_{\check{v}\alpha}^\beta - 2\xi_{j\beta}^\alpha \Gamma_{\check{v}\alpha}^\beta] \\ &\quad - \frac{1}{N - 1} [\xi_{j\beta}^\alpha \xi_{\alpha m}^\beta - 2\psi_\beta (\Gamma_{m\check{v}}^\beta + \xi_{mj}^\beta) - 2\xi_{m\beta}^\alpha (\Gamma_{\check{v}\alpha}^\beta + \xi_{\alpha j}^\beta)]. \end{aligned}$$

DEFINITION 4.2. The space GR_N is R -projectively flat if there exists an R -projective mapping of the space GR_N into a flat space.

For curvature tensors of the fourth kind (0.8) we get

$$\begin{aligned} \overline{R}_4^{ijmn} &= R_4^{ijmn} + P_{jm|n}^i - P_{nj|m}^i + P_{jm}^\alpha P_{n\alpha}^i - P_{nj}^\alpha P_{\alpha m}^i \\ &\quad + 2P_{mn}^\alpha \Gamma_{\check{v}\alpha}^i + 2P_{mn}^\alpha P_{\check{v}\alpha}^i \end{aligned}$$

i.e.

$$\begin{aligned} \overline{R}_4^{ijmn} &= R_4^{ijmn} + \delta_j^i (\psi_{mn} - \psi_{nm}) + \delta_m^i \psi_{jn} - \delta_n^i \psi_{jm} \\ &\quad + \psi_\alpha (\delta_n^i \xi_{jm}^\alpha - \delta_m^i \xi_{nj}^\alpha) + \xi_{jm|n}^i - \xi_{nj|m}^i + \xi_{jm}^\alpha \xi_{n\alpha}^i - \xi_{jn}^\alpha \xi_{\alpha m}^i \\ &\quad + 2\psi_n (\Gamma_{m\check{v}}^i + \xi_{mj}^i) + 2\psi_m (\Gamma_{n\check{v}}^i + \xi_{nj}^i) + 2\xi_{mn}^\alpha (\Gamma_{\check{v}\alpha}^i + \xi_{\alpha j}^i). \end{aligned}$$

Now analogously to the previous cases, we get an invariant of an R -projective mapping in the form

$$\begin{aligned} (4.1) \quad W(R)_4^i{}_{jmn} &= R_4^i{}_{jmn} + \frac{2}{N+1} \delta_j^i R_{m\check{v}}^n + \frac{1}{N^2-1} [(NR_{4jn} - R_{4nj})\delta_m^i \\ &\quad - (NR_{4jm} - R_{4mj})\delta_n^i] - \frac{2}{N+1} \delta_j^i \Gamma_{m\check{v}2}^\alpha - \frac{1}{N+1} \delta_m^i \Gamma_{j\check{v}2}^\alpha \\ &\quad + \frac{1}{N+1} \delta_n^i \Gamma_{j\check{v}2}^\alpha - \Gamma_{j\check{v}2}^i|_n + \Gamma_{n\check{v}1}^i|_m, \end{aligned}$$

i.e., the next theorems hold:

THEOREM 4.1. *The tensor (4.1) is an invariant of an R -projective mapping.*

THEOREM 4.2. *If GR_N is R -projectively flat then*

$$W(R)_4^i{}_{jmn} = 0.$$

5. R_5 -projective mappings

DEFINITION 5.1. A geodesic mapping $f : GR_N \rightarrow \overline{GR}_N$ is R_5 -projective if the following condition is satisfied

$$2\delta_m^i \xi_{j\beta}^\alpha \xi_{n\alpha}^\beta - 2\delta_n^i \xi_{j\beta}^\alpha \xi_{m\alpha}^\beta - (N-1)(\xi_{jm}^\alpha \xi_{\alpha n}^i - \xi_{jn}^\alpha \xi_{m\alpha}^i + \xi_{mj}^\alpha \xi_{n\alpha}^i - \xi_{nj}^\alpha \xi_{\alpha m}^i) = 0.$$

DEFINITION 5.2. The space GR_N is R_5 -projectively flat if there exists an R_5 -projective mapping of the space GR_N into a flat space.

For curvature tensors of the fifth kind (0.9) of the spaces GR_N and \overline{GR}_N we find the relation

$$\begin{aligned} \overline{R}_{5jmn}^i &= R_5^i jmn + \frac{1}{2}(P_{jm|_3}^i - P_{jn|_4}^i + P_{mj|_4}^i - P_{nj|_3}^i \\ &\quad + P_{jm}^\alpha P_{\alpha n}^i - P_{jn}^\alpha P_{m\alpha}^i + P_{mj}^\alpha P_{n\alpha}^i - P_{nj}^\alpha P_{\alpha m}^i) \end{aligned}$$

i.e.

$$\begin{aligned} \overline{R}_{5jmn}^i &= R_5^i jmn + \frac{1}{2}\delta_j^i (\psi_{mn} - \psi_{nm} + \psi_{mn} - \psi_{nm}) + \frac{1}{2}\delta_m^i (\psi_{jn} + \psi_{jn}) \\ (5.1) \quad &- \frac{1}{2}\delta_n^i (\psi_{jm} + \psi_{jm}) + \frac{1}{2}(\xi_{jm|_3}^i - \xi_{jn|_4}^i + \xi_{mj|_4}^i - \xi_{nj|_3}^i \\ &\quad + \xi_{jm}^\alpha \xi_{\alpha n}^i - \xi_{jn}^\alpha \xi_{m\alpha}^i + \xi_{mj}^\alpha \xi_{n\alpha}^i - \xi_{nj}^\alpha \xi_{\alpha m}^i). \end{aligned}$$

Putting

$$\psi_{jn} = \frac{1}{2}(\psi_{j_1 n} + \psi_{j_2 n})$$

we get from (5.1)

$$\begin{aligned} \overline{R}_{5jmn}^i &= R_5^i jmn + \delta_j^i (\psi_{mn} - \psi_{nm}) + \delta_m^i \psi_{jn} - \delta_n^i \psi_{jm} \\ (5.2) \quad &+ \frac{1}{2}(\xi_{jm|_3}^i - \xi_{jn|_4}^i + \xi_{mj|_4}^i - \xi_{nj|_3}^i + \xi_{jm}^\alpha \xi_{\alpha n}^i - \xi_{jn}^\alpha \xi_{m\alpha}^i + \\ &\quad + \xi_{mj}^\alpha \xi_{n\alpha}^i - \xi_{nj}^\alpha \xi_{\alpha m}^i). \end{aligned}$$

Eliminating ψ_{mn} from (5.2), analogously to the previous cases, we get

$$\overline{W}(\overline{R}_5)^i jmn = W(R_5)^i jmn,$$

where

$$\begin{aligned}
 W(R)_5^i{}_{jmn} &= R_5^i{}_{jmn} + \frac{2}{N+1} \delta_j^i R_5^m{}_{\nu n} + \frac{1}{N^2-1} [(NR_5^j{}_{jn} - R_5^nj) \delta_m^i \\
 &\quad - (NR_5^j{}_{jm} - R_5^mj) \delta_n^i] - \frac{1}{N+1} \delta_j^i (\Gamma_{m\nu 3}^\alpha|_\alpha + \Gamma_{nm 4}^\alpha|_\alpha) \\
 &\quad - \frac{1}{2(N+1)} \delta_m^i (\Gamma_{jn 3}^\alpha|_\alpha + \Gamma_{nj 4}^\alpha|_\alpha) + \frac{1}{2(N+1)} \delta_n^i (\Gamma_{jm 3}^\alpha|_\alpha + \Gamma_{mj 4}^\alpha|_\alpha) \\
 &\quad - \frac{1}{2} (\Gamma_{jm 3}^i|_n - \Gamma_{jn 4}^i|m + \Gamma_{mj 4}^i|n - \Gamma_{nj 3}^i|m).
 \end{aligned}
 \tag{5.3}$$

Hence:

THEOREM 5.1. *The tensor (5.3) is an invariant of a R -projective mapping.*

THEOREM 5.2. *If GR_N is R -projectively flat, then*

$$W(R)_5^i{}_{jmn} = 0.$$

Remark. If $GR_N(G\bar{R}_N)$ reduces to $R_N(\bar{R}_N)$, then the tensors $W(R)_\theta^i{}_{jmn}$ ($\theta = 1, \dots, 5$) reduce to Weyl's tensor (0.10).

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