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ON WELL-POSEDNESS OF QUADRATIC MINIMIZATION PROBLEM ON ELLIPSOID AND POLYHEDRON

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Abstract. We consider existence of solutions for quadratic minimization problem on an ellipsoid and on a polyhedron. In the case of polyhedron, we present a necessary and sufficient conditions for Tikhonov well-posedness of the problem.

1. We consider the following extremal problem:

$$J(u) = \|Au - f\|_F^2 \to \inf,\ u \in U,$$

where U is the ellipsoid

$$U = \{u \in H : \|Bu\|_G \leq R\}$$

or the polyhedron

$$U = \{u \in H : \langle c_i, u\rangle \leq \beta_i,\ i = 1, \dots, m\}.$$

Here H, F, G are real Hilbert spaces; $A : H \to F$, $B : H \to G$ are bounded linear operators; $f \in F$, $c_i \in H$, $c_i \neq 0$, $i = 1, \dots, m$ are fixed elements from the corresponding spaces; β_i, $i = 1, \dots m$ and $R > 0$ are given real numbers.

The results of this paper complete the results from [**1**]–[**3**]. Namely, in the case of an ellipsoid (1), (2), we get necessary conditions for the existence of solutions and show that these conditions are sufficient for normal solvable operators A and B; in the case of polyhedron (1), (3), we present the necessary and sufficient conditions for the existence of solutions as well as for the well-posedness.

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Let us introduce the following notation: $R(A)$ is the range space of the operator A, $AU = \{Au : u \in U\}$ is the image of U under the action of A, $\operatorname{Ker} A$ is the kernel of A, $A^* : F \to H$ is the adjoint operator of A, $\overline{M}$ is the closure of the set $M \subseteq H$ with respect to the norm of H, $L^{\perp}$ is the orthogonal complement of the subspace L, P is the orthogonal projector of H onto $\overline{R(A^*)}$.

The operators A and B generate the following orthogonal decompositions of H:

$$H = \overline{R(A^*)} \oplus \operatorname{Ker} A, \quad H = \overline{R(B^*)} \oplus \operatorname{Ker} B.$$

An operator A is called *normal solvable* if $R(A) = \overline{R(A)}$. This condition is equivalent to $\overline{R(A^*)} = R(A^*)$ [**4**].

Lemma 1. [**5**] *A linear bounded operator $A : H \to F$ is normal solvable if and only if*

$$\mu = \inf\{\|Au\| : u \perp \operatorname{Ker} A, \|u\| = 1\} > 0.$$

This lemma implies immediately

Lemma 2. *If a linear bounded operator $A : H \to F$ is not normal solvable, then there exists a sequence (p_n) such that*

$$p_n \in \overline{R(A^*)},\ \|p_n\| = 1,\ p_n \rightharpoonup 0,\ Ap_n \to 0 \text{ as } n \to \infty.$$

Let us notice that the set U in (2) and (3) is convex and closed with respect to the norm of H. If, moreover, the set U in (2) is bounded, then the existence of a solution for (1), (2) for each $f \in F$ follows by Weierstrass theorem [**6**]. If U is unbounded (that is always so for U in (3) when $\dim H = \infty$), then the problems (1), (2) and (1), (3) have solutions for each $f \in F$ if and only if AU is a closed set in F (see [**1**], [**2**]). We will use this existence criterion repeatedly in the sequel. Now we formulate the necessary conditions for the solvability of the problem (1), (2).

Theorem 1. *Suppose that the problem (1), (2) has a solution for each $f \in F$. Then at least one of the following conditions is satisfied:*

(i) $$R(A^*) \cap R(B^*B) = \{0\},$$

(ii) $$\operatorname{Ker} A + \operatorname{Ker} B = \overline{\operatorname{Ker} A + \operatorname{Ker} B}.$$

Proof. Assume that $R(A^*) \cap R(B^*B) \neq \{0\}$. The continuity of the operator A and the closure of the set AU imply that the set

$$A^{-1}(AU) = \operatorname{Ker} A + \operatorname{Ker} B + V_R,$$

where $V_R = \{u \in \overline{R(B^*)} : \|Bu\| \leq R\}$, is closed in the space H. Let $y \in R(A^*) \cap R(B^*B)$, $y \neq 0$. Take a point $z \in H$ such that $y = B^*Bz$. Present z, according to

(4), in the form $z = z_1 + z_2,\ z_1 \in \overline{R(B^*)},\ z_2 \in \operatorname{Ker} B$. Then $y = B^*Bz_1$, and, for the point $x_0 = \frac{R}{\|Bz_1\|} z_1$, we have

$$x_0 \in \overline{R(B^*)},\ B^*Bx_0 \in R(A^*) \cap R(B^*B),\ \|Bx_0\|^2 = R^2,$$

in particular, $x_0 \in V_R$. Now take any point $y_0 \in \overline{\operatorname{Ker} A + \operatorname{Ker} B}$. The point $y_0 + x_0$ is a limit point of the closed set $\operatorname{Ker} A + \operatorname{Ker} B + V_R$. Therefore, the point $y_0 + x_0$ is presentable as $y_0 + x_0 = p_0 + z_0$, where $p_0 \in \operatorname{Ker} A + \operatorname{Ker} B$ and $z_0 \in V_R$. Multiplying both sides of (6) by B^*Bx_0 and taking into account (5) and the orthogonality

$$R(A^*) \cap R(B^*B) \perp \overline{\operatorname{Ker} A + \operatorname{Ker} B},$$

we find that $R^2 = \|Bx_0\|^2 = \langle Bz_0, Bx_0\rangle$. Since $z_0 \in V_R$, we obtain

$$\|B(x_0 - z_0)\|^2 = \|Bx_0\|^2 - 2\langle Bx_0, Bz_0\rangle + \|Bz_0\|^2 \leq 0$$

and therefore $x_0 = z_0$. Now we have $y_0 = p_0 \in \operatorname{Ker} A + \operatorname{Ker} B$. Recalling that y_0 was an arbitrary point from $\overline{\operatorname{Ker} A + \operatorname{Ker} B}$, we finally get the condition (ii). This concludes the proof. □

The following example shows that the assumptions about normal solvability of both operators A and B do not guarantee the existence of solutions of the problem (1), (2) for all $f \in F$.

Example. Take $H = F = G = l_2$ and consider two closed subspaces of l_2:

$$L = \{x \in l_2 :\ x = (0, x_2, 0, x_4, 0, x_6, 0, \dots)\},$$
$$M = \{x \in l_2 :\ x = (0, x_2, x_2/2, x_4, x_4/4, x_6, x_6/6, \dots)\}.$$

Define A as the orthoprojector of l_2 onto $L^\perp$ and B as the orthoprojector of l_2 onto $M^\perp$. Then $A = A^*$, $B = B^* = B^*B$, $\operatorname{Ker} A = L$, $\operatorname{Ker} B = M$, operators A and B are normal solvable but both relations (i) and (ii) from Theorem 1 are violated:

$$x_0 = (1, 0, 0, \dots) \in R(A^*) \cap R(B^*B) = L^\perp \cap M^\perp \neq \{0\},$$
$$\operatorname{Ker} A + \operatorname{Ker} B = L + M \neq \overline{L + M} = \overline{\operatorname{Ker} A + \operatorname{Ker} B} = \{x_0\}^\perp.$$

It means that in this case the problem (1), (2) can not have a solution for each $f \in l_2$.

One can ask about additional conditions that normal solvable operators A and B should satisfy for the existence of a solution of the problem (1), (2) for each $f \in F$. In order to answer this question, we shall prove the following

LEMMA 3. *Let A be a normal solvable operator and let $V \subseteq H$ be a convex closed set. Then*

$$\overline{AV} = A(\overline{\operatorname{Ker} A + V}).$$

Proof. For each $y_0 \in \overline{AV}$ there exists a sequence $(u_n), u_n \in V$ such that the sequence $y_n = Au_n$ converges to y_0 as $n \to \infty$. According to (4) we can present u_n as

$$u_n = x_n + z_n,\ x_n \in R(A^*),\ z_n \in \operatorname{Ker} A.$$

Then

$$Ax_n = Au_n = y_n \to y_0,\ \ n \to \infty.$$

As the operator A is normal solvable, (7) implies that the sequence (x_n) is bounded. Therefore, (x_n) (or some its subsequence) converges weakly to some limit x_0 and also $x_n \in V + \operatorname{Ker} A$. The set $\overline{\operatorname{Ker} A + V}$ is weakly closed, thus $x_0 \in \overline{\operatorname{Ker} A + V}$ and

$$y_0 = \lim_{n\to\infty} Au_n = \lim_{n\to\infty} Ax_n = Ax_0 \in A(\overline{\operatorname{Ker} A + V}).$$

Therefore, we have proved the inclusion $\overline{AV} \subseteq A(\overline{\operatorname{Ker} A + V})$. Conversely, for each $y_0 \in A(\overline{\operatorname{Ker} A + V})$ there exists a sequence $u_n \in \operatorname{Ker} A + V$ such that the sequence $y_n = Au_n \to y_0$ as $n \to \infty$. Present the elements $u_n \in \operatorname{Ker} A + V$ in the form:

$$u_n = z_n + x_n,\ z_n \in \operatorname{Ker} A,\ x_n \in V.$$

Since $y_n = Au_n = Ax_n \in AV$, it follows that $y_0 \in \overline{AV}$. Thus we have proved the inclusion $A(\overline{\operatorname{Ker} A + V}) \subseteq \overline{AV}$, which completes the proof. □

Now we show that for normal solvable operators A and B the statement of Theorem 1 can be inverted.

THEOREM 2. *Let A and B be normal solvable operators. If at least one of the conditions (i) or (ii) from Theorem 1 is satisfied, then the problem (1), (2) has a solution for each $f \in F$.*

Proof. First consider the case (ii) when

$$\operatorname{Ker} A + \operatorname{Ker} B = \overline{\operatorname{Ker} A + \operatorname{Ker} B}.$$

Using Lemma 3 for $V = \operatorname{Ker} B$, we get

$$\overline{A(\operatorname{Ker} B)} = A(\overline{\operatorname{Ker} A + \operatorname{Ker} B}) = A(\operatorname{Ker} A + \operatorname{Ker} B) = A(\operatorname{Ker} B),$$

i.e., the set $A(\operatorname{Ker} B)$ is closed. Then, by Theorem 3 in [**2**], it follows that the problem (1), (2) has a solution.

Now consider the case (i) when $R(A^*) \cap R(B^*B) = \{0\}$. Since the operators A^*, B^*B are normal solvable and their ranges $R(A^*)$, $R(B^*B)$ are closed, we get

$$H = \{0\}^\perp = (R(A^*) \cap R(B^*B))^\perp = \overline{\operatorname{Ker} A + \operatorname{Ker} B},$$

i.e., the set $\operatorname{Ker} A + \operatorname{Ker} B$ is dense in H. Note that ellipsoid (2) has a nonempty interior (we consider $R > 0$), therefore $U + \operatorname{Ker} A + \operatorname{Ker} B = H$. On the other hand, $U = U + \operatorname{Ker} B$, hence $U + \operatorname{Ker} A = H$. Finally, we see that

$$AU = A(U + \operatorname{Ker} A) = AH = R(A),$$

i.e., the set AU is closed. This concludes the proof. □

Let us consider the existence problem for (1), (3). Suppose the operator $B : H \to R^m$ is defined by $Bu = (\langle c_1, u\rangle, \langle c_2, u\rangle, \dots, \langle c_m, u\rangle), \quad u \in H$. The operator B is normal solvable and

$$R(B^*) = \Big\{ \sum_{i=1}^{m} \lambda_i c_i : \ \lambda_i \in R^1, \ i = 1, \dots, m \Big\} = \mathcal{L}(c_1, c_2, \dots, c_m).$$

Since $H = R(B^*) \oplus \operatorname{Ker} B$, the constraints (3) can be presented in the form

(f1) $$U = V_\beta \oplus \operatorname{Ker} B,$$

where

$$V_\beta = \{v \in R(B^*) : \ \langle c_i, v\rangle \le \beta_j, \ j = 1, \dots, m\}.$$

THEOREM 3. *The problem (1), (3) has a solution for each $f \in F$ if and only if the operator A is normal solvable.*

Proof. The implication *normal solvability* ⇒ *existence* was proved in [**1**, p. 12]. Let us prove the converse implication. First observe that (f1) implies $AU = AV_\beta + A(\operatorname{Ker} B)$. We claim that $AU = AV_\beta + \overline{A(\operatorname{Ker} B)}$. Since by assumption the set AU is closed, we see that any point $y \in AV_\beta + \overline{A(\operatorname{Ker} B)}$ as a limit point of AU belongs to AU. So, we have obtained that $AV_\beta + \overline{A(\operatorname{Ker} B)} \subseteq AU$. It is obvious that the inverse inclusion is valid. Therefore

$$AV_\beta + A(\operatorname{Ker} B) = AV_\beta + \overline{A(\operatorname{Ker} B)}$$

is really true. Adding $A(R(B^*))$ to both sides, by the inclusion $V_\beta \subset R(B^*)$, we get

$$R(A) = A(R(B^*)) + A(\operatorname{Ker} B) = A(R(B^*)) + \overline{A(\operatorname{Ker} B)}.$$

To conclude the proof, it remains to note that the set $R(A)$ is closed as a sum of the finite-dimensional subspace $A(R(B^*))$ and the closed subspace $\overline{A(\operatorname{Ker} B)}$. □

2. Consider the question of well-posedness for the problem (1), (3) in Tikhonov sense.

Definition. [**1**] The problem (1) is well-posed in the space H in Tikhonov sense if the following three conditions hold: 1) $J_* = \inf\{J(u) : u \in U\} > -\infty$; 2) $U_* = \{u \in U : J(u) = J_*\} \ne \emptyset$; 3) each minimizing sequence (u_n) of the problem (1) converges strongly in H to the solution set U_*, i.e.,

$$d(u_n, U_*) = \inf\{\|u_n - u\| : u \in U_*\} \to 0 \text{ as } n \to \infty.$$

If at least one of the conditions 1), 2), 3) is not valid, then the problem is called *ill-posed*.

THEOREM 4. *The problem (1), (3) is well-posed in the sense of Tikhonov if and only if the operator A is normal solvable.*

Proof. Let A be a normal solvable operator and let u_n be an arbitrary minimizing sequence of the problem (1), (3). Present the elements u_n in the form $u_n = Pu_n + (I-P)u_n$ and note that

(f2) $$\|Pu_n - Pu_*\| \to 0 \text{ as } n \to \infty,$$

where $u_* \in U_*$ is a solution (for instance, normal) of the problem (1), (3). Consider the sequence $v_n = Pu_* + (I-P)u_n$. Then

$$J(v_n) = J(Pu_*) = J(u_*) = J_*$$

and

$$\langle c_i, v_n\rangle = \langle c_i, Pu_*\rangle + \langle c_i, (I-P)u_n\rangle = \langle c_i, u_n\rangle + \langle c_i, Pu_* - Pu_n\rangle,\ i = 1, 2, \dots, m.$$

Let us introduce the notation $\alpha_{in} = \langle c_i, Pu_* - Pu_n\rangle$. The last relation implies that

(f3) $$\langle c_i, v_n\rangle \le \beta_i + \alpha_{in},$$

and, moreover, according to (f2)

(f4) $$\alpha_{in} \to 0 \text{ as } n \to \infty,\ i = 1, 2, \dots, m.$$

Present the set U_* in the form: $U_* = (Pu_* + \operatorname{Ker} A) \cap U$ and notice that $v \in U_*$ if and only if $v = Pu_* + (I-P)v$ and

(f5) $$\beta_i \ge \langle c_i, v\rangle = \langle c_i, Pu_*\rangle + \langle c_i, (I-P)v\rangle,\ \ i = 1, \dots, m.$$

Take the finite-dimensional subspace

$$L = \mathcal{L}\{(I-P)c_1, (I-P)c_2, \dots, (I-P)c_m\}$$

and denote by Q the orthogonal projector of H onto L. Then we have

(f6) $$\langle (I-P)c_i, (I-P)v\rangle = \langle (I-P)c_i, Qv\rangle,\ \ i = 1, \dots, m.$$

According to (f5) and (f6), we get that for each $v \in U_*$

(f7) $$\langle (I-P)c_i, Qv\rangle \le \beta_i - \gamma_i,\ \ i = 1, 2, \dots, m,$$

where $\gamma_i = \langle c_i, Pu_*\rangle$. Using (f3), (f7), we obtain

(f8) $$\langle (I-P)c_i, Qv_n\rangle \le \beta_i - \gamma_i + \alpha_{in},\ \ i = 1, 2, \dots, m,\ \ n = 1, 2, \dots$$

In the subspace L define the set W by

$$W = \{w \in L : \langle (I-P)c_i, w\rangle \le \beta_i - \gamma_i,\ i = 1,2,\dots,m\}. \tag{f9}$$

According to (f7), $Qv \in W$ for all $v \in U_*$. By virtue of (f4), (f8), and Hoffman's lemma [**7**] we derive

$$d(Qv_n, W) = \inf\{\|Qv_n - w\| : w \in W\} \to 0,\ \ n \to \infty. \tag{f10}$$

Note that in (f10) the infimum is achievable for each $n = 1,2,\dots$ and take the elements $w_n \in W$ so that $d(Qv_n, W) = \|Qv_n - w_n\|$. Furthermore, consider the sequence $y_n = Pu_* + (I-Q)(I-P)u_n + w_n$, $n = 1,2,\dots$ Then, for all $n = 1,2,\dots$, $J(y_n) = J(Pu_*) = J(u_*) = J_*$, and using (f9) we get

$$\begin{aligned}\langle c_i, y_n\rangle &= \langle c_i, Pu_*\rangle + \langle (c_i, (I-Q)(I-P)u_n\rangle + \langle c_i, w_n\rangle \\ &= \gamma_i + \langle c_i, (I-P)u_n\rangle - \langle Qc_i, (I-P)u_n\rangle + \langle c_i, w_n\rangle \\ &= \gamma_i + \langle (I-P)c_i, (I-P)u_n\rangle - \langle (I-P)c_i, (I-P)u_n\rangle + \langle (I-P)c_i, w_n\rangle \\ &\le \gamma_i + \beta_i - \gamma_i = \beta_i.\end{aligned}$$

This means that (y_n) is a minimizing sequence for the problem (1), (3) (moreover, $y_n \in U_*$). Let us now note that

$$\begin{aligned}\|v_n - y_n\| &= \|Pu_* + (I-P)u_n - Pu_* - (I-Q)(I-P)u_n - w_n\| \\ &= \|Q(I-P)u_n - w_n\| = \|Qv_n - w_n\|.\end{aligned}$$

Finally, by (f10), we obtain

$$\begin{aligned}d(u_n, U_*) &\le \|u_n - y_n\| \le \|u_n - v_n\| + \|v_n - y_n\| \\ &= \|Pu_n - Pu_*\| + \|Qv_n - w_n\| \to 0,\ \ n \to \infty,\end{aligned}$$

hence, the well-posedness of the problem (1), (3) is proved.

Suppose conversely, that the problem (1), (3) is well-posed in the sense of Tikhonov. It is necessary to prove that the operator A is normal solvable. Let us suppose conversely that $R(A^*) \ne \overline{R(A^*)}$. Then, according to Lemma 2, there exists a sequence p_n such that

$$p_n \in \overline{R(A_*)},\ \|p_n\| = 1,\ p_n \rightharpoonup 0,\ Ap_n \to 0,\ \ n \to \infty. \tag{f11}$$

Let $c_1, \dots, c_k$ be some base of the system $c_1, \dots, c_m$. Define the sequences $(\lambda_{n_1}), \dots, (\lambda_{n_k})$ so that for the elements

$$v_n = u_* + p_n + \sum_{i=1}^{k} \lambda_{n_i} c_i$$

we have

$$\langle v_n, c_i\rangle = \langle u_*, c_i\rangle, \ \ i = 1, \dots, m.$$

These relations form a system of linear equations

$$\lambda_{n_1}\langle c_1, c_i\rangle + \lambda_{n_2}\langle c_2, c_i\rangle + \cdots + \lambda_{n_k}\langle c_k, c_i\rangle = -\langle p_n, c_i\rangle, \ \ i = 1, \dots, m.$$

This system is equivalent to the shortened system

$$\lambda_{n_1}\langle c_1, c_i\rangle + \lambda_{n_2}\langle c_2, c_i\rangle + \cdots + \lambda_{n_k}\langle c_k, c_i\rangle = -\langle p_n, c_i\rangle, \ \ i = 1, \dots, k. \tag{f12}$$

The system (f12) has a unique solution $\lambda_{n_1}, \dots, \lambda_{n_k}$; moreover, by virtue of (f11), we have

$$\lim_{n\to\infty} \lambda_{n_i} = 0, \ \ i = 1, \dots, k.$$

Thus we see that (v_n) is a minimizing sequence; however, by (f11), we derive that

$$d^2(v_n, U_*) \geq \|p_n\|^2 - \sum_{i=1}^{k} \lambda_{n_i}^2 \, \|c_i\|^2 \to 1 \text{ as } n \to \infty.$$

Therefore, we have constructed a minimizing sequence (v_n) that does not converge to the solution set U_*, but this is impossible under the above assumption of the well-posedness of the problem (1), (3). This completes the proof. □

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