

## REGULARLY VARYING SEQUENCES AND ENTIRE FUNCTIONS OF FINITE ORDER

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*Communicated by Stevan Pilipović*

ABSTRACT. We present a method for estimating the asymptotic behavior of:

$$f^\alpha(x) := \sum_{n=1}^{\infty} n^\alpha l_n a_n x^n, \quad x \rightarrow \infty, \quad \alpha \in R,$$

related to a given entire function  $f(x) := \sum_{n=1}^{\infty} a_n x^n$  of finite order  $\rho$ ,  $0 < \rho < +\infty$ ,  $a_n \geq 0$ ,  $n \in N$ ; where  $(l_n)$ ,  $n \in N$ , are slowly varying sequences in Karamata's sense.

### Preliminaries

**A.** Slowly varying functions  $l(x)$  in Karamata's sense are defined on a positive part of real axis, positive, locally bounded and satisfy:  $\lim_{x \rightarrow \infty} \frac{l(\lambda x)}{l(x)} = 1$ , for each  $\lambda > 0$ .

The class  $R_\alpha$  of regularly varying functions (r.v.f.) with index  $\alpha$  consists of all functions  $a(x)$  which can be represented as:  $a(x) = x^\alpha l(x)$ , for some  $\alpha \in R$ .

The theory of r.v.f. is very well developed and an excellent survey of results is given in [1] and [3].

Here we put special attention on a class  $SR_\alpha \subset R_\alpha$  (smoothly varying functions; [1, p. 44]) i.e.,  $b(x) \in SR_\alpha$  if it is a  $C^\infty$  r.v.f. of index  $\alpha$ , satisfying

$$x^n b^{(n)}(x)/b(x) \rightarrow \alpha(\alpha-1) \cdots (\alpha-n+1) = (\alpha)_n, \quad x \rightarrow \infty, \quad n \in N.$$

Some important properties of this class are:

If  $f \in SR_\alpha$ ,  $g \in SR_\beta$ , then

$$f \cdot g \in SR_{\alpha+\beta}; \quad f \circ g \in SR_{\alpha\beta}; \quad f' \in SR_{\alpha-1}, \quad \alpha \in R^+.$$

Also, for a given  $c(x) = x^{-\alpha} l(x)$ ,  $\alpha \in R^+$ , we consider its dual  $c^*(x)$  defined by

$$c^*(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xy} \frac{c(1/y)}{y} dy = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xy} y^{\alpha-1} l(1/y) dy; \quad x, \alpha \in R^+.$$

The next proposition is of crucial importance.

PROPOSITION 1. We have:  $c^*(x) \in SR_{-\alpha}$ ;  $c^*(x) \sim c(x)$ ,  $x \rightarrow \infty$ .

*Proof.* That  $c^*(x) \sim c(x)$ ,  $x \rightarrow \infty$ , is a consequence of Karamata's Tauberian theorem for Laplace transforms (with  $0^+$  and  $\infty$  reversed) (cf. [1, p. 43]).

By the same argument

$$(c^*(x))^{(n)} = \frac{(-1)^n}{\Gamma(\alpha)} \int_0^\infty e^{-xy} y^{\alpha+n-1} l\left(\frac{1}{y}\right) dy \sim \frac{(-1)^n \Gamma(\alpha+n)}{\Gamma(\alpha)} x^{-(\alpha+n)} l(x), \quad x \rightarrow \infty.$$

Hence,

$$\frac{x^n (c^*(x))^{(n)}}{c^*(x)} \sim \frac{(-1)^n \Gamma(\alpha+n)}{\Gamma(\alpha)} = (-\alpha)_n, \quad x \rightarrow \infty;$$

i.e.,  $c^*(x) \in SR_{-\alpha}$ .

We could treat regularly varying sequences (r.v.s.) as r.v.f. defined on  $N$  (see [2]) i.e.,  $(a_n)$  is a r.v.s. with index  $\alpha$  if it has the form

$$a_n = n^\alpha l_n; \quad l_n = l(n), \quad n \in N, \quad \alpha \in R,$$

for some slowly varying function  $l(x)$  defined for  $x \in R^+$ .

Examples of  $l_n$  are:

$$\ln^a 2n, \quad \ln^b (\ln 3n), \quad e^{(\ln n)^c}, \quad e^{\frac{\ln 2n}{\ln(\ln 3n)}}, \dots; \quad a, b \in R, \quad 0 < c < 1.$$

**B.** Denote by  $\mathcal{G} := \{g \mid g: R^+ \rightarrow R^+, g \in C^1\}$ , and define there an operator  $\widehat{g}$ ,

$$\widehat{g}(x) := \frac{xg'(x)}{g(x)}.$$

Some properties of this operator are ( $g, h \in \mathcal{G}$ ):

1.  $\widehat{cg} = \widehat{g}$ ,  $c \in R^+$ ;
2.  $\widehat{x^a} = a$ ,  $a \in R$ ;
3.  $\widehat{g+h} \leq \max(|\widehat{g}|, |\widehat{h}|)$ ;
4.  $\widehat{g \cdot h} = \widehat{g} + \widehat{h}$ ;
5.  $\widehat{g^a \cdot h^b} = a\widehat{g} + b\widehat{h}$ ,  $a, b \in R$ ;
6.  $\widehat{g \circ h} = (\widehat{g} \circ h) \cdot \widehat{h}$ ;
7.  $\widehat{g} \in \mathcal{G} \Rightarrow g \uparrow$ ,  $x \in R^+$ ;
8.  $(\widehat{g}(x) \rightarrow \alpha, x \rightarrow \infty, \alpha \in R) \Rightarrow g \in R_\alpha$ .

We also consider a set of entire functions  $F(z) = \sum_{k=0}^\infty a_k z^k$ , with non-negative coefficients and of finite order  $\rho$ ,  $0 < \rho < \infty$ . By definition:

$$\rho = \limsup_{x \rightarrow \infty} \frac{\ln \ln M_F(x)}{\ln x}$$

where  $M_F(x)$  denotes the maximum modulus of  $F(z)$  on the circle  $|z| = x$ .

In our case we have:

$$M_F(x) = \max_{|z|=x} |F(z)| = \max_{|z|=x} \left| \sum_{k=0}^\infty a_k z^k \right| = \sum_{k=0}^\infty a_k x^k = F(x), \quad x \in R^+.$$

Let us denote:  $f(x) := M_F(x) = F(x)$ ,  $x \in R^+$ . Hence:

$$\limsup_{x \rightarrow \infty} \frac{\ln \ln f(x)}{\ln x} = \rho, \quad \rho \in R^+, \quad (2)$$

$$f \in C^\infty; \quad f^{(n)} \in \mathcal{G}; \quad \widehat{f} \in \mathcal{G}.$$

PROPOSITION 2. We have  $\widehat{f} \in \mathcal{G}$ .

*Proof.* Taking into account properties (1) and (2), we have:

$$\widehat{f} = \widehat{xf'} - \widehat{f} = \frac{1}{\widehat{f}} \left( \frac{x(xf')'}{f} - \widehat{f}^2 \right).$$

Since

$$x(xf')' = \sum_{k=0}^{\infty} k^2 a_k x^k; \quad \widehat{f} = \frac{1}{f} \sum_{k=0}^{\infty} k a_k x^k,$$

we obtain:

$$\widehat{f} = \frac{1}{f\widehat{f}} \sum_{k=0}^{\infty} (k - \widehat{f})^2 a_k x^k > 0, \quad x \in R^+;$$

and the proof is over.

COROLLARY 1. The function  $\widehat{f}$  is monotone increasing on  $R^+$ .

PROPOSITION 3. We have:  $\limsup_{x \rightarrow \infty} \frac{\ln \widehat{f}(x)}{\ln x} = \rho$ .

*Proof.* Let

$$\delta := \limsup_{x \rightarrow \infty} \frac{\ln \widehat{f}(x)}{\ln x}, \quad \delta \in R^+.$$

From (2) it follows that, for each positive  $\epsilon$  and large enough  $x$ :

$$\ln f(x) < x^{\rho+\epsilon}, \quad x > x_0.$$

Corollary 1 gives

$$\ln f(ex) - \ln f(x) = \int_x^{ex} \frac{f'(t)}{f(t)} dt = \int_x^{ex} \widehat{f}(t) \cdot \frac{dt}{t} > \widehat{f}(x) \int_x^{ex} \frac{dt}{t} = \widehat{f}(x),$$

hence, for  $x > x_0$ , we get:

$$\widehat{f}(x) < \ln f(ex) < (ex)^{\rho+\epsilon},$$

i.e.,

$$\frac{\ln \widehat{f}(x)}{\ln x} < (\rho + \epsilon)(1 + 1/\ln x), \quad x > x_0.$$

Since  $\epsilon$  is arbitrarily small, we conclude  $\delta \leq \rho$ .

From the other side  $\ln \widehat{f}(x) < (\delta + \epsilon) \ln x$  for  $x > x_1$ , i.e.,

$$\widehat{f}(x) < x^{\delta+\epsilon}, \quad \frac{f'(x)}{f(x)} < x^{\delta-1+\epsilon}; \quad x > x_1.$$

It follows that

$$\ln f(x) = \int_{x_1}^x \frac{f'(t)}{f(t)} dt + \ln f(x_1) < \frac{x^{\delta+\epsilon}}{\delta+\epsilon} + O(1), \quad x > x_1;$$

i.e.,

$$\frac{\ln \ln f(x)}{\ln x} < \delta + \epsilon + o(1), \quad x \rightarrow \infty;$$

i.e.,

$$\rho \leq \delta.$$

Thus, we conclude:

$$\limsup_{x \rightarrow \infty} \frac{\ln \widehat{f}(x)}{\ln x} = \limsup_{x \rightarrow \infty} \frac{\ln \ln f(x)}{\ln x} = \rho.$$

**COROLLARY 2.** *The function  $\widehat{f}(x)$  is strictly increasing on  $R^+$  and  $\lim_{x \rightarrow \infty} \widehat{f}(x) = +\infty$ .*

We also consider the set of entire functions  $\{f_m\}$  generated from  $f$  by the recurrence relation

$$f_m(x) := x f'_{m-1}(x), \quad f_0(x) = f(x), \quad m \in N.$$

They are of the same order  $\rho$  and evidently satisfy:

$$\text{PROPOSITION 4.} \quad f_m(x) = \sum_k k^m a_k x^k; \quad \widehat{f}_m(x) \uparrow \infty; \quad \widehat{f}_m(x) > 0$$

$$f_m = f_{m-1} \widehat{f}_{m-1} = f \prod_1^m \widehat{f}_{k-1}; \quad \widehat{f}_m = \widehat{f}_{m-1} + \widehat{\widehat{f}}_{m-1} = \widehat{f} + \sum_1^m \widehat{\widehat{f}}_{m-1}.$$

### Main results

Now we come to our main subject, i.e., the investigation of the asymptotic behavior concerning functions  $f^\alpha(x) := \sum_{k=0}^{\infty} c_k a_k x^k$ , related to a given entire function  $f(x) := \sum_{k=0}^{\infty} a_k x^k$  considered before and where  $(c_k)$ ,  $(c_0 := 1)$  is any regularly varying sequence of index  $\alpha$ .

It is not difficult to prove that  $\{f^\alpha(x)\}$  are also entire functions of the same order  $\rho$  as  $f(x)$  (using, for example, the relation:  $\rho = \limsup_{n \rightarrow \infty} \frac{n \ln n}{\ln(1/|a_n|)}$ ).

The main idea of our method is to replace sequences  $(c_k)$  with asymptotically equivalent  $(c_k^*)$  achieving thus an integral representation for  $f^\alpha(x)$  (see also [5]). Then, using analytic properties of  $c^*(x)$  and  $f(x)$ , we establish the required asymptotic behavior in an almost elementary way.

THEOREM A. If  $\widehat{f}(x)$  is bounded from above, then:

$$\frac{f^\alpha(x)}{f(x)} \sim c_{[\widehat{f}(x)]}, \quad x \rightarrow \infty,$$

for any regularly varying sequence  $(c_k)$  of index  $\alpha$ ,  $\alpha < 0$ .

As we already explained, we first prove the theorem for a subclass of r.v.s. generated by  $c^*(x) \in SR_\alpha$ , i.e.,

PROPOSITION A1. Theorem A is valid for sequences  $(c_k^*)$  defined by

$$c_{k-1}^* := c^*(k), \quad k \in N; \quad c^*(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xt} t^{\alpha-1} l(1/t) dt, \quad x \in R^+.$$

*Proof.* With:  $u(\alpha, t) := \frac{1}{\Gamma(\alpha)} t^{\alpha-1} l(1/t)$  we produce an integral representation for:

$$\begin{aligned} f_*^{(\alpha)}(x) &:= \sum_{k=0}^\infty c_k^* a_k x^k = \sum_{k=0}^\infty a_k \int_0^\infty e^{-t} u(\alpha, t) (xe^{-t})^k dt = \\ &= \int_0^\infty e^{-t} u(\alpha, t) \left( \sum_{k=0}^\infty a_k (xe^{-t})^k \right) dt = \int_0^\infty e^{-t} u(\alpha, t) f(xe^{-t}) dt. \end{aligned}$$

The interchanging of the sum and the integral is justified since both converge for  $x \in R^+$ . Now:

$$\frac{f_*^{(\alpha)}(x)}{f(x)} = \left( \int_0^\xi + \int_\xi^\infty \right) \left( e^{-t} u(\alpha, t) \frac{f(xe^{-t})}{f(x)} dt \right) = T_1 + T_2$$

where  $\xi = \xi(x) := \widehat{f}(x)^{-1/2}$ .

For estimating  $T_1$  we use the following identity:

$$\ln \frac{f(xe^{-t})}{f(x)} + t \widehat{f}(x) = \int_0^t w \widehat{f}(a) \widehat{f}(a) dw, \quad a := xe^{w-t}.$$

Taking into account Proposition 2 and condition from Theorem A, we have:

$$0 < \widehat{f}(a) \leq M < +\infty,$$

where the constant  $M$  does not depend on  $a$ .

Also, since  $a \leq x$ , Corollary 1 gives  $\widehat{f}(a) \leq \widehat{f}(x)$ , i.e.,

$$0 < \int_0^t w \widehat{f}(a) \widehat{f}(a) dw \leq M \widehat{f}(x) \int_0^t w dw = \frac{M}{2} \widehat{f}(x) t^2.$$

Hence,

$$\ln \frac{f(xe^{-t})}{f(x)} = \widehat{f}(x)(-t + O(t^2)), \quad x \in R^+, \quad t \geq 0$$

where the constant in  $O$  is independent of  $x$  or  $t$ . From here it follows

$$T_1 = \int_0^\xi e^{-t} u(\alpha, t) \exp\left(\ln \frac{f(xe^{-t})}{f(x)}\right) dt = \int_0^\xi e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)} e^{O(\widehat{f}(x)t^2)} dt.$$

Since, for any  $B \in R^+$ ,  $e^B = 1 + O(Be^B)$  and, for  $t \in (0, \xi)$ ,  $\widehat{f}(x)t^2 = O(1)$ , we obtain:

$$\begin{aligned} T_1 &= \int_0^\xi e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)} dt + \int_0^\xi e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)} O(\widehat{f}(x)t^2) dt = \\ &= \int_0^\infty e^{-t(1+\widehat{f}(x))} u(\alpha, t) dt - \int_\xi^\infty e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)} dt \\ &\quad + O(\widehat{f}(x)) \int_0^\infty t^2 u(\alpha, t) e^{-t(1+\widehat{f}(x))} dt = T_{11} + T_{12} + T_{13}. \end{aligned}$$

Now:

$$\begin{aligned} T_{11} &= c^*(\widehat{f}(x) + 1) \sim c_{[\widehat{f}(x)]}^*, \quad x \rightarrow \infty; \\ |T_{12}| &= O(e^{-\xi\widehat{f}(x)} \int_0^\infty e^{-t} u(\alpha, t) dt) = O(e^{-\widehat{f}(x)^{1/2}}); \\ T_{13} &= O(\widehat{f}(x)) \cdot \frac{d^2 c^*(s)}{ds^2} \Big|_{s=1+\widehat{f}(x)} \\ &= O(\widehat{f}(x)) \cdot O\left(\frac{c^*(s)}{s^2}\right) \Big|_{s=1+\widehat{f}(x)} = O\left(\frac{c^*(\widehat{f}(x))}{\widehat{f}(x)}\right), \end{aligned}$$

since  $c^*(s) \in SR_\alpha$ . Hence, we conclude that:  $T_1 \sim c_{[\widehat{f}(x)]}^*$ ,  $x \rightarrow \infty$ .

For the estimation of the integral  $T_2$  the next lemma is necessary.

LEMMA A1. *Under the condition of Theorem A, i.e.,  $\sup \widehat{f}(x) \leq M < +\infty$ , for each  $x, t \in R^+$ :*

$$\frac{f(xe^{-t})}{f(x)} \leq \exp\left(\frac{e^{-Mt} - 1}{M} \widehat{f}(x)\right).$$

*Proof.* Write the condition as

$$D(\ln \widehat{f}(s)) \leq M D(\ln s), \quad s > 0. \quad (A1.1)$$

Integrating (A1.1) over  $[xe^{-u}, x]$ ,  $u \geq 0$ , we obtain

$$\widehat{f}(xe^{-u}) \geq \widehat{f}(x) \cdot e^{-Mu}. \quad (A1.2)$$

Integrating (A1.2) for  $u \in [0, t]$ , we come to the conclusion from the lemma. Therefore,

$$\begin{aligned} T_2 &= \int_{\xi}^{\infty} e^{-t} u(\alpha, t) \frac{f(xe^{-t})}{f(x)} dt \leq \int_{\xi}^{\infty} e^{-t} u(\alpha, t) \exp\left(\frac{e^{-Mt} - 1}{M} \widehat{f}(x)\right) dt \\ &< \exp\left(\frac{e^{-M\widehat{f}(x)^{-1/2}} - 1}{M} \widehat{f}(x)\right) \cdot \int_0^{\infty} e^{-t} u(\alpha, t) dt; \end{aligned}$$

i.e.,

$$T_2 = O(e^{-\widehat{f}(x)^{1/2}}), \quad x \rightarrow \infty;$$

so, Proposition A1 is proved.

The assertion of Theorem A follows using the fact  $c_n \sim c_n^*$ ,  $n \rightarrow \infty$  and a variant of Toeplitz's Limit Preservation Theorem (cf. [8, p. 36]) which says:

Let  $\{\phi_k(x)\}$ ,  $k = 0, 1, 2, \dots$ , be a set of non-negative functions defined on  $R^+$ , satisfying  $\sum_k \phi_k(x) = 1$ , and let  $(s_k)$ ,  $k = 0, 1, 2, \dots$  be any convergent sequence of positive reals,  $\lim s_k = s$ .

Then a necessary and sufficient condition for  $\sum_k s_k \phi_k(x) \rightarrow s$ ,  $x \rightarrow \infty$ , is  $\lim_{x \rightarrow \infty} \phi_k(x) = 0$ , for each fixed  $k \in N$ .

We are going to use this proposition by putting:

$$\phi_k(x) := \frac{c_k^* a_k x^k}{f_*^{(\alpha)}(x)}; \quad s_k := \frac{c_k}{c_k^*}, \quad k = 0, 1, 2, \dots$$

Then,

$$\sum_k \phi_k(x) = 1; \quad s = 1; \quad \sum_k s_k \phi_k(x) = \frac{f^{(\alpha)}(x)}{f_*^{(\alpha)}(x)},$$

and all we have to prove is  $\lim_{x \rightarrow \infty} \phi_n(x) = 0$  for fixed  $n$ .

For  $a_n \neq 0$  (otherwise, there is nothing to prove) write

$$\phi_n(x) = \frac{c_n^* a_n x^n}{f_*^{(\alpha)}(x)} = c_n^* \left( \frac{a_n x^n}{f(x/2)} \right) \left( \frac{f(x/2)}{f(x)} \right) \left( \frac{f(x)}{f_*^{(\alpha)}(x)} \right).$$

From Proposition A1:

$$\frac{f(x)}{f_*^{(\alpha)}(x)} \sim 1/c_{[\widehat{f}(x)]}^* = O(\widehat{f}(x)^{2|\alpha|}).$$

Lemma A1, for  $t = \ln 2$ , gives

$$\frac{f(x/2)}{f(x)} \leq \exp\left(-\frac{1 - 2^{-M}}{M} \widehat{f}(x)\right), \quad M > 0;$$

and, evidently:  $f(x/2) > a_n(x/2)^n$ . Hence,

$$\phi_n(x) = O\left(\widehat{f}(x)^{2|\alpha|} \exp\left(-\frac{1-2^{-M}}{M}\widehat{f}(x)\right)\right) = o(1), \quad x \rightarrow \infty, \quad \widehat{f}(x) \uparrow \infty,$$

i.e.,

$$f^{(\alpha)}(x) \sim f_*^{(\alpha)}(x) \sim f(x) c_{[\widehat{f}(x)]}^* \sim f(x) c_{[\widehat{f}(x)]}, \quad x \rightarrow \infty; \quad \alpha < 0;$$

therefore, Theorem A is valid.

Our task now is to extend the validity of Theorem A to non-negative indexes of r.v.s. ( $c_k$ ). First of all, we prove

PROPOSITION 5. *Under conditions of Theorem A, for any  $\alpha \geq 0$  we have*

$$\liminf_{x \rightarrow \infty} \frac{f^{(\alpha)}(x)}{(\widehat{f}(x))^{\alpha l} (\widehat{f}(x)) f(x)} \geq 1.$$

*Proof.* We use a form of Hölder's inequality:

$$\sum_k u_k w_k \geq \left(\sum_k u_k^p\right)^{1/p} \left(\sum_k w_k^q\right)^{1/q}, \quad u_k, w_k \in R^+, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad q < 0. \quad (7.1)$$

Putting there

$$u_k = k^{-1} (l_k a_k x^k)^{1/p}; \quad w_k = \begin{cases} k^{\alpha+1} (l_k a_k x^k)^{1/q}, & a_k > 0; \\ 0, & a_k = 0 \end{cases}, \quad k \in N,$$

we obtain:

$$f^{(\alpha)}(x) \geq (f^{(-p)}(x))^{1/p} (f^{(q(\alpha+1))}(x))^{1/q}.$$

Theorem A gives

$$\begin{aligned} \liminf_{x \rightarrow \infty} \frac{f^{(\alpha)}(x)}{(\widehat{f}(x))^{\alpha l} (\widehat{f}(x)) f(x)} &\geq \\ f &\geq \lim_{x \rightarrow \infty} \left( \frac{f^{(-p)}(x)}{(\widehat{f}(x))^{-p l} (\widehat{f}(x)) f(x)} \right)^{1/p} \cdot \lim_{x \rightarrow \infty} \left( \frac{f^{(q(\alpha+1))}(x)}{(\widehat{f}(x))^{q(\alpha+1) l} (\widehat{f}(x)) f(x)} \right)^{1/q} = 1. \end{aligned}$$

An extension of Theorem A is the following

THEOREM A'. *For any  $\epsilon > 0$ , Theorem A is valid for  $\alpha \leq 2 - \epsilon$ .*

*Proof.* Put in (7.1):

$$u_k^p = k^{2-\epsilon} l_k a_k x^k; \quad w_k^q = \begin{cases} k^2 a_k x^k, & a_k > 0 \\ 0, & a_k = 0 \end{cases}, \quad k \in N.$$



Since we could restrict  $0 < \epsilon \leq 2$ , by taking  $p = \epsilon/3$ , it follows that

$$\sum_k k^{2-\epsilon} l_k a_k x^k \leq f_2^{-p/q} \left( \sum_k k^{-1} l_k^{1/p} a_k x^k \right)^p, \quad k \in N.$$

But

$$f_2 = f \widehat{f} \widehat{f}_1 = f \widehat{f} (\widehat{f} + \widehat{\widehat{f}}) \sim f \widehat{f}^2, \quad x \rightarrow \infty,$$

and

$$\sum_k k^{-1} l_k^{1/p} a_k x^k \sim \widehat{f}^{-1} (l(\widehat{f}))^{1/p} f, \quad x \rightarrow \infty.$$

Hence,

$$\limsup_{x \rightarrow \infty} \frac{f^{(2-\epsilon)}(x)}{(\widehat{f}(x))^{2-\epsilon} l(\widehat{f}(x)) f(x)} \leq \lim_{x \rightarrow \infty} \frac{(f(x) \widehat{f}(x)^2)^{-p/q} (\widehat{f}(x))^{-1} (l(\widehat{f}(x)))^{1/p} f(x)^p}{(\widehat{f}(x))^{2-\epsilon} l(\widehat{f}(x)) f(x)} = 1.$$

This together with Proposition 5 proves the theorem.

Therefore we see that the Theorems A and A' provide the required asymptotic behavior of  $f^{(\alpha)}(x)$  for all regularly varying sequences  $(c_k)$  with index less than 2.

### Commentaries

The condition  $\sup \widehat{f} < +\infty$  seems a little ambiguous but is not very restrictive, as we are going to show.

The explicit representation

$$\widehat{f}(x) = f(1) \exp \left( \int_1^x \frac{\widehat{f}(t)}{t} dt \right); \quad 0 < \widehat{f}(t) \leq M,$$

means that  $\widehat{f}$  belongs to the class ER (Extended Variation, see [1, p. 74]).

Moreover, from  $\widehat{f}(x) = \widehat{f}_1(x) - \widehat{f}(x)$ , we see that  $\widehat{f}$  is of bounded variation (as a difference between two monotone increasing functions), i.e., bounded on finite intervals. Therefore, condition in Theorem A could be replaced by  $\limsup_{x \rightarrow \infty} \widehat{f}(x) < +\infty$ .

Strengthening this a bit, we obtain:

PROPOSITION 6. *If there exist  $\lim_{x \rightarrow \infty} \widehat{f}(x) = \delta$ , then  $\delta = \rho$  and*

$$\lim_{x \rightarrow \infty} \frac{\ln \ln f(x)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\widehat{f}(x)}{\ln x} = \lim_{x \rightarrow \infty} \widehat{f}(x) = \rho.$$

Moreover, in this case  $\widehat{f} \in R_\rho$ .

*Proof.* This follows from Proposition 2 and

$$\widehat{f}(x) = \frac{D(\ln \widehat{f}(x))}{D(\ln x)} = \frac{D(xD(\ln f(x)))}{D(\ln f(x))}.$$

The second part is Property 8 from (1) (cf. [1, p. 59]).

At this point we could connect asymptotically  $\widehat{f}$  with  $\ln f$ ; namely:

PROPOSITION 7. *The following are equivalent:*

- (i)  $\ln f(x) \sim a x^\rho b(x)$ ;
- (ii)  $\widehat{f}(x) \sim a \rho x^\rho b(x)$ ,  $x \rightarrow \infty$ ;  $b(x) \in R_0$ ,  $a, \rho \in R^+$ .

*Proof.* (ii)  $\Rightarrow$  (i): Since:  $\ln f(x) = \int_1^x \widehat{f}(t) \cdot dt/t + O(1)$ , the statement follows from r.v.f. Integration Theorem [3], i.e.,

$$\ln f(x) \sim a \rho b(x) \int_1^x t^\rho dt/t + O(1) \sim a x^\rho b(x), \quad x \rightarrow \infty.$$

(i)  $\Rightarrow$  (ii): Corollary 2 gives, for  $x \geq y > 0$ :

$$\ln f(x) - \ln f(y) = \int_y^x \widehat{f}(t) dt/t \begin{cases} \leq \widehat{f}(x) \ln x/y \\ \geq \widehat{f}(y) \ln x/y, \end{cases} \quad (5.1)$$

Putting in (5.1)  $x = \lambda y$ ,  $\lambda > 1$  and  $y = \lambda x$ ,  $\lambda < 1$ , we get

$$\widehat{f}(x) \begin{cases} \leq \frac{\ln f(\lambda x) - \ln f(x)}{\ln \lambda}, & \lambda > 1 \\ \geq \frac{\ln f(x) - \ln f(\lambda x)}{\ln 1/\lambda}, & 0 < \lambda < 1. \end{cases}$$

Therefore,

$$\limsup_{x \rightarrow \infty} \frac{\widehat{f}(x)}{a x^\rho b(x)} \leq \frac{1}{\ln \lambda} \left( \lim_{x \rightarrow \infty} \frac{\ln f(\lambda x)}{a x^\rho b(x)} - \lim_{x \rightarrow \infty} \frac{\ln f(x)}{a x^\rho b(x)} \right) = \frac{\lambda^\rho - 1}{\ln \lambda}, \quad \lambda > 1; \quad (5.2)$$

and analogously:

$$\liminf_{x \rightarrow \infty} \frac{\widehat{f}(x)}{a x^\rho b(x)} \geq \frac{1 - \lambda^\rho}{\ln 1/\lambda}, \quad 0 < \lambda < 1. \quad (5.3)$$

Since the right-hand side does not depend on  $x$ , putting  $\lambda \downarrow 1$  in (5.2) and  $\lambda \uparrow 1$  in (5.3), we obtain the statement from Proposition 7.

Further extension needs some smoothness condition on  $f$ , i.e.,

THEOREM B. *If  $\ln f(x) \in SR_\rho$ , then*

$$\frac{f^{(\beta)}(x)}{f(x)} \sim \rho^\beta c_{[\ln f(x)]}, \quad x \rightarrow \infty;$$

for any regularly varying sequence  $(c_k)$  of arbitrary index  $\beta \in R$ .

For justification of the condition from this theorem we cite an adapted version of Valiron's Proximate Order Theorem i.e., (cf. [1, p. 311]):

If  $f$  is an entire function of finite order  $\rho$ , then there always exists a  $g \in SR_\rho$ , with:

$$\limsup_{x \rightarrow \infty} \frac{\ln f(x)}{g(x)} = 1.$$

We prove first:

PROPOSITION B1. *If  $\ln f(x) \in SR_\rho$ , then  $\ln f_m(x) \in SR_\rho$  and  $\lim_{x \rightarrow \infty} \widehat{\widehat{f}}_m(x) = \rho$  for each  $m \in N$ .*

*Proof.* Suppose that  $\ln f_{m-1} \in SR_\rho$ , i.e.,

$$\frac{x^n (\ln f_{m-1})^{(n)}}{\ln f_{m-1}(x)} \rightarrow (\rho)_n, \quad x \rightarrow \infty.$$

Using properties of the class  $SR$  and Proposition 4 (see Preliminaries), we have  $D(\ln f_{m-1}(x)) \in SR_{\rho-1}$ , i.e.,  $\widehat{f}_{m-1}(x) = xD(\ln f_{m-1}(x)) \in SR_\rho$ . Since  $\ln x \in SR_0$ , we have  $\ln \widehat{f}_{m-1}(x) \in SR_0$ . Hence

$$x^n (\ln \widehat{f}_{m-1}(x))^{(n)} = o(\ln \widehat{f}_{m-1}(x)), \quad x \rightarrow \infty,$$

and

$$\begin{aligned} \frac{x^n (\ln f_m(x))^{(n)}}{\ln f_m(x)} &= \frac{x^n (\ln f_{m-1}(x) + \ln \widehat{f}_{m-1}(x))^{(n)}}{\ln f_{m-1}(x) + \ln \widehat{f}_{m-1}(x)} \sim \\ &\sim \frac{(\rho)_n \ln f_{m-1}(x) + o(\ln \widehat{f}_{m-1}(x))}{\ln f_{m-1}(x) + \ln \widehat{f}_{m-1}(x)} \sim (\rho)_n, \quad x \rightarrow \infty. \end{aligned}$$

Therefore,  $\ln f_m(x) \in SR_\rho$  and, analogously,  $\widehat{f}_m(x) \in SR_\rho$ . Also

$$\widehat{\widehat{f}}_m(x) = \frac{x(\widehat{f}_m(x))'}{\widehat{f}_m(x)} \rightarrow \rho, \quad x \rightarrow \infty;$$

and, since  $\ln f_0(x) = \ln f(x) \in SR_\rho$ , the proof is finished by induction.

This proposition and our former considerations show that we could apply Theorem A to  $f_m(x)$  for some fixed  $m \in N$ . We obtain:

$$f_m^{(\alpha)}(x) = f^{(m+\alpha)}(x) \sim f_m(x) \cdot (\widehat{f}_m(x))^{\alpha} l(\widehat{f}_m(x)), \quad \alpha < 0, \quad x \rightarrow \infty. \quad (\text{B.1})$$

But (Proposition 4),

$$\widehat{f}_k(x) = \widehat{f}(x) + \sum_1^k \widehat{f}_{l-1}(x) = \widehat{f}(x) + O(k\rho) \sim \widehat{f}(x), \quad x \rightarrow \infty, \quad k = 1, 2, \dots, m;$$

and

$$f_m(x) = f(x) \prod_1^m \widehat{f}_{l-1} = f(x) \prod_1^m (\widehat{f}(x) + O(1)) \sim f(x) (\widehat{f}(x))^m, \quad x \rightarrow \infty.$$

From (B.1) and Proposition 7 it follows that:

$$f^{(m+\alpha)}(x) \sim f(x) \cdot (\widehat{f}(x))^{m+\alpha} l(\widehat{f}(x)) \sim \rho^{m+\alpha} [(\ln f(x))^{m+\alpha} l(\ln f(x))], \quad \alpha < 0, \quad x \rightarrow \infty.$$

Putting  $m + \alpha = \beta$  we see that Theorem B is valid for  $\beta < m$ . Since  $m$  is an arbitrary integer, the proof is done.

Finally, for an illustration of our results, we give two characteristic examples.

EXAMPLE 1. Consider an entire function  $g$  of integer order  $p$  in the form:

$$g(x) := \exp P_p(x) = \sum_k a_k x^k,$$

where  $P_p(x) := b_p x^p + \dots$ ,  $b_p > 0$ , is a polynomial with nonnegative coefficients. Since  $\ln g(x) = P_p(x) \in SR_p$ , applying Theorem B we obtain:

PROPOSITION 8.  $e^{-P_p(x)} \sum_k c_k a_k x^k \sim (p b_p)^\beta c_{[x^p]}$ ,  $x \rightarrow \infty$ , for any r.v.s.  $(c_k)$  of index  $\beta \in R$ .

EXAMPLE 2. Let  $h(x) := \sum_k b_k x^k$ ,  $h(0) = 1$  be an entire function of order  $\rho$ ,  $0 < \rho < 1$  with negative zeros only. According to Hadamard's Factorization Theorem, we have the representation

$$h(x) = \prod_k \left(1 + \frac{x}{r_k}\right), \quad \sum_k \frac{1}{r_k} < \infty;$$

where  $\{-r_k\}$  are zeros of  $h(x)$  in decreasing order.

Denoting by  $n(x)$  zero-counting function of  $h$ , we get

$$\widehat{h}(x) = xD(\ln h(x)) = \sum_k \frac{x}{x+r_k} = \int_0^\infty \frac{x}{x+t} dn(t),$$

and

$$xD(\widehat{h}(x)) = \int_0^\infty \frac{xt}{(x+t)^2} dn(t) < \int_0^\infty \frac{x}{x+t} dn(t) = \widehat{h}(x),$$

i.e.,  $\widehat{\widehat{h}}(x) < 1$ . So, we can apply Theorem A' to  $h(x)$ .

There is more if we notice that the zeros of  $h(x)$  are separated by the zeros of  $h'(x)$ ; hence, all zeros of  $h_1(x)/x$  are negative and, by induction, the same is valid for  $h_n(x)/x$ ,  $n \in N$ . Therefore,

$$\widehat{\widehat{h}}_n(x) < \left(\widehat{\widehat{\frac{h_n(x)}{x}}}\right) < 1$$

and, reproducing the proof of Theorem B, we come to:

PROPOSITION 9. If  $h(x)$  is defined as before then, without any condition,

$$\sum_k c_k b_k x^k \sim c_{[\widehat{\widehat{h}}(x)]} h(x), \quad x \rightarrow \infty,$$

for r.v.s.  $(c_k)$  of arbitrary index.

More precisely, supposing the regular distribution of zeros, we get:

PROPOSITION 10. If  $n(x) \in R_\rho$  then:

$$\sum_k c_k b_k x^k \sim \left( \frac{\pi\rho}{\sin \pi\rho} \right)^\beta c_{[n(x)]} h(x), \quad x \rightarrow \infty;$$

for any regularly varying sequence  $(c_k)$  of index  $\beta \in R$ .

*Proof.* As we already showed,

$$\widehat{h}(x) = x \int_0^\infty \frac{dn(t)}{x+t}.$$

Karamata's Tauberian Theorem for the Stieltjes transform [1, p. 40] gives:

For  $0 < \rho \leq 1$ ;  $n(x) \sim x^\rho l(x)$ ,  $x \rightarrow \infty$  if and only if

$$\int_0^\infty \frac{dn(t)}{x+t} \sim \Gamma(1-\rho) \Gamma(1+\rho) x^{\rho-1} l(x), \quad x \rightarrow \infty.$$

Hence  $n(x) \in R_\rho$  implies  $\widehat{h}(x) \sim \frac{\pi\rho}{\sin \pi\rho} n(x)$ ,  $x \rightarrow \infty$ .

The rest is Proposition 9.

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(Received 27 10 1998)