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ON ONE HILBERT'S PROBLEM FOR THE LERCH ZETA-FUNCTION

R. Garunkštis and **A. Laurinčikas**

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Abstract. The functional independence of the Lerch zeta-function $L(\lambda, \alpha, s)$ is obtained. The cases of transcendental and rational α are considered.

1. Introduction

During the International Congress of Mathematicians in 1900 D. Hilbert raised a problem of algebraic-differential independence for functions given by Dirichlet series. Let s be a complex variable, and let, as usual, $\zeta(s)$ denote the Riemann zeta-function. D. Hilbert noted that an algebraic-differential independence of $\zeta(s)$ can be proved using the algebraic-differential independence of the Euler gamma-function $\Gamma(s)$ and the functional equation for $\zeta(s)$. He also conjectured that there is no algebraic-differential equation with partial derivatives which can be satisfied by the function

$$\zeta(s,x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}.$$

This conjecture was proved independently by D. D. Mordukhai-Boltovskoi [**6**] and by A. Ostrowski [**7**]. A. G. Postnikov [**8**] generalized the Hilbert problem for a system of Dirichlet series considering their differential independence. In [**9**] he dealt with the function

$$L(x,s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}\, x^m,$$

where $\chi(m)$ is a Dirichlet character modulo q, and proved that the equation

$$P\left(x, s, \frac{\partial^{l+r} L(x,s,\chi)}{\partial x^l\, \partial s^r}\right) \equiv 0$$

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can not be satisfied for any polynomial P. S. M. Voronin [**10**], [**12**] obtained the functional independence of the Riemann zeta-function, see also [**3**]. Let F_l, $l = 0, 1, \ldots, n$, be continuous functions, and let the equality

$$\sum_{l=0}^{n} s^l F_l\big(\zeta(s), \zeta'(s), \ldots, \zeta^{(N-1)}(s)\big) = 0$$

be valid identically for s. Then he proved that $F_l \equiv 0$ for $l = 0, 1, \ldots, n$. The functional independence of Dirichlet L-functions and of Dirichlet series with multiplicative coefficients was obtained in [**1**], [**11**], [**12**], and in [**2**], [**3**], respectively.

The aim of this note is to prove the functional independence of the Lerch zeta-function. We recall that the Lerch zeta-function $L(\lambda, \alpha, s)$, for $\sigma > 1$, is given by the following Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and otherwise by analytic continuation. Here λ and α, $0 < \alpha \leqslant 1$, are real parameters. If $\lambda \notin \mathbb{Z}$, where $\mathbb{Z}$ is the set of integer numbers, then $L(\lambda, \alpha, s)$ is an entire function. In the case $\lambda \in \mathbb{Z}$ the Lerch zeta-function reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$. In this note we will discuss the case $\lambda \notin \mathbb{Z}$ only. So we can suppose $0 < \lambda < 1$. Let N be a natural number.

Theorem 1. *Let α be a transcendental number. Let F_l, $l = 0, 1, \ldots, n$, be continuous functions, and let the equality*

$$\sum_{l=0}^{n} s^l F_l\big(L(\lambda, \alpha, s), L'(\lambda, \alpha, s), \ldots, L^{(N-1)}(\lambda, \alpha, s)\big) = 0$$

be valid identically for s. Then $F_l \equiv 0$ for $l = 0, 1, \ldots, n$.

The case when α is a rational number is more complicated. Let $\alpha = \frac{a}{q}$, $1 \leqslant a < q$, $(a, q) = 1$. In this case we suppose that the parameter λ is rational, too. Let $\lambda = \frac{l}{r}$, $1 \leqslant l < r$, $(l, r) = 1$. Moreover, we take $k = rq$, $d = (k, m)$, $\beta_m = \frac{lm}{k}$. Then, for $\sigma > 1$,

$$\sum_{\substack{m=1 \\ m \equiv a (\bmod q) \\ d=1}}^{\infty} \frac{e^{2\pi i \beta_m}}{m^s} = \frac{1}{\varphi(k)} \sum_{v=0}^{\varphi(k)-1} \eta_v \, L(s, \chi_v),$$

where

$$\eta_v = \sum_{\substack{m=1 \\ m \equiv a (\bmod q)}}^{k} e^{2\pi i \beta_m} \bar{\chi}_v(m), \qquad v = 0, 1, \ldots, \varphi(k) - 1.$$

Here χ_v denote the Dirichlet characters modulo k, $L(s,\chi_v)$ are the corresponding Dirichlet L-functions, and $\varphi(k)$, as usual, stands for the Euler function.

Theorem 2. *Let $\lambda = \frac{l}{r}$ and $\alpha = \frac{a}{q}$ be rational numbers. Suppose that there exists at least two primitive characters modulo k such that the corresponding numbers η_v are distinct from zero. Let the equality*

$$\sum_{l=0}^{n} s^l F_l\Big(q^{-s}L(\lambda,\alpha,s), \big(q^{-s}L(\lambda,\alpha,s)\big)', \dots, \big(q^{-s}L(\lambda,\alpha,s)\big)^{(N-1)}\Big) = 0$$

be valid identically for s. Then $F_l \equiv 0$ for $l = 0, 1, \dots, n$.

2. Auxiliary results

The proof of Theorem 1 and 2 is based on the universality property of the Lerch zeta-function. The universality of $L(\lambda,\alpha,s)$ was obtained in [**4**] and [**5**]. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ where $\mathbb{C}$ stands for the complex plane. Denote by meas $\{A\}$ the Lebesgue measure of the set A, and let, for $T > 0$,

$$\nu_T(\dots) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T], \dots\right\},$$

where instead of dots we write a condition satisfied by τ.

Lemma 1. *Let α be a transcendental number. Let K be a compact subset of the strip D with connected complement. Let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}\Big|L(\lambda,\alpha,s+i\tau) - f(s)\Big| < \varepsilon\right) > 0.$$

Proof of the lemma is given in [**4**].

Lemma 2. *Let $\lambda = \frac{l}{r}$ and $\alpha = \frac{a}{q}$ be rational numbers. Suppose there exist at least two primitive characters modulo k such that the corresponding numbers η_v are distinct from zero. Let $0 < R < \frac{1}{4}$, and let $f(s)$ be a continuous function on the disc $|s| \leqslant R$ and analytic in the interior of this disc. Then for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \nu_T\left(\max_{|s|\leqslant R}\Big|q^{-s-\frac{3}{4}-i\tau}L(\lambda,\alpha,s+\frac{3}{4}+i\tau) - f(s)\Big| < \varepsilon\right) > 0.$$

Proof of the lemma is given in [**5**].

We note that the proof of Lemma 1 is based on a limit theorem in the sense of the weak convergence of probability measures for $L(\lambda,\alpha,s)$ in the space of analytic functions $H(D)$. Let

$$\Omega_1 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$, $m = 0, 1, 2, \ldots$, and $\gamma\{s \in \mathbb{C} : |s| = 1\}$. Then Ω_1 is a compact Abelian topological group, and we have the probability space $\{\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S and m_{1H} is the Haar measure on $(\Omega_1, \mathcal{B}(\Omega_1))$. Let $\omega_1(m)$ stand for the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_m, and let

$$L_1(\lambda, \alpha, s, \omega_1) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m} \omega_1(m)}{(m+\alpha)^s} \qquad s \in D,\ \omega_1 \in \Omega_1.$$

Then $L_1(\lambda, \alpha, s, \omega_1)$ is an $H(D)$-valued random element defined on the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. Note that $\{\omega_1(m),\ m = 0, 1, \ldots\}$ is a sequence of independent random variables with respect to the measure m_{1H}. Now, using the linear independence over the field of rational numbers Q of the system $\{\log(m+\alpha),\ m = 0, 1, \ldots\}$ it is proved that the probability measure

$$P_T(A) \stackrel{def}{=} \nu_T\big(L(\lambda, \alpha, s + i\tau) \in A,\ A \in \mathcal{B}(H(D))\big),$$

converges weakly to the distribution of the random element $L_1(\lambda, \alpha, s, \omega_1)$ as $T \to \infty$. From this Lemma 1 easily follows.

In the case of rational α the system $\{\log(m+\alpha),\ m = 0, 1, \ldots\}$ is not linearly independent over Q, and we must consider the system $\{\log p,\ p \text{ is prime}\}$. In this case the torus Ω_1 is changed by

$$\Omega_2 = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p, and we obtain the probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$ denoting by m_{2H} the Haar measure on $(\Omega_2, \mathcal{B}(\Omega_2))$. Let $\omega_2(p)$ stand for the projection of $\omega_2 \in \Omega_2$ to the coordinate space γ_p, for a natural m

$$\omega_2(m) = \prod_{p^\alpha \| m} \omega_2^\alpha(p),$$

and let

$$L_2(\lambda, \alpha, s, \omega_2) = \omega_2(q) q^s e^{-(2\pi i \lambda a)/q} \sum_{\substack{m=1 \\ m \equiv a (\bmod q)}}^{\infty} \frac{e^{(2\pi i \lambda m)/q} \omega_2(m)}{m^s}, \quad \omega_2 \in \Omega_2, s \in D.$$

Then it is proved that the probability measure P_T converges weakly to the distribution of the $H(D)$-valued random element $L_2(\lambda, \alpha, s, \omega_2)$. Unfortunately, the random variables $\omega_2(m)$, $m = 0, 1, \ldots$, are not independent with respect to the measure m_{2H}, and therefore the above mentioned limit theorem for $L(\lambda, \alpha, s)$ in the space $H(D)$ cannot be used to obtain the universality theorem. To prove Lemma 2 we write $L(\lambda, \alpha, s)$ in the form of a linear combination of Dirichlet L-functions and then we apply a joint universality theorem for L-functions. However, this approach require a condition indicated in the statement of the lemma.

Lemma 3. *Suppose α is a transcendental number. Let the map $h:\ \mathbb{R}\to\mathbb{C}^N$ be defined by the formula*

$$h(t)=\Big(L(\lambda,\alpha,\sigma+it),L'(\lambda,\alpha,\sigma+it),\dots L^{(N-1)}(\lambda,\alpha,\sigma+it)\Big),\qquad \frac{1}{2}<\sigma<1.$$

Then the image of $\mathbb{R}$ is dense in $\mathbb{C}^N$.

Proof. It is easy to see that it is sufficient to prove that for each $\varepsilon>0$ and for arbitrary complex numbers $s_0,s_1,\dots,s_{N-1}$ there exists a number τ such that

$$\left|L^{(j)}(\lambda,\alpha,\sigma+i\tau)-s_j\right|<\varepsilon \tag{1}$$

for $j=0,1,\dots,N-1$. We consider a polynomial

$$p_N(s)=\frac{s_{N-1}s^{N-1}}{(N-1)!}+\frac{s_{N-2}s^{N-2}}{(N-2)!}+\dots+\frac{s_0}{0!}.$$

Then, clearly,

$$p_N^{(j)}(0)=s_j$$

for $j=0,1,\dots,N-1$. Let σ_1, $\frac{1}{2}<\sigma_1<1$, be fixed, and let K be a compact subset of the strip D such that σ_1 is an interior point of K. Denote by δ the distance of σ_1 from the boundary of K. Using Lemma 1, we find a number τ such that

$$\sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau)-p_N(s-\sigma_1)\right|<\frac{\varepsilon\delta^N}{2^N N!}\,. \tag{2}$$

Hence the Cauchy integral formula

$$L^{(j)}(\lambda,\alpha,\sigma_1+i\tau)-s_j=\frac{j!}{2\pi i}\int\limits_{|s-\sigma_1|=\delta/2}\frac{L(\lambda,\alpha,s+i\tau)-p_N(s-\sigma_1)}{(s-\sigma_1)^{j+1}}\,ds$$

together with (2) yield (1).

Lemma 4. *Let $\lambda=\frac{l}{r}$ and $\alpha=\frac{a}{q}$ be rational numbers. Suppose there exist at least two primitive characters modulo k such that the corresponding numbers η_v are distinct from zero. Let the map $h:\mathbb{R}\to\mathbb{C}^N$ be defined by the formula*

$$\begin{aligned}h(t)=&\Big(\big(q^{-\sigma-it}L(\lambda,\alpha,\sigma+it)\big),\big(q^{-\sigma-it}L(\lambda,\alpha,\sigma+it)\big)',\\ &\dots,\big(q^{-\sigma-it}L(\lambda,\alpha,\sigma+it)\big)^{(N-1)}\Big),\qquad \frac{1}{2}<\sigma<1.\end{aligned}$$

Then the image of $\mathbb{R}$ is dense in $\mathbb{C}^N$.

Proof of the lemma uses Lemma 2 and completely coincides with that of Lemma 3.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. We use the Voronin method [**1**], [**12**]. It is sufficient to prove that $F_n \equiv 0$. Let contrary to the assertion of the theorem $F_n \not\equiv 0$. Hence it follows there exists a bounded region $\mathcal{G}$ in $\mathbb{C}^N$ such that the inequality

$$\big|F_n(s_0, s_1, \ldots, s_{N-1})\big| > c > 0 \tag{3}$$

holds for all points $(s_0, s_1, \ldots, s_{N-1}) \in \mathcal{G}$. By Lemma 3 there exists a sequence $\{t_k\}$, $\lim\limits_{k\to\infty} t_k = \infty$, such that

$$\big(L(\lambda, \alpha, \sigma + it_k), L'(\lambda, \alpha, \sigma + it_k), \ldots, L^{(N-1)}(\lambda, \alpha, \sigma + it_k)\big) \in \mathcal{G}.$$

However this and (3) contradict the hypothesis of the theorem. Hence $F_n \equiv 0$.

Proof of Theorem 2 is similar to that of Theorem 1, and it uses Lemma 4.

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Department of Mathematics
Vilnius University
Naugarduko, 24
2006 Vilnius, Lithuania
antanas.laurincikas@maf.vu.lt

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