

## LECTURE NOTES ON QUANTUM COHOMOLOGY OF THE FLAG MANIFOLD

SERGEY FOMIN

This is an exposition of some recent developments related to the object in the title, particularly the computation of the Gromov-Witten invariants of the flag manifold [5] and the quadratic algebra approach [6]. The notes are largely based on the papers [5] and [6], authored jointly with S. Gelfand, A. N. Kirillov, and A. Postnikov. This is by no means an exhaustive survey of the subject, but rather a casual introduction to its combinatorial aspects.

### 1. CLASSICAL THEORY

Let us briefly review the standard facts from the Schubert calculus of the flag manifold; see [8] for details. Let  $\mathrm{Fl}_n$  be the variety of complete flags in  $\mathbb{C}^n$ . The cohomology ring  $H^*(\mathrm{Fl}_n, \mathbb{Z})$  can be described in two different ways. The first description, due to Borel [2], represents it as a quotient of a polynomial ring:

$$(1) \quad H^*(\mathrm{Fl}_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_n]/I_n,$$

where  $x_1, \dots, x_n \in H^2(\mathrm{Fl}_n, \mathbb{Z})$  are the first Chern classes of  $n$  standard line bundles on  $\mathrm{Fl}_n$ , and  $I_n$  is the ideal generated by symmetric polynomials in  $x_1, \dots, x_n$  without constant term<sup>1</sup>.

The second description is based on the decomposition of  $\mathrm{Fl}_n$  into Schubert cells, indexed by the elements of the symmetric group  $S_n$ . The corresponding cohomology classes  $\sigma_w$ ,  $w \in S_n$  (the Schubert classes) form an additive basis in  $H^*(\mathrm{Fl}_n, \mathbb{Z})$ .

The elements of the quotient ring  $\mathbb{Z}[x_1, \dots, x_n]/I_n$  which correspond to the Schubert classes under the isomorphism (1) were identified by Bernstein, Gelfand, and Gelfand [1] and Demazure [4]. Then Lascoux and Schützenberger [14] introduced

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<sup>1</sup>This result, as well as several others below, extends to the more general setup of the homogeneous space  $G/B$  for a complex semisimple Lie group  $G$ . In these notes, we only treat the type  $A$  case, with  $G = SL_n$ .

remarkable polynomial representatives of the Schubert classes called Schubert polynomials. These polynomials  $\mathfrak{S}_w$ ,  $w \in S_n$ , are defined as follows.

Let  $s_i$  denote the adjacent transposition  $(i \ i + 1)$ . For  $w \in S_n$ , an expression  $w = s_{i_1} s_{i_2} \dots s_{i_l}$  of minimal possible length is called a *reduced decomposition*. The number  $l = l(w)$  is the *length* of  $w$ . The symmetric group  $S_n$  acts on  $\mathbb{Z}[x_1, \dots, x_n]$  by  $wf = f(x_{w^{-1}(1)}, \dots, x_{w^{-1}(n)})$ . The *divided difference operator*  $\partial_i$  is defined by  $\partial_i f = (x_i - x_{i+1})^{-1}(1 - s_i)f$ . For any permutation  $w$ , the operator  $\partial_w$  is defined by  $\partial_w = \partial_{i_1} \partial_{i_2} \dots \partial_{i_l}$ , where  $s_{i_1} s_{i_2} \dots s_{i_l}$  is a reduced decomposition for  $w$ .

Let  $\delta = \delta_n = (n - 1, n - 2, \dots, 1, 0)$  and  $x^\delta = x_1^{n-1} x_2^{n-2} \dots x_n$ . For  $w \in S_n$ , the *Schubert polynomial*  $\mathfrak{S}_w$  is defined by  $\mathfrak{S}_w = \partial_{w^{-1}w_0} x^\delta$ , where  $w_0$  is the longest element in  $S_n$ . Equivalently,  $\mathfrak{S}_{w_0} = x^\delta$ , and  $\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w$  whenever  $l(ws_i) = l(w) - 1$ . The following result is immediate from [1].

**Theorem 1.** *The Schubert polynomials represent Schubert classes under Borel's isomorphism (1).*

## 2. QUANTUM COHOMOLOGY

The (small) *quantum cohomology ring*  $\mathrm{QH}^*(X, \mathbb{Z})$  of a smooth algebraic variety  $X$  is a certain deformation of the classical cohomology; see, e.g., [9] for references and definitions. The additive structure of this ring is usually rather simple. For example,  $\mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z})$  is canonically isomorphic, as an abelian group, to the tensor product  $\mathrm{H}^*(\mathrm{Fl}_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \dots, q_{n-1}]$ , where the  $q_i$  are formal variables (deformation parameters). The multiplicative structure of the quantum cohomology is however deformed comparing to  $\mathrm{H}^*(\mathrm{Fl}_n, \mathbb{Z})$ , and specializes to it in the classical limit  $q_1 = \dots = q_{n-1} = 0$ . The multiplication in  $\mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z})$  is given by

$$(2) \quad \sigma_u * \sigma_v = \sum_w \sum_{d=(d_1, \dots, d_{n-1})} q^d \langle \sigma_u, \sigma_v, \sigma_w \rangle_d \sigma_{w_0 w},$$

where the  $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d$  are the (3-point, genus 0) *Gromov-Witten invariants* of the flag manifold, and  $q^d = q_1^{d_1} \dots q_{n-1}^{d_{n-1}}$ . Informally, these invariants count equivalence classes of rational curves in  $\mathrm{Fl}_n$  which have multidegree  $d = (d_1, \dots, d_{n-1})$  and pass through given Schubert varieties. In order for an invariant to be nonzero, the condition  $l(u) + l(v) + l(w) = \binom{n}{2} + 2 \sum_{i=1}^{n-1} d_i$  has to be satisfied. The operation  $*$  defined by (2) is associative [15, 19], and obviously commutative.

The quantum analog of Borel's theorem was obtained by Givental and Kim [10, 11, 12, 13] and Ciocan-Fontanine [3] who showed that

$$(3) \quad \mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z}) \cong P_n / I_n^q,$$

where  $P_n = \mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n]$ , the  $x_i$  are the same as before, and  $I_n^q$  is the ideal generated by the coefficients  $E_1^n, \dots, E_n^n$  of the characteristic polynomial

$$(4) \quad \det(1 + \lambda G_n) = \sum_{i=0}^n E_i^n \lambda^i$$

of the matrix

$$(5) \quad G_n = \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix}.$$

(These coefficients are called *quantum elementary symmetric functions*.) More precisely, let us identify the polynomial  $x_1 + \cdots + x_i$  with the Schubert class  $\sigma_{s_i}$ . The quantum cohomology ring is then generated by the elements  $x_i$ , subject to the relations in the ideal  $I_n^q$ .

### 3. QUANTUM SCHUBERT POLYNOMIALS

The above description of  $\text{QH}^*(\text{Fl}_n, \mathbb{Z})$  does not tell which elements on the right-hand side of (3) correspond to the Schubert classes. The main goal of [5] was to give the quantum analogues of the Bernstein–Gelfand–Gelfand theorem and the Schubert polynomials construction of Lascoux and Schützenberger. This allowed us to design algorithms for computing the Gromov-Witten invariants for the flag manifold. Our approach relied on some of the most basic properties of the quantum cohomology, which can be expressed in elementary terms (see below).

Let  $A_n$  denote the vector space spanned by the classical Schubert polynomials. Another basis of  $A_n$  is formed by the monomials  $x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}}$  dividing the staircase monomial  $x^\delta$ . The space  $A_n$  is complementary to the ideal  $I_n$ , and also to the quantized ideal  $I_n^q$ .

The *quantum Schubert polynomial*  $\mathfrak{S}_w^q$  is defined as the unique polynomial in  $A_n$  that belongs to the coset modulo  $I_n^q$  representing the Schubert class  $\sigma_w$  under the canonical isomorphism (3). The primary goal of [5] was to algebraically identify these polynomials.

### 4. AXIOMATIC CHARACTERIZATION

The following properties of the quantum Schubert polynomials are directly implied by their definition.

**Property 1.**  $\mathfrak{S}_w^q$  is homogeneous of degree  $l(w)$ , assuming  $\deg(x_i) = 1$ ,  $\deg(q_j) = 2$ .

**Property 2.** Specializing  $q_1 = \cdots = q_{n-1} = 0$  yields  $\mathfrak{S}_w^q = \mathfrak{S}_w$ .

**Property 3.**  $\mathfrak{S}_w^q$  belongs to the span  $A_n$  of the classical Schubert polynomials.

It follows that the  $\mathfrak{S}_w^q$  form a linear basis in  $A_n$ , and that the transition matrices between the bases  $\{\mathfrak{S}_w^q\}$  and  $\{\mathfrak{S}_w\}$  are unipotent triangular, with respect to any linear ordering consistent with  $l(w)$ .

The next property reflects the fact that the Gromov-Witten invariants of the flag manifold are nonnegative integers.

**Property 4.** Consider any product of polynomials  $\mathfrak{S}_w^q$ . Expand it (modulo  $I_n^q$ ) in the linear basis  $\{\mathfrak{S}_w^q\}$ . Then all coefficients in this expansion are polynomials in the  $q_j$  with nonnegative integer coefficients.

The following result is a restatement of formula (3) in [3].

**Property 5.** For a cycle  $w = s_{k-i+1} \cdots s_k$ , we have  $\mathfrak{S}_w^q = E_i^k$ .

**Theorem 2.** [5] The polynomials  $\mathfrak{S}_w^q$  are uniquely determined by Properties 1–5.

We conjecture in [5] that Property 5, which is the only property stated above that does not trivially follow from the quantum-cohomology definition of the  $\mathfrak{S}_w^q$ , is not actually needed to uniquely determine the quantum Schubert polynomials.

The next two sections provide constructive descriptions of these polynomials.

## 5. QUANTUM POLYNOMIAL RING

For  $k = 1, 2, \dots$ , define the operator  $X_k$  acting in the polynomial ring by

$$(6) \quad X_k = x_k - \sum_{i < k} q_{ik} \partial_{(ik)} + \sum_{j > k} q_{kj} \partial_{(kj)},$$

where  $\partial_{(ij)}$  is the divided difference operator which corresponds to the transposition  $t_{ij}$ , and  $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$ . (We will always assume  $i < j$ .)

**Theorem 3.** [5] The operators  $X_i$  commute pairwise, and generate a free commutative ring. For any polynomial  $g \in P_n$ , there exists a unique operator  $G \in \mathbb{Z}[q_1, \dots, q_{n-1}][X_1, \dots, X_n]$  satisfying  $g = G(1)$ .

(Here  $G(1)$  denotes the result of applying  $G$  to the polynomial 1.)

For a polynomial  $g \in P_n$ , the polynomial  $G$  given by  $g = G(1)$  is called the *quantization* of  $g$ . The bijective correspondence  $g \leftrightarrow G$  between  $P_n$  and  $\mathbb{Z}[q_1, \dots, q_{n-1}][X_1, \dots, X_n]$  is by no means a ring homomorphism. Identifying the two spaces via this bijection, we obtain an alternative ring structure on  $P_n$ . The multiplication thus defined is called *quantum multiplication* and denoted by  $*$ ; it coincides with the usual multiplication in the classical limit.

Recall that  $I_n \subset P_n$  is the ideal generated by the elementary symmetric functions  $e_i = e_i(x_1, \dots, x_n)$ ,  $i = 1, \dots, n$ . It can be checked that  $I_n$  is also an ideal with respect to the quantum multiplication (i.e.,  $I_n$  is an invariant space for the operators  $X_1, \dots, X_n$  acting in  $P_n$ ).

We are now going to relate our quantum multiplication to the quantum cohomology of the flag manifold. First we verify that for  $i = 1, \dots, n$ , the quantization of the elementary symmetric function  $e_i(x_1, \dots, x_n)$ , is the quantum elementary symmetric function  $E_i^n$  defined by (4). As a corollary, the quantization map bijectively maps the ideal  $I_n$  onto the Givental-Kim ideal  $I_n^q$ . Thus the quotient  $P_n/I_n$ , with the quantum multiplication  $*$  defined above, is canonically isomorphic to the quotient ring  $P_n/I_n^q$  (hence to  $\text{QH}^*(\text{Fl}_n, \mathbb{Z})$ ). In fact, more is true.

**Theorem 4.** [5] The canonical isomorphism between the quotient space  $P_n/I_n$  and the classical cohomology of the flag manifold is also a ring isomorphism between

$P_n/I_n$ , endowed with quantum multiplication defined above in this section, and the quantum cohomology ring of the flag manifold.

In other words, the identification of the (classical) Schubert polynomials with the corresponding Schubert classes translates the quantum multiplication defined in this section into the multiplication in the quantum cohomology ring.

The quantum Schubert polynomial  $\mathfrak{S}_w^q$  is the quantization of the ordinary Schubert polynomial  $\mathfrak{S}_w$ , in the sense of the above construction. In other words,  $\mathfrak{S}_w^q$  is uniquely determined by  $\mathfrak{S}_w^q(X_1, X_2, \dots)(1) = \mathfrak{S}_w(x_1, x_2, \dots)$ . It follows that the quantum multiplication of ordinary Schubert polynomials translates into the ordinary multiplication of the corresponding quantum Schubert polynomials.

## 6. STANDARD MONOMIALS

Let  $e_i^k$  denote the elementary symmetric function of degree  $i$  in the variables  $x_1, \dots, x_k$ . The *standard elementary monomials* are defined by the formula

$$(7) \quad e_{i_1 \dots i_{n-1}} = e_{i_1}^1 \dots e_{i_{n-1}}^{n-1},$$

where we assume  $0 \leq i_k \leq k$  for all  $k$ . It is well known (and easy to prove) that the polynomials (7), for a fixed  $n$ , form a linear basis in the space  $A_n$  spanned by the Schubert polynomials for  $\text{Fl}_n$ . Each Schubert polynomial  $\mathfrak{S}_w$  is thus uniquely expressed as a linear combination of such monomials.

Let  $G_k$  denote the  $k$ th leading principal minor of the matrix  $G_n$  given by (5). The *quantum standard elementary monomial* is defined by

$$(8) \quad E_{i_1 \dots i_{n-1}} = E_{i_1}^1 \dots E_{i_{n-1}}^{n-1},$$

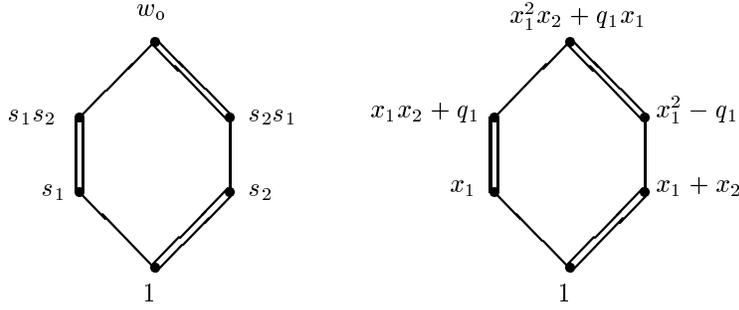
where  $E_i^k = E_i(X_1, \dots, X_k)$  denotes the coefficient of  $\lambda^i$  in the characteristic polynomial  $\chi(\lambda) = \det(1 + \lambda G_k)$  of  $G_k$ .

**Theorem 5.** [5] *The quantum Schubert polynomial  $\mathfrak{S}_w^q$  is obtained by replacing each standard monomial (7) in the expansion of  $\mathfrak{S}_w$  by its quantum analogue (8).*

The expansions of Schubert polynomials in terms of the standard monomials can be computed recursively top-down in the weak order of  $S_n$ , starting from  $\mathfrak{S}_{w_\circ} = e_{12\dots n-1}$ . Namely, use the basic divided difference recurrence for the  $\mathfrak{S}_w$  together with the rule for computing a divided difference of an elementary symmetric function, the Leibnitz formula for the  $\partial_i$ , and the corresponding straightening procedure. Having obtained such an expansion for  $\mathfrak{S}_w$ , “quantize” each term in it to obtain  $\mathfrak{S}_w^q$ . In the special case  $n = 3$ , this produces results shown in Figure 1.

## 7. COMPUTATION OF THE GROMOV-WITTEN INVARIANTS

The space  $A_n$  spanned by the Schubert polynomials for  $S_n$  can be described as the set of normal forms for the ideal  $I_n^q$ , with respect to certain term order. This allows one to use Gröbner basis techniques (see, e.g., [20]) to construct efficient algorithms for computing the Gromov-Witten invariants of the flag manifold.

FIGURE 1. Quantum Schubert polynomials for  $S_3$ 

**Definition 6.** Let us choose the *total degree – inverse lexicographic term order* on the monomials  $x_1^{a_1} \cdots x_n^{a_n}$ . In other words, we first order all monomials by the total degree  $\sum_i a_i$ , and then break the ties by using the inverse lexicographic order  $x_1 < x_2 < x_3 < \dots$ . This allows us to introduce the *normal*, or fully reduced form of any polynomial with respect to the ideal  $I_n^q$  and the term order specified above. This normal form can be found, e.g., via the Buchberger algorithm employing the corresponding Gröbner basis of  $I_n^q$ .

**Theorem 7.** [5] Choose a term order as in Definition 6. Then the reduced minimal Gröbner basis for the ideal  $I_n^q$  consists of the polynomials  $\det (E_{j-i+1}^{n-i+1})_{i,j=1}^k$ , for  $k = 1, \dots, n$ . The normal form of any polynomial  $F \in P_n$ , lies in the space  $A_n$ .

For a polynomial  $F \in P_n$ , let

$$\langle\langle F \rangle\rangle = \text{coefficient of } x^\delta \text{ in the normal form of } F.$$

Equivalently,  $\langle\langle F \rangle\rangle$  is the coefficient of  $\mathfrak{S}_{w_0}^q$  in the expansion of  $F$  (modulo  $I_n^q$ ) in the basis of quantum Schubert polynomials, since  $\mathfrak{S}_{w_0}^q$  is the only basis element that involves the staircase monomial  $x^\delta$ . The definition (2) implies that

$$\langle\langle \mathfrak{S}_{w_1}^q \cdots \mathfrak{S}_{w_k}^q \rangle\rangle = \sum_d q^d \langle \sigma_{w_1}, \dots, \sigma_{w_k} \rangle_d,$$

the generating function for the Gromov-Witten invariants. We thus arrived at the following result.

**Theorem 8.** [5] A Gromov-Witten invariant  $\langle \sigma_{w_1}, \dots, \sigma_{w_k} \rangle_d$  of the flag manifold is the coefficient of the monomial  $q^d x^\delta$  in the normal form (in the sense of Definition 6) of the product of quantum Schubert polynomials  $\mathfrak{S}_{w_1}^q \cdots \mathfrak{S}_{w_k}^q$ .

## 8. QUADRATIC ALGEBRAS

Another approach to the study of the cohomology ring—ordinary or quantum—of the flag manifold was suggested in [6], and further developed in [7, 17].

Let  $\mathcal{E}_n$  be the associative algebra generated by the symbols  $[ij]$ , for all  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ , subject to the convention  $[ij] + [ji] = 0$  and the relations

$$(9) \quad \begin{aligned} [ij]^2 &= 0, \\ [ij][jk] + [jk][ki] + [ki][ij] &= 0, \quad i, j, k \text{ distinct}, \\ [ij][kl] - [kl][ij] &= 0, \quad i, j, k, l \text{ distinct}. \end{aligned}$$

The algebras  $\mathcal{E}_n$  are naturally graded; the formulas for their Hilbert polynomials, for  $n \leq 5$ , can be found in [6]. The algebras  $\mathcal{E}_n$  are not Koszul for  $n \geq 3$  (proved by Roos [18]). It is unknown whether  $\mathcal{E}_n$  is generally finite-dimensional; it was proved in [7] that the Hilbert series of  $\mathcal{E}_n$  divides that of  $\mathcal{E}_{n+1}$ .

The ‘‘Dunkl elements’’  $\theta_1, \dots, \theta_n \in \mathcal{E}_n$  are defined by

$$(10) \quad \theta_j = - \sum_{i < j} [ij] + \sum_{j < k} [jk].$$

**Theorem 9.** [6] *The complete list of relations satisfied by the Dunkl elements  $\theta_1, \dots, \theta_n \in \mathcal{E}_n$  is given by  $\theta_i \theta_j = \theta_j \theta_i$  (for any  $i$  and  $j$ ) and  $e_i(\theta_1, \dots, \theta_n) = 0$  (for  $i = 1, \dots, n$ ). Thus these elements generate a commutative subring canonically isomorphic to  $P_n/I_n$ , and to the cohomology ring of the flag manifold.*

Let  $s_{ij} \in S_n$  denote the transposition of elements  $i$  and  $j$ . Consider the ‘‘Bruhat operators’’  $[ij]$  acting in the group algebra of  $S_n$  by

$$(11) \quad [ij] w = \begin{cases} ws_{ij} & \text{if } l(ws_{ij}) = l(w) + 1; \\ 0 & \text{otherwise.} \end{cases}$$

One easily checks that these operators satisfy the relations (9). We thus obtain an (unfaithful) representation of the algebra  $\mathcal{E}_n$ , called the *Bruhat representation*. This representation has an equivalent description in the language of Schubert polynomials. Let us identify each element  $w \in S_n$  with the corresponding Schubert polynomial  $\mathfrak{S}_w$ . Then the generators of  $\mathcal{E}_n$  act in  $\mathbb{Z}[x_1, \dots, x_n]/I_n$  by

$$(12) \quad [ij] \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_{ij}} & \text{if } l(ws_{ij}) = l(w) + 1; \\ 0 & \text{otherwise.} \end{cases}$$

The following result is a restatement of the classical Monk’s rule [16].

**Theorem 10.** *In the representation (12) of the quadratic algebra  $\mathcal{E}_n$  in the quotient ring  $\mathbb{Z}[x_1, \dots, x_n]/I_n$ , a Dunkl element  $\theta_j$  acts as multiplication by  $x_j$ , for  $j = 1, \dots, n$ . In other words,  $x_j f = \theta_j f$ , for any coset  $f \in \mathbb{Z}[x_1, \dots, x_n]/I_n$ .*

## 9. STRUCTURE CONSTANTS AND NONNEGATIVITY CONJECTURE

Let  $c_{uv}^w$  denote the coefficient of  $\mathfrak{S}_w$  in the product  $\mathfrak{S}_u \mathfrak{S}_v$ . Equivalently,  $c_{uv}^w$  is the number of points in the intersection of the general translates of three (dual) Schubert cells labelled by  $u$ ,  $v$ , and  $w$ , respectively. Thus all the  $c_{uv}^w$  are non-negative integers. The problem of finding a combinatorial interpretation for  $c_{uv}^w$  is one of the central open problems in Schubert calculus. In fact, no elementary proof

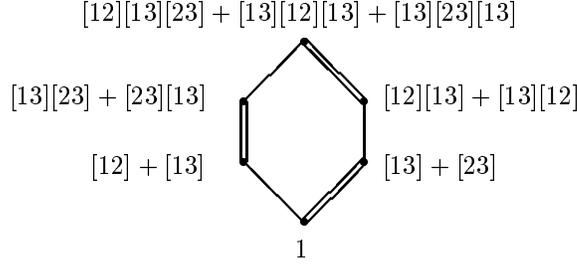


FIGURE 2. Evaluations of Schubert polynomials at Dunkl elements

of the fact that  $c_{uv}^w \geq 0$  is known. Much less is known about the more general Gromov-Witten invariants of the flag manifold.

Let  $\mathcal{E}_n^+ \subset \mathcal{E}_n$  be the cone of all elements that can be written as nonnegative integer combinations of noncommutative monomials in the generators  $[ij]$ , for  $i < j$ .

**Conjecture 11.** [6] (Nonnegativity conjecture) *For any  $w \in S_n$ , the Schubert polynomial  $\mathfrak{S}_w$  evaluated at the Dunkl elements belongs to the positive cone  $\mathcal{E}_n^+$ :*

$$(13) \quad \mathfrak{S}_w(\theta) = \mathfrak{S}_w(\theta_1, \dots, \theta_{n-1}) \in \mathcal{E}_n^+ .$$

Let us now explain why Conjecture 11 implies nonnegativity of the structure constants  $c_{uv}^w$ , and why furthermore a combinatorial description for the evaluations  $\mathfrak{S}_w(\theta)$  would provide a combinatorial rule describing the  $c_{uv}^w$ .

The action (12) of  $\mathcal{E}_n$  on the quotient ring  $\mathbb{Z}[x_1, \dots, x_n]/I_n$  is defined in such a way that every noncommutative monomial in the generators  $[ij]$ ,  $i < j$ , when applied to a Schubert polynomial  $\mathfrak{S}_v$ , gives either another Schubert polynomial or zero. It follows that, for any  $z \in \mathcal{E}_n^+$ , the polynomial  $z\mathfrak{S}_v$  is *Schubert-positive*, i.e., is a nonnegative linear combination of Schubert polynomials. In particular, if Conjecture 11 holds, then the polynomial  $\mathfrak{S}_u(\theta)\mathfrak{S}_v(x)$  is Schubert-positive (here  $x$  stands for  $x_1, \dots, x_n$ ). Since, according to Theorem 10,

$$(14) \quad \mathfrak{S}_u(\theta)\mathfrak{S}_v(x) = \mathfrak{S}_u(x)\mathfrak{S}_v(x) ,$$

we conclude that  $\mathfrak{S}_u\mathfrak{S}_v$  is Schubert-positive, i.e., the structure constants  $c_{uv}^w$  are nonnegative. Now suppose we have a combinatorial description for  $\mathfrak{S}_u(\theta)$ . By (14),

$$(15) \quad c_{uv}^w = \langle \text{coefficient of } w \text{ in } \mathfrak{S}_u(\theta) v \rangle ,$$

where the action of  $\mathfrak{S}_u(\theta)$  on  $v \in S_n$  is the Bruhat representation action (11). Thus (15) would provide a combinatorial rule for  $c_{uv}^w$ .

The following conjecture, if proved, would provide an alternative description of the basis of Schubert cycles.

**Conjecture 12.** [6] *The evaluations  $\mathfrak{S}_w(\theta)$  are the additive generators of the intersection of the cone  $\mathcal{E}_n^+$  with the commutative subalgebra generated by the Dunkl elements.*

10. QUANTUM DEFORMATION OF THE QUADRATIC ALGEBRA

The *quantum deformation*  $\mathcal{E}_n^q$  of the quadratic algebra  $\mathcal{E}_n$  is defined by replacing the relation  $[ij]^2 = 0$  in (9) by

$$(16) \quad [ij]^2 = \begin{cases} q_i & \text{if } j = i + 1 ; \\ 0 & \text{otherwise .} \end{cases}$$

The ‘‘quantum Bruhat operators’’  $[ij]$ , acting in the  $\mathbb{Z}[q_1, \dots, q_{n-1}]$ -span of the symmetric group  $S_n$  by

$$(17) \quad [ij]w = \begin{cases} ws_{ij} & \text{if } l(ws_{ij}) = l(w) + 1 ; \\ q_{ij}ws_{ij} & \text{if } l(ws_{ij}) = l(w) - l(s_{ij}) ; \\ 0 & \text{otherwise ,} \end{cases}$$

provide a representation of  $\mathcal{E}_n^q$ , which degenerates into the ordinary Bruhat representation in the classical limit. The operators (17) can be viewed as acting in the quotient space  $\mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n]/I_n^q$  by

$$(18) \quad [ij] \mathfrak{S}_w^q = \begin{cases} \mathfrak{S}_{ws_{ij}}^q & \text{if } l(ws_{ij}) = l(w) + 1 ; \\ q_{ij} \mathfrak{S}_{ws_{ij}}^q & \text{if } l(ws_{ij}) = l(w) - l(s_{ij}) ; \\ 0 & \text{otherwise .} \end{cases}$$

The Dunkl elements  $\theta_j \in \mathcal{E}_n^q$  are defined by the same formula (10) as before.

**Theorem 13.** [5] (Quantum Monk’s formula) *In the representation (18) of  $\mathcal{E}_n^q$ , a Dunkl element  $\theta_j$  acts as multiplication by  $x_j$ , for  $j = 1, \dots, n$ .*

The following result is a corollary of Theorem 13.

**Corollary 14.** *As an element of the quotient ring  $P_n/I_n^q$ , a quantum Schubert polynomial  $\mathfrak{S}_w^q$  is uniquely defined by the condition that, in the quantum Bruhat representation (17), it acts on the identity permutation 1 by  $w = \mathfrak{S}_w^q(\theta_1, \dots, \theta_n)(1)$ .*

The quantum analogue of Theorem 9 stated below was conjectured in [6] and proved by A. Postnikov in [17].

**Theorem 15.** *The commutative subring generated by the Dunkl elements in the quadratic algebra  $\mathcal{E}_n^q$  is canonically isomorphic to the quantum cohomology ring of the flag manifold. The isomorphism is defined by  $\theta_1 + \dots + \theta_j \mapsto \sigma_{s_j}$ .*

The following statement strengthens and refines Conjecture 11.

**Conjecture 16.** [6] *For any  $w \in S_n$ , the evaluation  $\mathfrak{S}_w^q(\theta_1, \dots, \theta_n)$  can be written as a linear combination of monomials in the generators  $[ij]$ , with nonnegative integer coefficients.*

It is not even clear a priori that the evaluations  $\mathfrak{S}_w^q(\theta)$  can be expressed as linear combinations of monomials with coefficients not depending on the quantum parameters  $q_1, \dots, q_{n-1}$ .

A reformulation of (2) in the language of quantum Schubert polynomials gives

$$(19) \quad \mathfrak{S}_u^q \mathfrak{S}_v^q = \sum_{w \in S_n} \sum_d q^d \langle \sigma_u, \sigma_v, \sigma_w \rangle_d \mathfrak{S}_{w \circ v}^q .$$

In view of Theorem 13, one obtains the following analogue of (15).

**Corollary 17.** [6] For  $u, v, w \in S_n$  and  $d = (d_1, \dots, d_{n-1}) \in \mathbb{Z}_+^{n-1}$ , we have

$$(20) \quad \langle \sigma_u, \sigma_v, \sigma_w \rangle_d = \langle \text{coefficient of } q^d w \circ w \text{ in } \mathfrak{S}_u^q(\theta) v \rangle,$$

where  $\mathfrak{S}_u^q(\theta)$  acts on  $v$  according to the quantum Bruhat representation (17).

Assuming Conjecture 16 holds, one would like to have a combinatorial rule for a nonnegative expansion of  $\mathfrak{S}_w^q(\theta)$ . Such a rule would immediately lead to a direct combinatorial description of the Gromov-Witten invariants  $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d$  of the flag manifold, given by (20).

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Department of Mathematics  
 Massachusetts Institute of Technology  
 Cambridge, MA 02139