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ON COMMUTING GENERALIZED INVERSES IN SEMIGROUPS

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Abstract. Let S be a semigroup and let $a \in S$. In this note we describe all possible commuting generalized inverses of a (in the sense of Definitions 1 and 2). It turns out that the Drazin inverse of a is the only generalized inverse of a which commutes with a.

We consider systems of equations in x of the form

$$t_i(a,x) = t_i'(a,x) \quad (i = 1, \dots, s+1;\ s \in \mathbb{N}) \tag{1}$$

where $t_i,\ t_i'$ are semigroup terms made up only from a and x.

Definition 1. We say that the system (1) defines a generalized inverse of a if:

(i) in any semigroup S the system (1) for a given $a \in S$ cannot have more than one solution;

(ii) in the case when S is a monoid in which a has its true inverse a^{-1} (that is to say, its group inverse with respect to the subgroup whose unity is the unity of the entire monoid S), the system (1) for $x = a^{-1}$ reduces to a system of identities;

(iii) there exists a monoid S in which the system (1) is consistent for at least one $a \in S$ which does not have its true inverse.

Definition 2. If the system (1) defines a generalized inverse and if one of the equations (1) is $ax = xa$, we say that (1) defines a commuting generalized inverse of a.

In this note we describe all possible commuting generalized inverses, thereby extending the results from [**1**] and [**2**].

The equation $ax = xa$ implies that any term made up from a and x has the form $a^m x^n$, where $m, n \in \mathbb{N}_0$, $m + n \geq 1$, and $a^m x^n$ is, by definition, equal to x^n

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if $m = 0$ and to a^m if $n = 0$. Furthermore, having in mind the condition (ii) of Definition 1, we see that if (1) is to define a commuting generalized inverse it must have the form

$$ax = xa,\ a^{m_i+r_i}x^{n_i+r_i} = a^{m_i}x^{n_i} \quad (m_i, n_i \in \mathbb{N}_0,\ r_i \in \mathbb{N},\ i = 1, \dots, s). \tag{2}$$

If $m_i > 0$ or $n_i > 1$ for all $i = 1, \dots, s$, the system (2) can have more than one solution, as shown by the following example. In the matrix semigroup let $a = 0$ and let A be the subset of $\{n_1, \dots, n_s\}$ defined by: $n_i \in A \Longleftrightarrow m_i = 0$. If $A = \emptyset$, the system (2) is satisfied by arbitrary x. If $A \neq \emptyset$, the system (2) is satisfied by any x such that $x^n = 0$ where $n = \min A$ (> 1).

Therefore, in view of the condition (i) of Definition 1 we have to suppose that there exists $i \in \{1, \dots, s\}$ such that $m_i = 0$ and $n_i = 1$. We may take $m_1 = 0$, $n_1 = 1$, $r_1 = k$ and the system (2) becomes

$$ax = xa,\ a^k x^{k+1} = x,\ a^{m_i+r_i}x^{n_i+r_i} = a^{m_i}x^{n_i} \quad (i = 2, \dots, s). \tag{3}$$

If $n_i > 0$ for all $i = 2, \dots, s$, the system (3) can again have more than one solution. Indeed, in the matrix semigroup if a is nonsingular, the system (3) has at least two solutions, given by: $x = 0$ and $x = a^{-1}$. We therefore take $n_2 = 0$, we put $m_2 = m$, $r_2 = r$ and we obtain the system

$$ax = xa,\ a^k x^{k+1} = x,\ a^{m+r}x^r = a^m,\ a^{m_i+r_i}x^{n_i+r_i} = a^{m_i}x^{n_i} \ (i = 3, \dots, s). \tag{4}$$

We now prove a number of lemmas concerning the equations which appear in (4). For positive integers $q_1, \dots, q_s$ we denote their highest common factor by $(q_1, \dots, q_s)$.

Lemma 1. *If $m, r, k \in \mathbb{N}$, there exists a positive integer q such that $qk - r \neq k$ and that*

$$a^{m+r}x^r = a^m,\ a^k x^{k+1} = x \ \Rightarrow \ a^k x^{k+1} = a^{qk-r}x^{qk-r+1}.$$

Proof. If $k = 1$ we may take $q = r + 2$. Let $k > 1$. If $k \geq m + r$, then

$$a^k x^{k+1} = a^{k-m-r}a^{m+r}x^r x^{k-r+1} = a^{k-r}x^{k-r+1},$$

and so $q = 1$. If $1 < k < m + r$, there exists a $q \in \mathbb{N}$ such that $qk > m + r$ and $(q-1)k > r$. We then have

$$a^k x^{k+1} = a^{qk}x^{qk+1} = a^{qk-m-r}a^{m+r}x^r x^{qk-r+1} = a^{qk-r}x^{qk-r+1}.$$

Lemma 2. *If $p, k \in \mathbb{N}$ and if $n = (p, k)$, then the system*

$$a^p x^{p+1} = x,\ a^k x^{k+1} = x \tag{5}$$

is equivalent to the equation

$$a^n x^{n+1} = x. \tag{6}$$

Proof. Suppose that $p > k$. Since $n = (p,k)$, there exist positive integers λ and μ such that $\lambda p - \mu k = n$, and so

$$\begin{aligned} x = a^p x^{p+1} &= a^{\lambda p} x^{\lambda p+1} = a^{\lambda p-k} a^k x^{k+1} x^{\lambda p-k} \\ &= a^{\lambda p-k} x^{\lambda p-k+1} = \cdots = a^{\lambda p-\mu k} x^{\lambda p-\mu k+1} = a^n x^{n+1}, \end{aligned}$$

which means that (5) implies (6). The converse is trivial.

Corollary 1. *If $p_1, \dots, p_s \in \mathbb{N}$ and if $n = (p_1, \dots, p_s)$, then the system*

$$a^{p_i} x^{p_i+1} = x \quad (i = 1, \dots, s)$$

is equivalent to the equation $a^n x^{n+1} = x$.

Lemma 3. *If $m, k, r \in \mathbb{N}$ and if $n = (k, r)$, the systems*

$$a^{m+r} x^r = a^m, \; a^k x^{k+1} = x \tag{7}$$

and

$$a^{m+n} x^n = a^m, \; a^n x^{n+1} = x \tag{8}$$

are equivalent.

Proof. We only prove that (7) implies (8), the converse being trivial. By Lemma 1 there exists a $q \in \mathbb{N}$ such that

$$x = a^k x^{k+1} = a^{qk-r} x^{qk-r+1},$$

and so, since $(qk-r, k) = (r, k) = n$, by Lemma 2 we get $a^n x^{n+1} = x$. Furthermore, for some $t \in \mathbb{N}$ we have $r = tn$, and so

$$\begin{aligned} a^{m+r} x^r = a^{m+tn} x^{tn} &= a^{m+(t-1)n} a^n x^{n+1} x^{(t-1)n-1} \\ &= a^{m+(t-1)n} x^{(t-1)n} = \cdots = a^{m+n} x^n. \end{aligned}$$

Lemma 4. *If $p, q \in \mathbb{N}_0$, $n, r \in \mathbb{N}$ and if $h = (n, r)$, the system*

$$a^n x^{n+1} = x, \; a^{p+r} x^{q+r} = a^p x^q \tag{9}$$

is equivalent to the equation

$$a^h x^{h+1} = x. \tag{10}$$

Proof. Again, (9) follows from (10) trivially. Conversely, let $\lambda \in \mathbb{N}$ be such that $\lambda n > \max(p, q)$. We then have

$$\begin{aligned} x &= a^n x^{n+1} = a^{\lambda n} x^{\lambda n+1} = a^{\lambda n-p} a^p x^q x^{\lambda n-q+1} \\ &= a^{\lambda n-p} a^{p+r} x^{q+r} x^{\lambda n-q+1} = a^r a^{\lambda n} x^{\lambda n+1} x^r = a^r x^{r+1} \end{aligned}$$

and using Lemma 2 we get (10).

Corollary 2. *If $m_i, n_i \in \mathbb{N}_0$, $n, r_i \in \mathbb{N}$ and if $h = (n, r_2, \dots, r_s)$ the system*

$$a^n x^{n+1} = x, \; a^{m_i+r_i} x^{n_i+r_i} = a^{m_i} x^{n_i} \quad (i = 2, \dots, s)$$

is equivalent to the equation $a^h x^{h+1} = x$.

Lemma 5. *If $p, q \in \mathbb{N}_0$, $m, n, r \in \mathbb{N}$ and if $h = (n, r)$, the system*

$$a^{m+n} x^n = a^m, \; a^n x^{n+1} = x, \; a^{p+r} x^{q+r} = a^p x^q \tag{11}$$

is equivalent to the system

$$a^{m+h} x^h = a^m, \; a^h x^{h+1} = x. \tag{12}$$

Proof. By Lemma 4 the last two equations of (11) are equivalent to the second equation of (12). Let $n = \lambda h$. Starting with (11) we get

$$\begin{aligned} a^m &= a^{m+n} x^n = a^{m+n-h} a^h x^{h+1} x^{n-h-1} \\ &= a^{m+n-h} x^{n-h} = \cdots = a^{m+n-(\lambda-1)h} x^{n-(\lambda-1)h} = a^{m+h} x^h. \end{aligned}$$

Corollary 3. *If $m_i, n_i \in \mathbb{N}_0$, $m, n, r_i \in \mathbb{N}$ and if $h = (n, r_3, \dots, r_s)$, the system*

$$a^{m+n} x^n = a^m, \; a^n x^{n+1} = x, \; a^{m_i+r_i} x^{n_i+r_i} = a^{m_i} x^{n_i} \quad (i = 3, \dots, s)$$

is equivalent to the system

$$a^{m+h} x^h = a^m, \; a^h x^{h+1} = x.$$

Lemma 6. *If $m_i, n_i \in \mathbb{N}_0$, $k, m, r, r_i \in \mathbb{N}$ and if $h = (k, r, r_3, \dots, r_s)$, the system*

$$a^k x^{k+1} = x, \; a^{m+r} x^r = a^m, \; a^{m_i+r_i} x^{n_i+r_i} = a^{m_i} x^{n_i} \quad (i = 3, \dots, s) \tag{13}$$

is equivalent to the system

$$a^{m+h}x^h = a^m,\ a^h x^{h+1} = x. \tag{14}$$

Proof. The implication (14) $\Rightarrow$ (13) is trivial. Conversely, if $n = (k, r)$, by Lemma 3 the system (13) is equivalent to the system

$$a^n x^{n+1} = x,\ a^{m+n}x^n = a^m,\ a^{m_i+r_i}x^{n_i+r_i} = a^{m_i}x^{n_i}\ (i = 3, \dots, s)$$

and since $(k, r, r_3, \dots, r_s) = (n, r_3, \dots, r_s)$, by Corollary 3, this system is equivalent to (14).

We are now able to prove the following

Theorem 1. *The system* (1) *defines a commuting generalized inverse of a if and only if: it has the form* (4) *and* $(k, r, r_3, \dots, r_s) = 1$.

Proof. Unless it has the form (4), the system (1) according to Definitions 1 and 2 cannot define a commuting generalized inverse of a. By Lemma 6, the system (4) is equivalent to the system

$$ax = xa,\ a^{m+h}x^h = a^m,\ a^h x^{h+1} = x \tag{15}$$

where $h = (k, r, r_3, \dots, r_s)$. If $h > 1$ the system (15) may have more than one solution. An example is provided by the matrix semigroup where a is nonsingular. For $h = 1$ the system (15) becomes the Drazin system, introduced in [3], and it is well known that it defines a commuting generalized inverse of a.

Hence, we arrive at the following conclusion:

Theorem 2. *The Drazin inverse of a, defined by the system*

$$ax = xa,\ a^{m+1}x = a^m,\ ax^2 = x$$

is the only possible commuting generalized inverse of a.

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Errata for the paper

SOME REMARKS ON POSSIBLE GENERALIZED INVERSES IN SEMIGROUPS

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An uncorrected version of the above paper was printed by mistake, and so there are two serious errors.

1. A part of the text was omitted from the Abstract, which should read as follows:

Abstract. For a given element a of a semigroup S it is possible that the system of equations in x: $axa = a$, $ax = xa$ is inconsistent and that one or both systems (S_k): $a^{k+1}x = a^k$, $ax = xa$ and (Σ_k): $axa = a$, $a^k x = xa^k$ are consistent for some positive integer k, in which case they have more than one solution. Some relations between those two systems are established. However, the chief aim of this note is to investigate the possibilities of extending (S_k) by adding new balanced equations, so that this new system has unique solution. It is proved that if the extended system has unique solution, then the generalized inverse of a, defined by it, must be the Drazin inverse. It is also shown that the system $(\Sigma_2) \wedge ax^2 = x^2 a \wedge xax = x$ cannot be extended into a system with unique solution.

2. At the bottom of page 36 the important formula $(B_{m,n,r})$ is incorrect. It should be:

$$(B_{m,n,r}) \begin{cases} a^{m+r}x^{n+r} = a^m x^n \\ m \in \{0, 1, \dots, k-1\},\ r \in \{1, 2, \dots, k\},\ 1 \le m + r \le k,\ n \in \mathbb{N} \end{cases}$$

3. There are also some other misprints which are not so important.