PUBLICATIONS DE l'INSTITUT MATHEMATUQUE
Nouvelle série, tome 64 (78), 1998, 107–132

USER'S GUIDE TO EQUIVARIANT METHODS IN COMBINATORICS II

Rade T. Živaljević

Communicated by Mileva Prvanović

1 Introduction

This paper is a natural continuation of the article [**53**] which was itself written as a continuation of the chapter "Topological methods", from CRC Handbook of Discrete and Computational Geometry, [**51**]. Recall that the objectives of [**53**] were to present a user friendly and reasonably detailed exposition of some ideas and tools of equivariant Topology which have proven to be useful in Combinatorics. We assume familiarity with the notation, definitions and basic results of [**53**], in particular we freely use the *key words and phrases* from that paper. Examples are *configuration spaces* $X_{\mathcal{P}}$, *test spaces* $V_{\mathcal{P}}$, *group actions, G-spaces, equivariant maps, index theory, (deleted) joins* etc.

The immediate objectives of this article are threefold. We continue the development of the *index theory* started in [**53**]. Our objective is to introduce and discuss the applicability of the so called *ideal valued (cohomological) index theory* (IVIT), essentially following the ideas of Fadell and Husseini, [**18**]. The main difference compared to [**53**] is that the *complexity* or the *index* $\mathrm{Ind}_G(X)$ of a G-space X is no longer an integer. Instead, it is a polynomial or more generally an ideal in a cohomology ring, which obviously permits a more subtle classification of G-spaces and has important implications on the existence of equivariant maps.

The second objective is to introduce and discuss combinatorial geometric applications of the elementary *obstruction theory.* We focus on the examples from combinatorial practise which serve both as an illustration of the theory and exemplify the role of special ideas characteristic for combinatorial applications.

[0]*AMS Subject Classification* (1990): Primary 52A35; Secondary 05C10, 55M20, 55N91, 55R25

Finally, we give a brief review and discuss the current status of the well known *topological Tverberg problem*. Our main objective is to review the fundamental, unfortunately unpublished, contribution of M. Özaydin, [**34**], and discuss its relation with subsequent developments.

2 Ideal valued cohomological index theory

A key property of the numerical index function $\mathrm{Ind}_G(\cdot)$ as presented in [**53**] is the so called *monotonicity property* saying that if there exists a G-equivariant map $f : X \to Y$, then $\mathrm{Ind}_G(X) \leq \mathrm{Ind}_G(Y)$. Informally speaking, this means that the space X must be of smaller "complexity" of the space Y if a G-equivariant map from X to Y should exist. In the ideal–valued, cohomological index theory (IVIT), the complexities $\mathrm{Ind}_G(X)$ of spaces are typically polynomials, or more generally (polynomial) ideals. The monotonicity in this case means that if there is a G-equivariant map $f : X \to Y$, then $\mathrm{Ind}_G(X) \supseteq \mathrm{Ind}_G(Y)$ which in the special case of principal ideals $(p) := \mathrm{Ind}_G(X), (q) := \mathrm{Ind}_G(Y)$ simply means that the polynomial q is divisible by p.

Key words and phrases:

- **Classifying space BG, classifying bundle** $G \to EG \to BG$: These are the fundamental objects associated to a topological group G serving as a basis for any equivariant cohomology theory, [**4**], [**10**], [**12**], [**18**].
- **ideal valued cohomological index** $\mathrm{Ind}_G(X)$**:** A complexity function $\mathrm{Ind}_G : G-\mathrm{Top} \to \mathcal{I}_G$ assigning to a G-space X a polynomial p or more generally an ideal in the ring $R_{G,k} := H^*(BG, k)$.
- **fundamental poset** $(\mathcal{I}_G, \subset)$ **of** G**-degrees of complexity**: This is the poset which takes the role of the *fundamental poset* $\mathcal{A}_G$ from [**53**], section 3. Elements of $\mathcal{I}_G$ are ideals in the ring $R_{G,k} := H^*(BG, k)$. Recall that the product $\wp_1 \cdot \wp_2$ of two ideals is defined by $\wp_1 \cdot \wp_2 := \{p \cdot q \mid p \in \wp_1, q \in \wp_2\}$.

2.1 Axioms for IVIT

- (*monotonicity*) If there exists a G-equivariant map $f : X \to Y$ then

$$\mathrm{Ind}_G(X) \supset \mathrm{Ind}_G(Y)$$

- (*additivity*) If $\{X_1 \cup X_2, X_1, X_2\}$ is an excisive pair of spaces, e.g. if X_1, X_2 are either both open in $X_1 \cup X_2$ or both are CW-subspaces of a CW-complex $X_1 \cup X_2$, then

$$\mathrm{Ind}_G(X_1) \cdot \mathrm{Ind}_G(X_2) \subset \mathrm{Ind}_G(X_1 \cup X_2)$$

- (*continuity*) If $A \subset X$ is a closed G-invariant subspace of X then for some open, G-invariant space $U \supset A$,

$$\mathrm{Ind}_G(\overline{U}) = \mathrm{Ind}_G(A)$$

- (*index theorem*) Let $f : X \to Y$ be a G-map, $B \subset Y$ a closed G-invariant subspace of Y and $A := f^{-1}(B) \subset X$. Then

$$\mathrm{Ind}_G(A) \cdot \mathrm{Ind}_G(Y \setminus B) \subset \mathrm{Ind}_G(X)$$

Comments: For each group G there exists the associated *classifying space* BG and the universal G-bundle $EG \to BG$. Recall that EG is characterized, up to the G-homotopy type, as a contractible CW-complex with a free G-action, while BG is the associated orbit space EG/G. The key property of the G-bundle $EG \to BG$ is that it classifies all (principal) G-bundles which means that for each free, CW-complex X and the associated G-bundle $X \to X/G$ there is a (homotopically) unique map $\alpha_X : X/G \to BG$ such that the bundle $X \to X/G$ is isomorphic to the bundle obtained by "pulling back" the bundle $EG \to BG$ along α_X. In combinatorial applications we usually assume that the group G is finite. Otherwise it may be a Lie group, for example one of the classical matrix groups $O(n), SO(n), U(n)$. For a not necessarily free G-space one defines the homotopic orbit space $X_G := EG \times_G X$. As before there exists a homotopically unique map $\alpha_X : X_G \to BG$. Given a field (or just a ring) of coefficients k, let $R_G = R_{G,k} := H^*(BG, k)$ be the fundamental ring associated to G and k. Then there exists a natural homomorphism $\beta_X = H^*(\alpha) : H^*(BG, k) \to H^*(X_G, k)$ and by definition,

$$\mathrm{Ind}_G(X) = \mathrm{Ind}_{G,k}(X) := \mathrm{Ker}(\beta_X).$$

Hence, $\mathrm{Ind}_G(X)$ is an ideal in the ring R_G. Typically, $R_G = R_{G,k}$ is a polynomial ring $k[z_1, \ldots, z_n]$ and $\mathrm{Ind}_G(X) = \{f_1, \ldots, f_m\}$ is an ideal generated by a collection of polynomials. If $m = 1$ then $\mathrm{Ind}_G(X) = (f_1)$ is a principal ideal, i.e. the polynomial f_1 alone serves as a measure of complexity of the G-space X. Details can be found in [**18**], [**21**], [**52**].

2.2 Computations of the index

Both an expert in algebraic topology and the reader who has just a *nodding* acquaintance with the topological concepts used in section 2.1 is invited to take these axioms for granted and apply the new index freely on different combinatorial geometric problems. The reader familiar with applications of the numerical index given in [**53**], section 4, will agree that the usual proof scheme outlined in that paper and the formal properties of the index function Ind_G, are just a beginning of an interesting game of applications. In order to "sharpen" our tools we collect in this section some of the most useful index computations.

Proposition 2.1 *If $G = Z/2$ is the cyclic group and $k = F_2$ the field of two elements then $R_{G,k} := H^*(BZ/2, F_2) = F_2[t]$ is the polynomial ring with one generator in dimension 1. The $(Z/2)$-index of a sphere S^n with the antipodal Z_2-action is just the principal ideal generated by the polynomial t^{n+1},*

$$\mathrm{Ind}_{Z/2}(S^n) = (t^{n+1}) \subset F_2[t].$$

Proposition 2.2 *Let $G = Z/p$ be a cyclic group where p is an odd prime, and F_p be, as before, the same object seen as a field of coefficients. Then $R_{G,k} := H^*(BZ/p, F_p) = F_p[a,b]/(a^2)$ is the polynomial ring with $\deg(a) = 1$, $\deg(b) = 2$ and one relation $a^2 = 0$. The unit sphere $S^{2n-1} \subset C^n$ is clearly a Z/p-space if Z/p is interpreted as a subgroup of $S^1 \subset C$ and S^1 acts on S^{2n-1} by complex multiplication. Then,*

$$\mathrm{Ind}_{Z/p}(S^{2n-1}) = (b^n) \subset F_p[a,b]/(a^2).$$

Proposition 2.3 *Let $G = S^1 = U(1)$ be the circle group, seen as the subgroup of $C \setminus \{0\}$. Let $k = Z$ be the ring of integers. Then $R_{G,k} := H^*(BU(1), Z) = Z[t]$ is the polynomial ring with one generator t of degree 2. The unit sphere $S^{2n-1} \subset C^n$ is a $U(1)$-space and*

$$\mathrm{Ind}_{U(1)}(S^{2n-1}) = (t^n) \subset Z[t].$$

All three propositions above are subsumed by the following more general statement.

Proposition 2.4 *Let G be one of the groups from the preceding three propositions and let k the associated ring of coefficients. Let X be a E_n-space from Definition 3.5 in* **[53]***, i.e. a finite, n-dimensional, $(n-1)$-connected CW-complex which is a free G-space. Then,*

- $\mathrm{Ind}_G(E_n) = (t^{n+1})$ *if* $G = Z/2$ *and* $k = F_2$
- $\mathrm{Ind}_G(E_{2n}) = (ab^n)$ *if* $G = Z/p$ *and* $k = F_p$
- $\mathrm{Ind}_G(E_{2n-1}) = (b^n)$ *if* $G = Z/p$ *and* $k = F_p$
- $\mathrm{Ind}_G(E_{2n-1}) = (t^n)$ *if* $G = U(1)$ *and* $k = Z$

Finally, for any finite group G we have the following proposition.

Proposition 2.5 *Let G is a finite group and let X be a space of type E_n, cf. Proposition 2.4. Then for each element $p \in R_G = H^*(BG, k)$ if $\deg(p) \leq n$ then $p \notin \mathrm{Ind}_G(X)$.*

Proof: Since X is a space of type E_n, it can be extended to a space Y of type EG by adding only G-cells of dimension at least $(n+1)$. This means that $X/G \subset Y/G$ contains the n–skeleton of the complex $Y/G = BG$ which implies the injectivity of the map $H^j(BG,k) \rightarrow H^j(X/G,k)$ for all $j \leq n$. This completes the proof since for a free G-space X, the spaces X/G and $X_G = EG \times_G X$ have the same homotopy type. □

Example 2.6 *If G is a finite group then the m-fold join $G * G * \ldots * G = G^{*(m)}$ is a $(m-2)$-connected, $(m-1)$ dimensional simplicial complex. This complex has an obvious free, simplicial action of the group G so we conclude that $G^{*(m)}$ is a E_{m-1}-space and Propositions 2.4 and 2.5 apply in this case.*

Proposition 2.7 [**18**] *Suppose that X and Y are G_1 and G_2 spaces respectively so that $X \times Y$ is seen as a $(G_1 \times G_2)$-space. Assume that $H^*(BG_1, k) \cong k[x_1, \ldots, x_k]$ and $H^*(BG_2, k) \cong k[y_1, \ldots, y_l]$, where k is a field of coefficients. Suppose that $p_1 = \mathrm{Ind}_{G_1}(X) = \{f_1, \ldots, f_m\}$ and $p_2 = \mathrm{Ind}_{G_2}(Y) = \{g_1, \ldots, g_n\}$. Then the index*

$$p := \mathrm{Ind}_{G_1 \times G_2}(X \times Y) = \{f_1, \ldots, f_m, g_1, \ldots, g_n\}$$

is the ideal in $k[x_1, \ldots, x_k, y_1, \ldots, y_l]$ generated by all polynomials f_i, g_j.

Corollary 2.8

- (a) *Let $T^n = T^n_{Z/2} := Z/2 \times \ldots \times Z/2 = (Z/2)^{\oplus(n)}$ and let $S^{m_1} \times \ldots \times S^{m_n}$ be a T^n-space with the product action described in Proposition 2.7. Then,*

$$\mathrm{Ind}_{T^n}(S^{m_1} \times \ldots \times S^{m_n}) = \{t_1^{m_1+1}, t_2^{m_2+1}, \ldots, t_n^{m_n+1}\} \subset F_2[t_1, t_2, \ldots, t_n].$$

- (b) *Let $T^n = T^n_{Z/p} := Z/p \times \ldots \times Z/p = (Z/p)^{\oplus(n)}$, where p is an odd prime, and let $S^{2m_1-1} \times \ldots \times S^{2m_n-1}$ be a T^n-space again with the product action described in Proposition 2.7. Then,*

$$\mathrm{Ind}_{T^n}(S^{2m_1-1} \times \ldots \times S^{2m_n-1}) = \{b_1^{m_1}, b_2^{m_2}, \ldots, b_n^{m_n}\} \subset F_p[b_1, b_2, \ldots, b_n].$$

2.3 What is the index of a sphere $S(V)$?

A problem of central interest is to compute the index of the unit sphere $S(V)$ associated to a linear (orthogonal) G-representation V. Recall that a linear representation V of G is a homomorphism $\rho : G \to GL(V)$ where $GL(V)$ is the group of all linear, invertible endomorphisms of V. In other words, a representation is just a linear action of G on V. It turns out that the index of $S(V)$ is frequently a principal ideal (p) in a polynomial ring. In order to simplify the presentation we often call the polynomial p itself the index of the space $S(V)$ and write $\mathrm{Ind}_G(S(V)) = p$. For all basic definitions and facts related to the theory of linear representations of (finite) groups the reader is refereed to [**19**]. For example we freely use the fact that each representation is equivalent to an orthogonal representation.

Proposition 2.9 *Let $S(W) = \{x \in W \mid |x| = 1\}$ be the unit sphere in an Euclidean space W. Suppose that W is decomposed into an orthogonal sum $W = U \oplus V$. Then,*

$$S(W) = S(U \oplus V) \cong S(U) * S(V)$$

Moreover, if U, V, W are G-representations then this decomposition is a homeomorphism of G-spaces.

The following proposition shows that in principle, the computation of the index of a G-sphere of the form $S(W)$ is reduced to the case when W is an irreducible representation.

Proposition 2.10 *Suppose U and V are two representations of the group G and let $W := U \oplus V$. Assume that vector bundles $U \to U_G \to BG$ and $V \to V_G \to BG$ are orientable over a field of coefficients k. Let $S(U), S(V), S(W)$ be the corresponding spheres seen as G-spaces. Let $H^*(BG,k) = R$, $H^*(S(U)_G,k) \cong R/(f)$ and $H^*(S(V)_G,k) \cong R/(g)$ which implies that $\mathrm{Ind}_G(S(U)) = (f)$ and $\mathrm{Ind}_G(S(V)) = (g)$. Then,*

$$\mathrm{Ind}_G(S(W)) = \mathrm{Ind}_G(S(U \oplus V)) = (f \cdot g) \subset R$$

Proof: From the *additivity* axiom for the ideal valued index theory (IVIT) and the assumption $f \in \mathrm{Ind}_G(S(U))$, $g \in \mathrm{Ind}_G(S(V))$ we deduce that $f \cdot g \in \mathrm{Ind}_G(S(W))$. Note that $f \cdot g \in H^{m+n}(BG,k)$ where $m = \dim(U)$ and $n = \dim(V)$. This means that in the spectral sequence of the sphere bundle $S(W) \to S(W)_G \to BG$ the element $f \cdot g$ of the base must be hit by the generator $e \in H^{m+n-1}(S(W))$ of the fibre. From here we immediately deduce that $H^*(S(W)_G,k) \cong R/(f \cdot g)$ and $\mathrm{Ind}_G(S(W)) = (f \cdot g)$. □

It is well known that if G is a finite product of cyclic groups, each real representation W of G can be decomposed into a sum of one-dimensional and two-dimensional representations. We illustrate the relevant ideas in the case of the n-dimensional $Z/2$-torus, $T^n = T^n_{Z/2} := (Z/2)^{\oplus(n)}$. Recall that each irreducible representation V of T^n is one–dimensional and it corresponds to a homomorphism (character) $\chi : T^n \to (Z/2,\cdot)$ where $(Z/2,\cdot)$ is the multiplicative group $\{+1,-1\}$. If ω_i is the generator of the i^{th} copy of $Z/2$ in T^n, then the character can be reconstructed from the vector $\chi_V = (\chi(\omega_1),\dots,\chi(\omega_n)) \in (Z/2,\cdot)^n$. Let $\alpha = (\alpha_1,\dots,\alpha_n) \in F_2^n$ be the 0–1 vector defined by $\chi(\omega_i) = (-1)^{\alpha_i}$.

Proposition 2.11 *Let V be the one–dimensional representation of the group $T^n = (Z/2)^{\oplus(n)}$ associated to a vector $\alpha = (\alpha_1,\dots,\alpha_n) \in F_2^n$. Then,*

$$\mathrm{Ind}_{T^n}(S(V)) = \alpha_1 t_1 + \ldots + \alpha_n t_n \in F_2[t_1,\dots,t_n].$$

Proof: The case $\alpha_1 = 1$ and $\alpha_j = 0$ for $j \geq 2$ follows from Propositions 2.1 and 2.7. The general case is reduced to the special case by an appropriate automorphism of T^n.

Corollary 2.12 *Suppose that W is a $T^n = (Z/2)^{\oplus(n)}$ representation which admits a decomposition $W \cong V_1 \oplus \ldots \oplus V_k$ where V_i is the one–dimensional representation associated to the vector $\alpha^i = (\alpha^i_1,\dots,\alpha^i_n) \in F_2^n$. Then,*

$$\mathrm{Ind}_{T^n}(S(W)) = \prod_{i=1}^{k}(\alpha^i_1 t_1 + \ldots + \alpha^i_n t_n).$$

Remark 2.13 Edgar Ramos introduced an invariant (see Theorem 3.1 in [**37**]) which he used to established a general Borsuk-Ulam type theorem for a product of balls. This invariant arises in the context of a matrix A with coefficients in F_2 which can be seen as a matrix $A = (\alpha_j^i)$ associated to a representation W. Then it is not difficult to check that the invariant introduced by Ramos is just the associated polynomial $\mathrm{Ind}_{T^n}(S(W))$ from Corollary 2.12 evaluated at $(t_1, \ldots, t_n) = (1, \ldots, 1)$.

It is clear how the constructions above should be modified to include representations of other abelian groups. For example if $G = (Z/p)^n$ then each irreducible, complex, one–dimensional representation V of G is labeled by a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in F_p^n$. More precisely, suppose $\chi : G \to C$ is the character of V, $\zeta = e^{2\pi i/p}$, and ω_i is the generator of the i^{th} copy of Z/p in $(Z/p)^n$. Then the vector $\alpha = (\alpha_i) \in F_p^n$ is characterized by the equality $\omega_i \cdot v = \zeta^{\alpha_i} v$ where $v \in V$ and $\alpha_i \in \{0, 1, \ldots, p-1\} = F_p$.

We record for the future reference the Z/p-analog of Proposition 2.11 and Corollary 2.12.

Proposition 2.14 *Let us assume that V is the one–dimensional, complex representation of the group $T^n = T^n_{Z/p} = (Z/p)^{\oplus(n)}$ associated to a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in F_p^n$. Then,*

$$\mathrm{Ind}_{T^n}(S(V)) = \alpha_1 t_1 + \ldots + \alpha_n t_n \in F_p[b_1, \ldots, b_n].$$

Moreover, if W is a representation which admits a decomposition $W \cong V_1 \oplus \ldots \oplus V_k$ where V_i is the one–dimensional, complex representation of $(Z/p)^n$ associated to the vector $\alpha^i = (\alpha_1^i, \ldots, \alpha_n^i) \in F_p^n$ then,

$$\mathrm{Ind}_{(Z/p)^n}(S(W)) = \prod_{i=1}^{k} (\alpha_1^i b_1 + \ldots + \alpha_n^i b_n).$$

At the end of this section we briefly analyze the case of *standard* and *regular* representations which frequently occur in combinatorial applications.

Definition 2.15 *Let G be a finite group. Let $\mathrm{Reg}_G := \oplus_{g \in G}\ k \cdot \{e_g\}$ be a vector space over k which has a basis vector e_g for each $g \in G$. The vector space Reg_G is called the* regular *representation of the group G if the action is defined by $g \cdot e_h := e_{gh}$.*

Definition 2.16 *Let k be a field of real or complex numbers. The* tautological *(permutation) representation $P(q) = P_k(q)$ (over k) of the symmetric group S_q is the vector space $P(q) = k^q$ where S_q acts by permuting the coordinates. Let $M(m, q) = M_k(m, q)$ be the vector space of all $(m \times q)$–matrices with entries in k. Then S_q acts on $M_k(m, q)$ by permuting the columns. Obviously $P(q) \cong M_k(1, q)$. The representation $M_k(1, n)$ is the sum of a trivial, 1-dimensional representation and the so called* standard *representation $W(1, q) = W_k(1, q)$. The representation $W(1, q)$ is explicitly described as the collection of row-vectors $a = [a_1, \ldots, a_q] \in M(1, q)$ such that $a_1 + \ldots + a_q = 0$. Let $W(m, q) := W(1, q)^{\oplus m}$ be the sum of m standard representations.*

The standard representation $W(m,q)$ naturally arises in the well known *topological Tverberg problem.* An elementary but important Proposition 2.18 shows a connection of the tautological representation of the symmetric group S_q, where $q = p^k$ is a power of a prime, with the regular representation of a subgroup of the form $(Z/p)^k$.

Remark 2.17 Note that S_q can be viewed as the group of all permutations of the vector space F_p^k. Then the group of all translations of F_p^k, seen as a subgroup of S_q, is clearly isomorphic to $T^k_{Z/p}$.

Proposition 2.18 *Suppose that $q = p^k$ is a power of a prime number, including the case $p = 2$. Let $T^k = T^k_{Z/p}$ be the subgroup of S_q described in the Remark 2.17. Then the restriction of the tautological permutation representation $P(q)$ on the subgroup T^k is isomorphic to the regular representation of the group $T^k = (Z/p)^k$.*

Proof: The proof follows easily from Definitions 2.15 and 2.16.

Corollary 2.19 *Let $W(1,q) = W_C(1,q)$ be the standard representation (over C) of the group S_q (Definition 2.16) and let $q = p^k$ be the power of a prime number. Then the T^k-index of the sphere $S(W(1,q))$ is given by the formula*

$$\mathrm{Ind}_{T^k}(S(W(1,p^k))) = \prod_{\alpha \in F_p^k \setminus \{0\}} (\alpha_1 b_1 + \ldots + \alpha_k b_k) \in F_p[b_1, \ldots, b_k]. \qquad (1)$$

Proof: By Proposition 2.18, the tautological permutation representation $P(q)$ of S_q, viewed as a representation of T^k, is isomorphic to the regular representation Reg_{T^k}. It is well known [**19**], that the regular representation Reg_{T^k} of T^k decomposes over C as follows

$$\mathrm{Reg}_{T^k} \cong \oplus_{\alpha \in F_p^k} V_\alpha$$

where V_α is the irreducible, one–dimensional T^k–representation, associated to the vector $\alpha \in F_p^k$. Since the standard representation $W(1,q)$ is obtained from $P(q)$ by removing the trivial representation, we observe that, as a T^k–representation, $W(1,q)$ admits the following decomposition,

$$W(1,q) \cong \oplus_{\alpha \in F_p^k \setminus \{0\}} V_\alpha.$$

The proof now follows from the Proposition 2.14. □

Remark 2.20 The Corollary 2.19 is true in the case $p = 2$ even if we work over the field of real numbers. More precisely, let $S(W_R(1,2^k))$ be the unit sphere in the standard (real) representation $W_R(1,2^k)$ of the symmetric group S_{2^k}. Let $(Z/2)^k \subset S_{2^k}$ be the subgroup described in the Remark 2.17 ($p = 2$). Then the permutation representation $P_R(2^k)$ is again isomorphic to the regular representation Reg_{T^k}. Since the regular representation has the decomposition $\mathrm{Reg}_{T^k} \cong \oplus_{\alpha \in F_p^k} V_\alpha$

already over real numbers, we conclude that, as T^k–representations, $W(1,q) \cong \oplus_{\alpha\in F_p^k\setminus\{0\}} V_\alpha$ and finally

$$\mathrm{Ind}_{(Z/2)^k}(S(W_R(1,2^k))) = \prod_{\alpha\in F_2^k\setminus\{0\}} (\alpha_1 b_1 + \ldots + \alpha_k b_k). \tag{2}$$

2.4 How to decide whether or not $p \in I$?

Let us suppose that $\mathrm{Ind}_G(X) = I_1$ and $\mathrm{Ind}_G(Y) = I_2$ where I_1 and I_2 are two ideals in a polynomial ring $k[z_1,\ldots,z_n]$. According to the general theory, we would be able to show that there does not exist a G-equivariant map $f : X \to Y$ if $I_2 \not\subset I_1$. In other words we are supposed to find a polynomial $p \in I_2$ which is not in I_1. Here is a proposition which demonstrates one of the possible approaches to this problem.

Proposition 2.21 *Let $P := F_2[x_1,\ldots,x_k,y_1,\ldots,y_l]$ be the ring of all polynomials in $k+l$ variables $x_1,\ldots x_k,y_1,\ldots,y_l$ with coefficients in F_2. Let L be the subring of P generated by all elementary symmetric polynomials $w_i := \sigma_i(x_1,\ldots,x_k), \widetilde{w}_j := \sigma_j(y_1,\ldots,y_l)$. Let I be the ideal in L generated by the symmetric polynomials $\sigma_1,\sigma_2,\ldots,\sigma_{k+l}$ in all $(k+l)$ variables $x_1,\ldots,y_l$. Then*

$$(w_k)^l = (x_1x_2\ldots x_k)^l \notin I$$

Proof: Let k be a field and let $\mathcal{K} := k(z_1,\ldots,z_n)$ be the field of all rational functions in n variables $z_1,z_2,\ldots,z_n$ with coefficients in k. Let $\mathcal{S} := \mathcal{K}^{S_n}$ be the subfield of all symmetric rational functions, i.e. the subfield of all invariants with respect to the obvious action of the symmetric group S_n. Then, see [**2**],

- Each element $h \in \mathcal{K}$ can be uniquely represented as a linear combination of the form
$$h = \Sigma_{\nu\in J}\ \alpha_\nu X_\nu \tag{3}$$
where $\alpha_\nu \in \mathcal{S}$, J is the collection of all sequences $\nu = (\nu_1,\nu_2,\ldots,\nu_n)$ of natural numbers where $\nu_i \leq i-1$ for all i, and X_ν are the associated monomials of the form
$$X_\nu := z_1^{\nu_1} z_2^{\nu_2} \ldots z_n^{\nu_n}.$$

Suppose that $(w_k)^l \in I$ which means that

$$(w_k)^l = \beta_1\sigma_1 + \beta_2\sigma_2 + \ldots + \beta_{k+l}\sigma_{k+l} \tag{4}$$

for some $\beta_i \in L$. Let us assume that $k = F_2$ and that the variables $z_1,z_2,\ldots,z_n$ are interpreted as $y_1,\ldots,y_l,x_1,\ldots,x_k$. Each of the polynomials β_i has a unique representation described in (3). We observe next that $(w_k)^l = (x_1\ldots x_k)^l$ is already one of the polynomials of the form X_ν, the one corresponding to the sequence $\nu_1 = \ldots = \nu_l = 0$, $\nu_{l+1} = \ldots \nu_{k+l} = l$. This means that its unique representation of the form (3) is $(w_k)^l = 1\cdot(w_k)^l$ which immediately implies that the equation (4) is not possible.

Remark 2.22 Obviously the proof of Proposition 2.21 is valid not only for the ring L but for the whole polynomial ring $F_2[x_1, \ldots, x_k, y_1, \ldots, y_l]$. In other words, $(w_k)^l = (x_1 x_2 \ldots x_k)^l \notin I'$ where I' is the ideal in $F_2[x_1, \ldots, x_k, y_1, \ldots, y_l]$ generated by all symmetric polynomials $\sigma_1, \sigma_2, \ldots, \sigma_{k+l}$ in all $(k+l)$ variables $x_1, \ldots, y_l$. The reason we chose the formulation above is that, according to well known results of A. Borel, [**31**], the cohomology ring $H^*(G_k(R^d), F_2)$ of the Grassmann manifold $G_k(R^d)$ of all linear, k-dimensional subspaces in R^d is isomorphic to the quotient ring L/I. The ring $F_2[x_1, \ldots, x_k, y_1, \ldots, y_l]/I'$ also has a geometric interpretation. It is isomorphic to the cohomology ring of the flag manifold $\tilde{V}_{d,k} := V_{d,k}/T^k$ where $V_{d,k}$ is the Stiefel manifold of all orthonormal k-frames in R^d. The combinatorial geometric consequences of these results are collected in section 3.4.

3 Applications

The reader can find many examples of combinatorial problems which can be reduced to the question of (non)existence of a G-equivariant map $f : X \to Y$. Review papers [**5**], [**8**], [**29**], [**43**], [**51**], [**53**] are primary source of information also serving as a guide for original research articles. Usually Y is a sphere of the form $S(V)$ where V is a linear representation of G. In this case, the nonexistence of an equivariant map f implies that each continuous, G-equivariant map $g : X \to V$ has a zero in X. Of course all this is a special case of the scheme outline in the section "How to solve it" in [**53**].

3.1 Topological Tverberg problem

Our first example is the famous topological Tverberg problem. Recall that the well known Tverberg theorem, [**44**], [**41**], [**51**], says that every collection $C \subset R^d$ of $N := (q-1)(d+1)+1$ points can be partitioned into q nonempty subcollections, $C = C_1 \cup \ldots \cup C_q$, so that the associated convex hulls have a nonempty intersection, $\bigcap_{i=1}^q \mathrm{conv}(C_i) \neq \emptyset$. The starting point of [**7**] was the observation that Tverberg theorem can be reformulated as the following statement. For any *affine* map $f : \Delta^N \to R^d$, where Δ^N is a N-dimensional simplex, there exist disjoint faces $\Delta_1, \ldots, \Delta_q$ of Δ^N such that the intersection $f(\Delta_1) \cap \ldots \cap f(\Delta_q)$ is nonempty. The topological Tverberg problem is the question whether the Tverberg theorem is still true if f is an arbitrary (not necessarily affine) continuous map.

Theorem 3.1 provides the most general, known result in this direction and at the time of writing of this paper it is still unknown if the condition on q being a power of prime can be removed. The Theorem 3.1 was in the case $k = 1$ proved by Bárány, Shlosman, and Szücs in [**7**] and extended to the case $q = p^k$ by M. Özaydin in an unpublished preprint, [**34**]. The proof in [**34**] is also based on the idea of cohomological index although in somewhat less explicit form involving the localization theorem of A. Borel.

Theorem 3.1 *Let $q = p^k$ be a power of a prime number. Let $f : \Delta^N \to R^d$ be a continuous map from a N-dimensional simplex where $N = (q-1)(d+1)$. Then there* *exist disjoint faces* $\Delta_1, \ldots, \Delta_q$ *of* Δ^N *such that the intersection* $f(\Delta_1) \cap \ldots \cap f(\Delta_q)$ *is nonempty.*

By the usual method based on the deleted product (deleted join) technique, [**7**], [**38**], [**39**], [**49**], [**53**], Theorem 3.1 can be deduced from the following result.

Theorem 3.2 *Suppose that* $q = p^k$ *is a power of prime. Let* $W_R(1,q)$ *be the standard, real* S_q*–representation (Definition 2.16). Let* X *be a space of type* E_M *where* $M = (q-1)d$*, i.e.* X *is a free,* $(M-1)$*–connected,* S_q*–space. Then there does not exist a* S_q*–equivariant map*

$$f : X \to S(W_R(1,q)^{\oplus d}).$$

Actually, the proof will show that the map f *cannot be equivariant already with respect to the subgroup* $(Z/p)^k \subset S_q$ *described in Remark 2.17.*

Proof: We assume that $d = 2d_1$ is an even integer. This simplifies the proof a little and illustrates all relevant ideas. We observe now that there exist the following isomorphisms of S_q or $(Z/p)^k$–representations,

$$W_R(1,q)^{\oplus d} \cong W_R(d,q) \cong W_C(d_1,q) \cong W_C(1,q)^{\oplus d_1}.$$

It follows from Proposition 2.18 and Corollary 2.19 that $\mathrm{Ind}_{(Z/p)^k}(S(W_R(1,q)^{\oplus d}))$ is a polynomial $P \in F_p[b_1, \ldots, b_k]$ of degree $(q-1)d_1$. Since $\deg(b_i) = 2$, the polynomial P represents a cohomology class of dimension $M = (q-1)d$. The theorem follows from the monotonicity axiom and the Proposition 2.5 which says that X, being a space of type E_M $(M = (q-1)d)$, has the property $P \notin \mathrm{Ind}_{(Z/p)^k}(X)$.

Remark 3.3 Note that for the odd case $d = 2d_1 + 1$ we actually need slightly more general forms of Proposition 2.14 and Corollary 2.19 applicable not only for complex but for all real, irreducible $(Z/p)^k$–representations. Alternatively, one can use Proposition 2.5. from [**26**], which says that the topological Tverberg theorem in the case (q,d) is true provided it is true in the case $(q, d+1)$.

Historical comments: Theorem 3.1 was proved in the case $k = 1$ by I. Bárány, S.B. Shlosman and A. Szücs in [**7**]. This important paper opened new perspectives for applications of topological methods in geometric combinatorics. Murad Özaydin has extended this result to the case $q = p^k$ in the preprint [**34**] which has been known for more than ten years but has never been published. Perhaps the reason why [**34**] was not better publicized was a feeling that Theorem 3.1 should be true without any restriction on q and that a proof of this was within reach. At present we still don't know if the topological Tverberg theorem is true in full generality. There have been several erroneous proofs in the meantime. For example the result of Kriz, [**24**] (Theorem 2.6), used by some authors as a basis for a proof of the general topological Tverberg theorem, is unfortunately false, [**25**]. Theorem 3.1 has been in the last few years rediscovered by several authors. The proof of Volovikov, [**46**], is based on a similar idea as the proof of Özaydin. These two proofs both rely on a localization lemma of A. Borel from [**9**] and [**20**] respectively. A good more recent reference for this localization result is [**12**], Proposition 3.14. A proof based on the Chern class computations, very similar in spirit to the index theory proof given in this paper, can be found in the preprint [**42**] of K. Sarkaria. A more detailed exposition of this proof, which also contains much of the background material, can be found in the preprint [**26**]. Note that there are other results which should be considered as close relatives of Theorem 3.1. A good example is Corollary 2 from the paper of Bartsch, [**3**], which says that if V and W are two orthogonal representations of $G = (Z/p)^k$ without trivial subrepresentations, and if there exists a G-equivariant map $f; S(V) \to S(W)$ between the associated unit spheres, then $\dim(W) \geq \dim(V)$.

3.2 Colored Tverberg theorem

Definition 3.4 *A coloring of a set $C \subset R^d$ by $k+1$ colors is a function $\phi : C \to \langle k \rangle$ where $\langle k \rangle := \{0, 1, \ldots, k\}$. It is always assumed that ϕ is an epimorphism and that $k \leq d$. A $(k+1)$-element set $B \subset C$ is* multicolored *if it contains a point from each of the colors, i.e. if $\phi(B) = \langle k \rangle$. If B is multicolored then the set* conv(B) *is called a (possibly degenerate)* rainbow simplex, [**51**]. *Sometimes it is more convenient to describe a coloring ϕ of C by the associated partition $\mathcal{C} := \{C_0, \ldots, C_k\}$ of C where $C_i := \phi^{-1}(i)$. The collection $\mathcal{C}$ is called the* coloring family *for C and individual sets $C_i = \phi^{-1}(i)$ are often referred to as the colors of C,* [**51**], [**5**].

Colored Tverberg theorems are relatives of Tverberg theorem where we start with a set $C \subset R^d$ which is colored by a family $\mathcal{C} = \{C_0, \ldots, C_k\}$ of colors and ask if one can find "many" disjoint, multicolored sets $B_1, \ldots, B_r$ so that the associated rainbow simplices $\sigma_i := \mathrm{conv}(B_i)$ have a nonempty intersection. More precisely, let $T(r, k, d)$ be the minimum number t such that for any set $C \subset R^d$ of size $t(k+1)$ and every coloring $\phi : C \to \langle k \rangle$ with the property that the size of the set $C_i := \phi^{-1}(i)$ is t, there exist r disjoint, multicolored sets B_j $(j = 1, \ldots, r)$ such that

$$\bigcap_{j=1}^{r} \mathrm{conv}(B_j) \neq \emptyset.$$

Problem 3.5 *Colored Tverberg Problem*
Determine the number $T(r, k, d)$ for different values of integers r, k, d where $d \geq k$ or find a good upper bound for the function $T(r, k, d)$.

The reader can find more information about partial solutions of the colored Tverberg problem in review papers [**51**], [**5**] or in the original articles [**6**], [**55**], [**30**], [**48**]. Here are some of the highlights. It is known that $T(3, 3, 2) = 3$ and $T(3, 3, 3) = 5$. It is also known that $T(r, 2, 2) = r$ and $T(2, d, d) = 2$ and there is a conjecture that $T(r, d, d) = r$ for all r and d. It is not difficult to see that in the case $k < d$, the number $T(r, k, d)$ is well defined only if $r \leq d/(d-k)$. In the case

$k = d-1$ the necessary condition says that $r \leq d$. Under this condition, one proves that $T(r, d-1, d) = 2r-1$ if r is a prime number. Similarly, if r is a prime, it is known that $T(r, d, d) \leq 2r-1$. Since the function $T(r, d, d)$ is monotone in d, this leads, in light of the fact that there is always a prime number in the interval $[n, 2n-1]$, to the estimate $T(r, d, d) \leq 4r-3$.

Our objective in this section is to show that there exists a *topological* version of the colored Tverberg theorem parallel to Theorem 3.1 and that it is also a consequence of Theorem 3.2.

Definition 3.6 *Let $\Delta_0, \Delta_1, \ldots, \Delta_k$ be a collection of simplices of dimension $t-1$ where Δ_i is viewed as a simplex colored by the color $i \in \langle k \rangle$. Let $\Delta := \Delta_0 * \Delta_1 * \ldots * \Delta_k$ be the join of simplices Δ_i. A k-dimensional face σ of Δ is called a rainbow face if $\sigma \cap \Delta_i \neq \emptyset$ for each $i \in \langle k \rangle$.*

Problem 3.7 *Topological Colored Tverberg Problem*
Evaluate or estimate the function $TC(r, k, d)$ defined as the minimum number t such that for any continuous map

$$f : \Delta \to R^d$$

there exist r disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of Δ such that

$$f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset.$$

It is clear that if we assume that $f : \Delta \to R^d$ is an affine map, then the topological colored Tverberg problem reduces to colored Tverberg problem.

Theorem 3.8 *Suppose that $r = p^l$ is a power of a prime number. Then,*

(1) $TC(r, k, d) \leq 2r-1$ *if* $k < d$ *and* $r \leq d/(d-k)$,

(2) $TC(r, d, d) \leq 2r-1$.

Proof: (outline) The analogues of inequalities (1) and (2) for the function $T(r, k, d)$ were proved, in the case when r is a prime number, in [**48**] and [**55**] respectively. Let us show on the example of the first inequality how these proofs should be modified to yield more general statements (1) and (2) above.

Let $\Delta \to R^d$ be an arbitrary continuous function, where $\Delta := \Delta_0 * \Delta_1 * \ldots * \Delta_k$ is the simplex described in Definition 3.6 and each of the simplices Δ_i has $t = 2r-1$ vertices. The map f induces a continuous map $F_1 : \Delta^{*(r)} \to (R^d)^{*(r)}$, where $X^{*(r)} := X * \ldots * X$ is the join of r copies of the space X. Each element of $X^{*(r)}$ has the form $x = t_1 x_1 + \ldots + t_r x_r$, where $x_i \in X, t_i \geq 0$ and $t_1 + \ldots + t_r = 1$. If X is a simplicial complex, then $X^{*(r)}$ also has a natural simplicial structure with simplices of the form $\theta = \theta_1 * \ldots * \theta_r$, where θ_i are simplices in X. Let K be the subcomplex of $\Delta^{*(r)}$, defined as the union of all simplices of the form $\theta = \theta_1 * \ldots * \theta_r$ where θ_i are *multicolored*, pairwise disjoint simplices in Δ. Let $F_2 : K \to (R^d)^{*(r)}$ be the restriction of the map F_1 on K. It is convenient to embed the abstract join $(R^d)^{*(r)}$ in the Euclidean space $V := (R \oplus R^d)^{\oplus(r)}$ by the embedding $t_1v_1 + \ldots + t_rv_r \mapsto (t_1, v_1, t_2, v_2, \ldots, t_r, v_r)$. Let $D_1 := \{v = t_1v_1 + \ldots + t_rv_r \in (R^d)^{*(r)} \mid v_1 = v_2 = \ldots = v_r\}$ be the "diagonal" in $(R^d)^{*(r)}$. If D is the actual $(d+1)$-dimensional diagonal in $V = (R^{d+1})^r$, then $D_1 = D \bigcap (R^d)^{*(r)}$. Let $F = \pi \circ F_2 : K \to L$ be the composition of F_2 and $\pi : (R \oplus R^d)^{\oplus(r)} \to L$ where $L := D^\perp \cong V/D$ is the orthogonal complement of D in V. It turns out that K is isomorphic to the configuration space $A_{r,k,t}$ from the proof of Theorems 4 in [**48**] which is shown there to be s-connected for $s = (r-1)(k+1) + k - 1$. All maps F_1, F_2, F are S_r-equivariant with the obvious S_r-actions on $\Delta^{*(r)}, K = A_{r,k,t}, V = (R \oplus R^d)^{\oplus(r)}$ and $L = V/D$. Note that V is, as a S_r-representation, isomorphic to the matrix representation $M_R(d+1, r)$ from Definition 2.16 and that in the decomposition $M_R(d+1, r) \cong W(d+1, r) \oplus D$ the subspace D corresponds to the part with the trivial S_r-action. This means that L is isomorphic to $W(d+1, r)$ as a S_r-representation. The dimension of the space $W(d+1, r)$ is $(d+1)(r-1)$ and it is easy to check that $(d+1)(r-1) \leq s+1 = (r-1)(k+1) + k$ is equivalent to the assumption $r \leq d/(d-k)$. By Theorem 3.2, the map $F : K \to L$ must have a zero and the result follows. □

3.3 Equipartitions of masses by hyperplanes

A Lebesgue measurable set $A \subset R^d$ is equipartitioned by a collection $\mathcal{H} = \{H_i\}_{i=1}^k$ of k hyperplanes if each of the 2^k orthants associated to $\mathcal{H}$ contains the fraction $1/2^k$ of the total measure of A. More generally, a triple of integers (d, j, k) is admissible if for each collection $\mathcal{A} = \{A_1, \ldots, A_j\}$ of j measurable sets in R^d, there exists a collection $\mathcal{H} = \{H_1, \ldots, H_k\}$ of k hyperplanes which is an equipartition for each of the sets A_i. In order to simplify the presentation we restrict our attention to Lebesgue measurable sets in R^d. Actually, all results in this section have obvious generalizations to arbitrary (bounded) σ-additive Borel measures in R^d.

E. Ramos gave a detailed study of admissible triples in [**37**]. He showed that $d \geq j(2^k-1)/k$ is a necessary condition for a triple (d, j, k) to be admissible and that this condition is sufficient in many interesting special cases. Among other results he showed that $(5,1,4), (5,3,2), (9,5,2), (9,3,3)$ have this property and also described some infinite series of triples which are admissible. Ramos does not formally use the index theory approach although it is not difficult to recognize some of his ideas in this context. His central general invariant, see the Remark 2.13, was described as the parity of a certain function (related to the permanent) of a 0–1 matrix. Our objective in this section is to clarify the connection with the index theory approach. A detailed study of the equipartition problem from this point of view will be presented in the forthcoming paper [**36**].

Proposition 3.9 *Let $T^k = (Z/2)^k$ be the usual $(Z/2)$–torus and let Reg_{T^k} be the associated, real, (2^k)–dimensional regular representation. Let W_{T^k} be the (2^k-1)–dimensional representation, obtained from Reg_{T^k} by subtracting the trivial, one–dimensional representation V_0. Then a sufficient condition for a triple (d, j, k) to* *be admissible is the nonexistence of a T^k–equivariant map*

$$f : (S^d)^k \to S(W_{T^k}^{\oplus j}).$$

Proof: The ideas of Ramos, [**37**], can be expressed in the usual *configuration space–test map* scheme described in [**53**]. The configuration space $X_{\mathcal{P}}$ is naturally the manifold of all ordered collections $\mathcal{H} = (H_1, \ldots, H_k)$ of oriented hyperplanes in R^d. In order to obtain a compact space we go one dimension up and embed R^d in R^{d+1}, say as the hyperplane with the equation $x_{d+1} = 1$. Then each oriented hyperplane H in $R^d \cong \{x \in R^{d+1} \mid x_{d+1} = 1\}$ is a trace of an oriented hyperplane $H' \subset R^{d+1}$ which passes through the origin. The oriented hyperplane H' is determined by the corresponding orthogonal unit vector $u \in S^d \subset R^{d+1}$ so the natural environment for all collections $\mathcal{H} = (H_1, \ldots, H_k)$ is the manifold $(S^d)^k$. Now we define the test space $V_{\mathcal{P}}$ and the test map $C : X_{\mathcal{P}} \to V_{\mathcal{P}}$. Let $\mathcal{H} = (H_1, \ldots, H_k)$ be a collection of oriented hyperplanes in $X_{\mathcal{P}}$ which is determined by a collection $(u_1, \ldots, u_k) \in (S^d)^k$ of unit vectors. The collection $\mathcal{H}$ divides R^d into (2^k)–orthants Ort_β which are naturally indexed by 0–1 vectors $\beta = \beta_{\mathcal{H}} \in F_2^k$. Let $b_\beta^j(\mathcal{H})$ be the measure of the set $A_j \cap Ort_\beta$. Then b_β^j can be extended to a continuous function $B_\beta^j : (S^d)^k \to R^1$ and the functions B_β^j together define a T^k–equivariant, continuous map $B^j : (S^d)^k \to R^{2^k}$. The test space $V_{\mathcal{P}} = R^{2^k}$ is found to be isomorphic, as a T^k–representation, to the regular representation Reg_{T^k}. The "zero" subspace $Z_{\mathcal{P}} \subset V_{\mathcal{P}}$ is identified as the trivial subrepresentation V_0 of Reg_{T^k}. In other words, the collection $\mathcal{H} = (H_1, \ldots, H_k)$, labeled by the vector $u \in (S^d)^k$, is an equipartition for the measurable set A_j iff $B^j(u) \in V_0$. Let $C^j : (S^d)^k \to W_{T^k}$ be the (T^k)–equivariant map obtained by composing B^j with the orthogonal projection $\mathrm{Reg}_{T^k} \to W_{T^k}$. Then $\mathcal{H}$ is an equipartition for A_j iff $C^j(u) = 0$ and $\mathcal{H}$ is an equipartition for all measurable sets $A_1, \ldots, A_j$ iff $u \in (S^d)^k$ is a zero of the map $C = (C^1, \ldots, C^j) : (S^d)^k \to W_{T^k}^{\oplus j}$ which implies the desired result.

Corollary 3.10 *Let D_k be the product of all nonzero polynomials in $F_2[t_1, \ldots, t_k]$ of the form $(\alpha_1 t_1 + \ldots + \alpha_k t_k)$. Then the triple (d, j, k) is admissible if the polynomial $(D_k)^j$ is not in the ideal generated by monomials $t_1^{d+1}, \ldots, t_k^{d+1}$,*

$$(D_k)^j = \prod_{\alpha \in F_2^k \setminus \{0\}} (\alpha_1 t_1 + \ldots + \alpha_k t_k)^j \notin \{t_1^{d+1}, \ldots, t_k^{d+1}\}.$$

Proof: The proof follows from the monotonicity property of the index function, Corollary 2.8, and the observation (Remark 2.20) that $\mathrm{Ind}_{(Z/2)^k}(S(W_R(1, 2^k))) = \mathrm{Ind}_{(Z/2)^k}(S(W_{T^k}))$ is the product of all nonzero, linear polynomials in the ring $F_2[t_1, \ldots, t_k]$.

3.4 Center transversal theorem

The following theorem, [**54**], [**51**], is called the *center transversal theorem.*

Theorem 3.11 *Let $A_0, A_1, \ldots, A_k$, $0 \leq k \leq d-1$, be a collection of Lebesgue measurable sets in R^d. Then there exists a k-dimensional affine subspace $D \subseteq R^d$ such that for every closed halfspace $H(v,\alpha) := \{x \in R^d \mid \langle x, v\rangle \leq \alpha\}$ and every $i \in \{0,1,\ldots,k\}$,*

$$D \subseteq H(v,\alpha) \Longrightarrow \mu(A_i \cap H(v,\alpha)) \geq \frac{1}{n-k+1}\,\mu(A_i).$$

The theorem is usually formulated for more general measures which include the counting measures (finite sets), continuous mass distributions etc. It is shown in [**54**] that theorem 3.11 follows from the following general topological principle.

Proposition 3.12 *Let $R^k \to \mathcal{E}_k \to G_k(R^d)$ be the canonical k–plane bundle over the Grassmann manifold of all k–dimensional subspaces of R^d. Then $(w_k)^{n-k} \neq 0$ where $w_k \in H^*(G_k(R^d), Z/2)$ is the top Stiefel–Whitney class of the canonical bundle $\mathcal{E}_k$. Since $(w_k)^{n-k}$ is the top Stiefel–Whitney class of the Whitney sum $\mathcal{E}_k^{\oplus(n-k)}$, this implies that each continuous, cross–section of $\mathcal{E}_k^{\oplus(n-k)} \to G_k(R^d)$ must have a zero. As a consequence, each collection $\mathcal{C} = \{c_0, c_1, \ldots, c_k\}$ of k continuous cross–sections of the bundle $\mathcal{E}_k \to G_k(R^d)$ must have a point of coincidence, i.e. a point $p \in G_k(R^d)$ such that $c_0(p) = c_1(p) = \ldots = c_k(p)$.*

For other applications of this principle and its extensions the reader is referred to [**17**], [**27**], [**47**]. It is well known, see for example [**47**] and [**12**] (section I.7), that there is a one–to–one correspondence between continuous cross-sections of a vector bundle $R^k \to \mathcal{E} \to B$ and $O(k)$-equivariant maps $V_k(\mathcal{E}) \to R^k$ where $V_k(\mathcal{E}) = \bigcup_{b\in B} \mathrm{Iso}(R^k, \mathcal{E}_b)$ is the Stiefel-type manifold of all orthonormal k-frames in $\mathcal{E}$. In particular $\mathcal{E} \to B$ does not admit a nowhere zero, continuous, cross–section if and only if there does not exist a $O(k)$–equivariant map $V_k(\mathcal{E}) \to R^k \setminus \{0\}$. This shows that Proposition 3.12 can be approached from the point of view of equivariant maps and index theory. For the future reference we formulate this as an independent guiding principle.

"Sections are equivariant maps" principle: Suppose that X and Y are G-spaces and that the action on X is free. Consider the fibre bundle $Y \to X \times_G Y \to X/G$. It turns out that G-equivariant maps $f : X \to Y$ are in one–to–one correspondence with the sections $s : X/G \to X \times_G Y$ of this bundle. In particular, all arguments involving Stiefel-Whitney or Chern classes for proving the nonexistence of a continuous, cross–section of a vector bundle are equivalent to a the nonexistence of an equivariant map of the form $X \to S(V)$ and can be carried on in the context of the index theory (IVIT). Conversely, at least in the case when $Y = S(V)$ is a representation sphere, the index theory can be interpreted as a calculus with characteristic classes.

After these preliminaries, we are naturally led to the questions which index computations are "responsible" for Theorem 3.11 and Proposition 3.12. The following result [**18**] provides an answer.

Proposition 3.13 *Let $V_{d,k}$ be the Stiefel manifold of all orthonormal k-frames in R^d. The manifold $V_{d,k} = \mathrm{Iso}(R^k, R^d)$ can be viewed as the manifold of all linear isometries $I : R^k \to R^d$. From here we see that $V_{d,k}$ is a (right) $O(k)$-space, where $O(k) = \mathrm{Iso}(R^k, R^k)$ is the orthogonal group and the action is given by $I \cdot g := I \circ g$. Let R^k be the tautological representation of $O(k)$. Then there does not exist a $O(k)$–equivariant map*

$$f : V_{d,k} \to S((R^k)^{\oplus(d-k)}).$$

Proof: We will actually prove more by showing that there does not exist a $T^k = (Z/2)^k$ equivariant map $f : V_{d,k} \to S((R^k)^{\oplus(d-k)})$, where $T^k \subset O(k)$ is the set of all diagonal matrices with entries in $\{1, -1\}$. Proposition 2.10, applied on T^k-representations R^k and $(R^k)^{\oplus(d-k)}$ respectively, leads to the following formulas

$$\begin{aligned} \mathrm{Ind}_{T^k}(S(R^k)) &= (t_1 t_2 \dots t_k) \\ \mathrm{Ind}_{T^k}(S((R^k))^{\oplus(d-k)}) &= ((t_1 t_2 \dots t_k)^{d-k}) \end{aligned}$$

If is sufficient to show that $(t_1 t_2 \dots t_k)^{d-k} \notin \mathrm{Ind}_{T^k}(V_{d,k})$. There are at least three different ways to establish this fact. The argument in [**18**], Theorem 3.16, is inductive. It is based on a careful analysis of the cohomology of the flag manifold $\tilde{V}_{d,k} := V_{d,k}/T^k$. The second way is to use the Pieri formula from the Schubert calculus. This idea works at least for the weaker statement $(w_k)^{d-k} \notin \mathrm{Ind}_{O(k)}(V_{d,k})$. Finally, one can rely on Proposition 2.21 (Remark 2.22). Indeed, the cohomology of the flag manifold $\widetilde{V}_{d,k}$ is isomorphic to the quotient ring of the polynomial ring $F_2[t_1, \dots, t_k, s_1, \dots, s_{d-k}]$ by the ideal I' generated by all symmetric polynomials in all d variables $t_1, \dots, t_k, s_1, \dots, s_{d-k}$. The desired result is now, in light of the Remark 2.22, a consequence of Proposition 2.21.

4 Index theory versus obstruction theory

Key words and phrases:

- **Equivariant homotopy classes** $[X, Y]_G$**:** These are the homotopy classes of G–equivariant maps where two maps are equivalent if they are G–homotopic.

- **equivariant cohomology groups** $H^*_G(X, M)$**, equivariant cochains:** If X is a (free) G–space then the cellular (singular etc.) cochain complex $C^*(X)$ is a complex of G-modules. Then the equivariant cohomology with the coefficients in a G–module M is the cohomology of the chain complex $\mathrm{Hom}_G(C^*(X), M)$, see [**12**].

4.1 Obstruction exact sequence

The basic idea of the *obstruction theory* is to associate a computable, cohomological invariant to the problem of extending a continuous map. For example suppose we want to check the existence of an equivariant map $f : X \to Y$ of two, G-CW–complexes X and Y. We can try to solve this problem by constructing inductively an equivariant map $f_n : X^{(n)} \to Y$ from the n–skeleton of X and then check whether this map can be equivariantly extended to the skeleton $X^{(n+1)}$ of X. In this way arises an *obstruction cocycle* $c = c(f_n)$ whose cohomology class $[c(f_n)]$ belongs to an appropriate equivariant cohomology group $H_G^{n+1}(X, \pi_n(Y))$. The vanishing of this class is a necessary and sufficient condition for the existence of an equivariant map $g : X^{n+1} \to Y$ which agrees with f_n on the skeleton X^{n-1}. In other words, vanishing of $[c(f_n)]$ guarantees that f_n can be extended to $X^{(n+1)}$, after a suitable modification on the n-skeleton of X. More formally and more precisely these ideas can be, following [**12**], recast in the form of the following statement.

Theorem 4.1 *Suppose that X is a free G-CW–complex and that Y is n–simple G-space (which holds for example if the fundamental group $\pi_1(Y)$ is trivial). Then for each integer $n \geq 1$ there exists an exact obstruction sequence,*

$$[X^{(n+1)}, Y]_G \xrightarrow{\theta} \mathrm{Im}([X^{(n)}, Y]_G \longrightarrow [X^{(n-1)}, Y]_G) \xrightarrow{\tau} H_G^{n+1}(X, \pi_n(Y)) \qquad (5)$$

The "exactness" of this sequence of sets means that $\mathrm{Im}(\theta) = \tau^{-1}(0)$. *More explicitly this means that a G-equivariant map $f : X^{n-1} \to Y$, which can be extended G-equivariantly to X^n, admits an equivariant extension to X^{n+1} iff certain equivariant, obstruction class $\tau([f])$ vanishes.*

In the majority of interesting examples the space Y is a G–sphere $S(V) \cong S^n$ where V is a linear representation of the group G. In this case Theorem 4.1 has the following very useful corollary.

Corollary 4.2 *Suppose that X is a free, $(n+1)$–dimensional, G-complex where $n \geq 2$. Let $Y = S(V) = S^n$ be an n-dimensional G-sphere, associated to a linear G–representation V, and let $\{*\}$ be a one–element set. Then the sequence* (5) *reduces to*

$$[X, S^n]_G \longrightarrow \{*\} \xrightarrow{\tau} H_G^{n+1}(X, \mathcal{Z}) \qquad (6)$$

where $\mathcal{Z}$ is the group $Z \cong \pi_n(S^n)$ viewed as a G–module. The sequence (6) *should be interpreted as the statement that the existence of a G–equivariant map $f : X \to S(V)$ depends solely on the question whether a functorially defined element $\tau(*) \in H_G^{n+1}(X, \pi_n(Y))$ vanishes or not.*

Proof: We have to check that both $[X^{n-1}, S^n]_G$ is a one-element set $\{*\}$, i.e. that any two G-equivariant maps $f, g : X^{n-1} \to S^n$ are G-homotopic, and that $[X^n, S^n]_G \neq \emptyset$. Both statements are consequences of the fact that the first obstruction to extending (equivariantly) a map $h : Y^{(m)} \to S^n$ to Y^{m+1}, where Y is a free, G-complex, arises if $m = n$. □

Remark 4.3 Note that there exists a relative version of Theorem 4.1, cf. [**12**], where an equivariant map is already prescribed in advance on a G–subspace $A \subset X$, the action is free only on $X \setminus A$ and all homotopies are relative A. In this case the obstructions "live" in relative equivariant cohomology groups $H_G^{n+1}(X, A; \mathcal{Z})$. Also, if $\mathcal{Z} = \pi_n(Y)$ is a trivial G-module, the equivariant cohomology groups coincide with the usual (nonequivariant) cohomology groups $H_G^{n+1}(X/G, A/G; \pi_n(Y))$ of the pair of quotient spaces $(X/G, A/G)$.

Example 4.4 *Borsuk–Ulam Theorem*

One of the equivalent forms of the well-known Borsuk-Ulam theorem is the statement that there does not exist a $Z/2$-equivariant map $f : S^n \to S^{n-1}$. In different notation it says that $[S^n, S^{n-1}]_{Z/2}$ is an empty set. A more usual reformulation is that each $Z/2$-equivariant map $f : S^n \to R^n$ must have a zero. Borsuk-Ulam theorem is of course an easy consequence of both the numerical [**53**] and the ideal-valued index theory. We outline here a different proof which serves as a good illustration of the use of Theorem 4.1 and Corollary 4.2.

The sphere $S^n \subset R^{n+1}$ is seen as a CW-complex with the usual (minimal) CW-structure, invariant under the antipodal $Z/2$-action. This structure has two cells in each dimension and the $(n-1)$-skeleton of S^n is just the equatorial sphere S^{n-1}. Denote by e_+ and e_- the top dimensional cells of S^n, where e_+ is the upper and e_- the lower hemisphere of S^n. By Corollary 4.2, in order to show that $[S^n, S^{n-1}]_{Z/2} = \emptyset$, it would be sufficient to prove that $\tau(*) \neq 0$ in $H^n_{Z/2}(S^n, \mathcal{Z})$. The modulo 2 reduction leads to a homomorphism $H^n_{Z/2}(S^n, \mathcal{Z}) \to H^n_{Z/2}(S^n, Z/2)$ of equivariant cohomology groups and, by a slight abuse of notation, we assume that obstruction cochains and classes are taken with $Z/2$ rather than Z coefficients. Hence, it is sufficient to show that $\tau_0 := \tau(*)$ is nonzero as an element in the group $H^n_{Z/2}(S^n, Z/2)$. This is very convenient since this way we don't have to work with local coefficients $\mathcal{Z}$. The idea is to use a concrete, simple $Z/2$-equivariant map $f : S^n \to R^n$ to find the obstruction cochain $c(g) = c(f|S^{n-1}) \in C^n_{Z/2}(S^{n-1}, Z/2)$, where $g := f|S^{n-1}$ is the restriction of f on the $(n-1)$-skeleton of S^n. We choose f to be the orthogonal projection of S^n to the equatorial plane R^n. One observes that on the level of cellular chains, $c(g)(e_+) = c(g)(e_-) = 1$. Indeed, $c(g)(e_+)$ is by definition the degree of the map $g : S^{n-1} \to R^n \setminus \{0\}$ and this degree is just the parity of the number of zeros of f in e_+; the same holds for $c(g)(e_-)$. It remains to check that $[c(g)] \in H^n_{Z/2}(S^n, Z/2) \cong Z/2$ represents the generator. This immediately follows from the geometric interpretation of the Poincaré duality $H^n(RP^n, Z/2) \cong H_0(RP^n, Z/2) \cong Z/2$ and the observation that $H^n_{Z/2}(S^n, Z/2) \cong H^n(RP^n, Z/2)$. □

In the following two sections we demonstrate some of the applications of Theorem 4.1 and Corollary 4.2 on interesting geometric problems which are typically not tractable by the methods of index theory.

4.2 Borsuk-Ulam theorems for the cyclic group

In our first application we construct an example which shows why the condition that the order of the group is a power of prime often plays an important role in Borsuk-Ulam type theorems.

Example 4.5 Let $G = Z/6$. Let S^1 be the circle presented in the form of a regular hexagon, which is the most economical $Z/6$-invariant, simplicial structure on S^1. Alternatively we can view S^1 as a unit circle in the complex plane subdivided by the vertices $b_k := \exp(2\pi\sqrt{-1}k/6)\,(k = 0, \dots, 5)$ of the regular hexagon. Let $S^5 = S^1 * S^1 * S^1$ be the 5-sphere, viewed as the join of three copies of S^1 with the induced simplicial structure and the induced $Z/6$ action. Let $W(1,6)$ be the standard representation of the symmetric group S_6, Definition 2.16, also viewed as a representation of $Z/6$. Let $S^4 := S(W(1,6))$ be the corresponding sphere. Note that $W(1,6)$ can be described as follows. Suppose ω is a generator of $Z/6$ and let $v \in W(1,6) \setminus \{0\}$. Then $v_0, v_1, \dots, v_5$, where $v_i := \omega^i \cdot v$, are vertices of a regular simplex $\Delta^5 = \text{conv}\{v_i \mid 0 \leq i \leq 5\}$. It follows that the representation $W(1,6)$ is isomorphic to the 5-dimensional Euclidean space with the S_6 action arising from all isometries of a regular 5-simplex $\Delta^5 \subset R^5$.

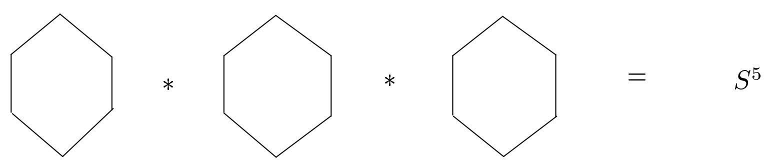

Figure 1:

Proposition 4.6 *Suppose that $S^5 := S^1 * S^1 * S^1$ and $S^4 := S(W(1,6))$ are $Z/6$–spheres described in the Example 4.5. Then the set $[S^5, S^4]_{Z/6}$ of equivariant homotopy classes is nonempty or in other words, there* does *exist a $(Z/6)$–equivariant map*

$$\phi : S^5 \to S^4.$$

Proof: Let $g : S^1 \to W(1,6) \setminus \{0\}$ be defined as the unique simplicial map such that $g(b_i) = v_i$. Let $F : S^5 \to W(1,6)$ be the map defined by

$$F(t_1x_1 + t_2x_2 + t_3x_3) := t_1g(x_1) + t_2g(x_2) + t_3g(x_3)$$

where on the right-hand side is the genuine convex combination while on the left-hand side is the "convex" (join) decomposition of a point $x \in S^5 = S^1 * S^1 * S^1$. The sphere S^5, as a simplicial complex, consists of 6^3-simplices of dimension 5 of the form $\sigma = \theta_1 * \theta_2 * \theta_3$, where θ_i are edges (1-simplices) of S^1.

Observation $(*)$: If the first edge θ_1 is fixed, then there exist *exactly two* choices for the ordered pair (θ_2, θ_3) such that the origin is contained in $F(\sigma)$ where $\sigma = \theta_1 * \theta_2 * \theta_3$. In other words, there are exactly two $Z/6$-orbits, each consisting of six 5-dimensional simplices, whose F-images contain the origin $0 \in R^5$.

Recall that there exists a (minimal) $Z/6$-invariant CW-decomposition of S^5 which has exactly 6 cells, $e^i, \omega(e^i), \dots, \omega^5(e^i)$, in each dimension $i = 0, \dots, 5$. It is described as the CW-structure arising if in the decomposition $S^5 = S^1 * S^1 * S^1$, only the first circle is assumed to be a hexagon. Here we rely on the fact that the join $\sigma^p * S^q$ of a p-simplex with a q-sphere, is a cell of dimension $p + q + 1$. Let

$$F^{(4)} : (S^5)^{(4)} \to W(1,6) \setminus \{0\}$$

be the restriction of F on the 4-skeleton of S^5. The obstruction cocycle $c(F^{(4)})$ for extending $F^{(4)}$ to a map $F' : S^5 \to W(1,6) \setminus \{0\}$, can be described, in light of the Observation $(*)$, as the equivariant cochain which takes value 2 (or -2) on each of the top-dimensional cells in the minimal $Z/2$-equivariant CW-decomposition of S^5. The associated cellular chain complex (resolution) has the usual form

$$Z(Z/6) \stackrel{1-\omega}{\longrightarrow} Z(Z/6) \stackrel{N}{\longrightarrow} \dots \stackrel{N}{\longrightarrow} Z(Z/6) \stackrel{1-\omega}{\longrightarrow} Z(Z/6)$$

where $Z(Z/6)$ is the group ring of $Z/6$ and $N = 1 + \omega + \dots + \omega^5$. The obstruction cocycle $c(F^{(4)})$ is an element of $H_{Z/6}(S^5, \mathcal{Z})$ where the $Z/6$-module $\mathcal{Z}$ is identified as $H_4(W(1,6) \setminus \{0\}, Z) \cong H_4(S^4, Z) \cong Z$ where the generator $\omega \in Z/6$ acts by $\omega(u) = -u$. The cochain complex computing the equivariant cohomology with $\mathcal{Z}$ coefficients is obtained from the chain complex above by an application of the functor $\mathrm{Hom}_{Z/6}(\cdot, \mathcal{Z})$ and has the following form

$$Z \stackrel{2}{\longleftarrow} Z \stackrel{0}{\longleftarrow} \dots \stackrel{0}{\longleftarrow} Z \stackrel{2}{\longleftarrow} Z$$

We conclude from here that the obstruction cocycle $c(F^{(4)})$ is a coboundary which means that $F^{(4)}$ can be first modified and then extended to an equivariant map from S^5 to $W(1,6) \setminus \{0\}$. □

The following (unpublished) result of Özaydin [**34**], is a very useful criterion for the existence or nonexistence of equivariant maps. The proof of this result is based on the important idea of *transfer* for equivariant cohomology groups.

Proposition 4.7 *Let X be a $(n+1)$–dimensional free G–complex, and let Y be a $(n-1)$–connected G-complex where $n \geq 2$. Then there exists a G-equivariant map $f : X \to Y$ if and only if for each prime p there exists a G_p–equivariant map $f_p : X \to Y$, where G_p is Sylow p–subgroup of G.*

Let us note that Proposition 4.6 can be deduced from Proposition 4.7. Indeed both $Z/2$ and $Z/3$, viewed as Sylow subgroups of $Z/6$, do not act transitively on

S^4. This means that there exist (constant) equivariant maps $S^5 \to S^4$ for both $Z/2$ and $Z/3$ and Proposition 4.6 follows from Proposition 4.7. On the other hand the simple geometric idea of the proof of Proposition 4.6 can be used in some cases where Proposition 4.7 is not applicable. An example is the following proposition.

Proposition 4.8 *Suppose $G = Z/4$. Let $S^3 = S^1 * S^1$ be the $(Z/4)$–simplicial complex obtained as a join of two circles (regular squares) and let $S^2 = S(W(1,4)$ be the unit sphere in the standard $(Z/4)$–representation $W(1,4)$. Then there does not exist a $(Z/4)$–equivariant map*

$$f : S^3 \longrightarrow S^2.$$

Proof: The proof is an easy modification of the proof of Proposition 4.6 so we leave the details to the reader.

Remark: The reader may find it interesting to find all integers n for which there exists a (Z/n)–equivariant map

$$f : E^{Z/n}_{n-1} \to S(W(1,n))$$

where $E^{Z/n}_m$ is a $(m-1)$–connected, free (Z/n)–complex and $W(1,n)$ is the standard representation of Z/n. For those familiar with [**34**], note that already Proposition 4.8 is in disagreement with the (second) half of Theorem 4.2 from that paper.

4.3 Equipartitions of masses in R^3

We want to prove that for any Lebesgue measurable set $A \subset R^3$ there exist three planes H_1, H_2, H_3 partitioning A into eight parts of equal measure. In other words, using the terminology of section 3.3, we want to show that $(3,1,3)$ is an admissible triple. According to the Proposition 3.9, it would be sufficient to show that there does not exist a $T^3 = (Z/2)^{\oplus 3}$ equivariant map

$$f : (S^3)^3 \to S(W_{T^3}).$$

Unfortunately the Corollary 3.10 cannot be used since the polynomial

$$D_3 = \prod_{\alpha \in F_2^3 \setminus \{0\}} (\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3) = \det \begin{bmatrix} t_1 & t_1^2 & t_1^4 \\ t_2 & t_2^2 & t_2^4 \\ t_3 & t_3^2 & t_3^4 \end{bmatrix}$$

is in the ideal $\{t_1^4, t_2^4, t_3^4\}$. One of the reasons this happened is that the appropriate group of symmetries in the equipartition problem above is not T^3! Rather it is the semi-direct product $G = T^3 \leftthreetimes S_3$. We will prove that the desired equipartition exists by showing that an obstruction theoretic argument applies to the following sharper statement. More detailed proofs of more general statements will appear in [**36**].

Proposition 4.9 *Let A, B, C be a collection of three Lebesgue measurable sets in R^3. Then there exist three hyperplanes H_1, H_2, H_3 which form an equipartition of A, such that the hyperplane H_1 is a common bisector of both B and C.*

Proof: (outline) The configuration space $X_{\mathcal{P}}$ associated to this problem is as before $(S^3)^3$. The target space is $V_{\mathcal{P}} = R^7 \oplus R^1 \oplus R^1$ where R^7 corresponds to 7 functions testing if the collection H_1, H_2, H_3 is an equipartition of A and the two remaining functions test if H_1 is a halving plane for B and C. The group of symmetries in this problem is $G = Z/2 \oplus ((Z/2)^{\oplus 2} \leftthreetimes S_2)$ where $(Z/2)^{\oplus 2} \leftthreetimes S_2$ is the semi–direct product of $(Z/2)^{\oplus 2}$ and S_2. Hence, the result is a consequence of the fact that there does not exist a G–equivariant map $f : (S^3)^3 \to S(R^9)$. Note that the action of G on $(S^3)^3$ is not free on the set $D := \{(x_1, x_2, x_3) \in (S^3)^3 \mid x_2 = x_3 \text{ or } x_2 = -x_3\}$. This means that we need a relative form of Theorem 4.1 (cf. Remark 4.3). Let us also note that $G = Z/2 \oplus ((Z/2)^{\oplus 2} \leftthreetimes S_2)$ acts on $R^7 \oplus R^1 \oplus R^1$ as a group of orientation preserving linear maps. This means that $\mathcal{Z} = \pi_8(S(R^9)) \cong Z$ is a trivial G–module. In other words, the equivariant cohomologies from Theorem 4.1 and Corollary 4.2 can be computed, in light of Remark 4.3, as the usual cohomologies of the pair $((S^3)^3/G, D/G)$. These cohomologies are, via Poincaré duality after reducing modulo 2, isomorphic to $Z/2$. The usual geometric interpretation of the Poincaré duality, already used in Example 4.4, says that a criterion for an equivariant cochain $c \in C^9_G((S^3)^3, D; Z/2)$ to represent a generator of this group is that the total sum of values of c on all representatives of G–equivariant cells of dimension 9 in $(S^3)^3$ is an odd number. In particular this criterion is satisfied if there exists *exactly one* G orbit where the cochain is nonzero. Recall that, while the obstruction cohomology class $\tau = [c]$ is uniquely defined, the obstruction cocycle c is very sensitive to the choice of the equivariant map which is supposed to be extended. In other words, each choice of measurable sets A, B, C leads to a G–equivariant map $f : (S^3)^3 \to R^9$ which can be used for computing the associated obstruction cocycle! We carry on this idea for the following special choice of measurable sets A, B and C.

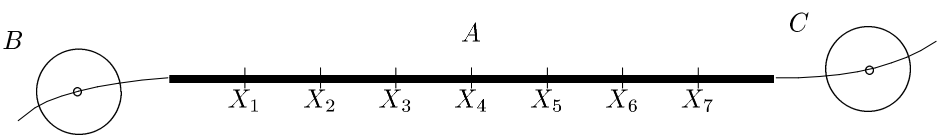

Figure 2:

Let B and C be disjoint unit balls with centers at the moment curve $\Gamma = \{(t, t^2, t^3) \mid t \in R\}$. Let the set A be an "infinitesimal" tube around an interval $I = \{(t, t^2, t^3) \in \Gamma \mid 0 \leq t \leq 1\}$ which is assumed to be disjoint from B and C. Then the only collections $\{H_1, H_2, H_3\}$ of planes which form a desired equipartition of A, B and C are described as follows. The plane H_1 is spanned by the centers of balls B and C and the center of the interval I. Let $X_1, X_2, \ldots, X_7$ be the points in the interval I where $X_i = \Gamma(i/7)$. If $U = \{X_1, X_3, X_6\}$ and $V = \{X_2, X_5, X_7\}$ then either $U \subset H_2$ and $V \subset H_3$ or vice versa. This implies that the "orbit criterion" is satisfied by the obstruction cocycle τ in this case which completes the proof. □

References

[1] N. Alon, *Some recent combinatorial applications of Borsuk-type theorems,* in: *Algebraic, Extremal and Metric Combinatorics,* M.M. Deza, P. Frankl, D.G Rosenberg, editors, Cambridge Univ. Press, Cambridge, 1988, 1–12.

[2] E. Artin, *Galois Theory* (second edition), Univ. of Notre Dame Press, Notre Dame, London 1971.

[3] T. Bartsch. *A simple proof of the degree formula for Z/p-equivariant maps,* Math. Zeitschrift **212** (1993), 285–292.

[4] M.F. Atiyah, R. Bott, *The moment map and equivariant cohomology,* Topology **23** (1984), 1–23.

[5] I. Bárány, *Geometric and combinatorial applications of Borsuk's theorem,* in: *New Trends in Discrete and Computational Geometry*, János Pach, ed., Algorithms and Combinatorics 10, Springer-Verlag, Berlin, 1993.

[6] I. Bárány, D.G. Larman, *A colored version of Tverberg's theorem,* J. London Math. Soc. (2) **45** (1992), 314–320.

[7] I. Bárány, S.B. Shlosman, A. Szücs, *On a topological generalization of a theorem of Tverberg,* J. London Math. Soc. (2, **23** (1981), 158–164.

[8] A. Björner, *Topological methods,* in: R. Graham, M. Grötschel, and L. Lovász (editors), *Handbook of Combinatorics*, North-Holland, Amsterdam, 1995.

[9] A. Borel at al. *Seminar on Transformation Groups*, Ann. Math. Studies 46, Princeton Univ. Press, 1960.

[10] G.E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York 1972.

[11] P.E. Conner, E.E. Floyd, *Fixed point free involutions and equivariant maps,* Bull. Amer. Math. Soc. **66** (1960), 416–441.

[12] T. tom Dieck, *Transformation Groups*, de Gruyter stud. in math. 8, Berlin, 1987.

[13] J. Dieudonné, *A History of Algebraic and Differential Topology*, Birkhäuser, 1989.

[14] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, 1972.

[15] A. Dold, *Simple proofs of some Borsuk-Ulam results,* Contemporary Math. **19** (1983), 65–69.

[16] A. Dold, *Parametrized Borsuk-Ulam theorems,* Comment. Math. Helv. **63** (1988), 275–285.

[17] V.L. Dol'nikov, *Transversals of families of sets in R^n and a connection between the Helly and Borsuk theorems,* Russian Acad. Sci. Sb. Math. **79**:1 (1994),

[18] E. Fadell, S. Husseini, *An ideal-valued cohomological index, theory with applications to Borsuk-Ulam and Bourgin-Yang theorems,* Ergod. Th. and Dynam. Sys. **8*** (1988), 73–85.

[19] W. Fulton, J. Harris, *Representation Theory,* Graduate texts in mathematics 129, Springer-Verlag, 1991.

[20] W. Y. Hsiang. *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Math. und ihrer Grenzgebiete 85, Springer-Verlag, 1975.

[21] S.Y. Husseini, *The equivariant J-homomorphism for arbitrary S^1-actions,* Indiana Univ. Math. J. **33** (1984), 491–494.

[22] M. Izydorek, S. Rybicki, *On parametrized Borsuk-Ulam theorem for free Z_p-action,* in: *Proc. Barcelona Conf. on Algebraic Topology* 1990, pp. 227–234.

[23] J. Jaworowski, *A continuous version of the Borsuk-Ulam theorem,* Proc. Amer. Math. Soc. **82** (1981), 112–114.

[24] I. Kriz, *Equivariant cohomology and lower bounds for chromatic numbers,* Trans. Amer. Math. Soc. **333**, 2 (1992), 567–577.

[25] I. Kriz, *Correction to [24],* Trans. Amer. Math. Soc. (to appear).

[26] M. de Longueville, *The topological Tverberg theorem for prime powers,* (preprint), TU-Berlin, January, 1998.

[27] E. Makai, S. Vrećica, R. Živaljević, *Plane sections of convex bodies of maximal volume*, to appear in Discrete and Comp. Geometry.

[28] V.V. Makeev, *Knaster problem on continuous, maps of a sphere into euclidean spaces*, J. Sov. Math. **52** (1990), 2854–2860.

[29] J. Matoušek, *Topological methods in Combinatorics and Geometry,* Lecture notes, KAM Series (Tech. Report), Charles University, Prague, 1994.

[30] J. Matoušek, *Note on the colored Tverberg theorem,* J. Comb. Theory Ser. B **66** (1996), 146–151.

[31] J. Milnor and J.D. Stasheff, *Characteristic Classes,* Princeton University Press, Princeton, 1974.

[32] L. Montejano, *Recognizing sets by means of some of their sections*, Manuscripta Math. **76** (1992), 227–239.

[33] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo-Park, 1984.

[34] M. Özaydin, *Equivariant maps for the symmetric group*, (unpublished preprint), Univ. of Wisconsin-Madison, 1987.

[35] J. Pach (Ed.), *New Trends in Discrete and Computational Geometry,* Algorithms and Combinatorics 10, Springer-Verlag, 1993.

[36] Z. Petrović, S. Vrećica, R. Živaljević, *Combinatorics and topology of partitions of masses by hyperplanes,* (in preparation).

[37] E. Ramos, *Equipartitions of mass distributions, by hyperplanes,* Discrete Comput. Geom. **15** (1996), 147–167.

[38] K.S. Sarkaria, *A generalized Kneser conjecture,* J. Comb. Theory B **49** (1990), 236–240.

[39] K.S. Sarkaria, *Kuratowski complexes*, Topology, **30** (1991), 67–76.

[40] K.S. Sarkaria, *A generalized van Kampen-Flores theorem,* Proc. Amer. Math. Soc. **111** (1991), 559–565.

[41] K.S. Sarkaria, *Tverberg's theorem via number fields*, Israel J. Math. **79** (1992), 317–320.

[42] K.S. Sarkaria, *Tverberg partitions and Borsuk–Ulam theorems*, (preprint), Panjab University.

[43] H. Steinlein, *Borsuk's antipodal theorem and its generalizations, and applications, a survey*, in: *Topological Methods in Nonlinear Analysis*, Sém. Math. Sup. 95, Presses Univ. Montréal, 1985, 166–235.

[44] H. Tverberg, *A generalization of Radon's theorem*, J. London Math. Soc. **41** (1966), 123–128.

[45] H. Tverberg, S. Vrećica, *On generalizations of Radon's theorem and the ham sandwich theorem*, Europ. J. Combinatorics **14** (1993), 259–264.

[46] A. Yu. Volovikov. *On a topological generalization of the Tverberg theorem*, Math. Notes **59**:3, 1996.

[47] S. Vrećica, R. Živaljević, *The ham sandwich theorem revisited*, Israel J. Math. **78** (1992), 21–32.

[48] S. Vrećica, R. Živaljević, *New cases of the colored Tverberg theorem*, in: *Jerusalem Combinatorics '93*, H. Barcelo, G. Kalai (eds.), Contemporary mathematics, Amer. Math. Soc., Providence, 1994, 325–334.

[49] A. Vučić, R. Živaljević, *Note on a conjecture of Sierksma*, Discrete Comput. Geom. **9** (1993), 339–349.

[50] C.T. Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson, I and II,* Ann. of Math., **60** (1954), 262–282, and **62** (1955), 271–283.

[51] R.T. Živaljević, *Topological methods*, in: *CRC Handbook of Discrete and Computational Geometry*, J.E. Goodman, J. O'Rourke, eds. CRC Press, Boca Raton, 1997.

[52] R.T. Živaljević, *The Tverberg–Vrećica problem and the combinatorial geometry on vector bundles*, Israel J. Math., (to appear).

[53] R.T. Živaljević, *User's guide to equivariant methods in combinatorics*, Publ. Inst. Math. (Beograd) **59(73)** (1996), 114–130.

[54] R. Živaljević, S. Vrećica, *An extension of the ham sandwich theorem*, Bull. London Math. Soc. **22** (1990), 183–186.

[55] R. Živaljević, S. Vrećica, *The colored Tverberg's problem and complexes of injective functions*, J. Combin. Theory, Ser. A **61 (2)** (1992), 309–318.

Matematički institut
Kneza Mihaila 35
11001 Beograd, p.p. 367
Yugoslavia
rade@mi.sanu.ac.yu

(Received 30 09 1998)
(Revised 29 11 1998)