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STABILITY AND ASYMPTOTIC BEHAVIOR FOR CERTAIN SYSTEMS OF DELAY DIFFERENCE EQUATIONS

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Abstract. We give a condition for systems of difference equations to have solutions tending to a constant vector.

Introduction

We consider systems of difference equations for which each constant function is a solution and give a simple condition which guarantees that each solution of these equations tends to a constant limit as $n \to \infty$ and that the zero solution of these equations is uniformly stable. Most of these results are quite easy to state. However, for good introductory level material on difference equations, we refer the reader to [**1, 2, 3**].

Notations and preliminaries

Denote by $\mathcal{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, where n_0 is a natural number or zero; $\mathbb{R}^k$ – the k-dimensional real euclidean space with the norm, $|x| = \sum_{i=1}^{k} |x_i|$, $x = (x_1, ...x_k)$, M^k the space of all $k \times k$ matrices $A = (a_{ij})$ with the norm $|A| = \max_j \sum_{i=1}^{k} |a_{ij}|$. If $u : \mathcal{N}(n_0) \to \mathbb{R}$ ($\mathbb{R}$–the real numbers) is a discrete function, $\{u(n)\}$ is the sequence denoted by $u(n)$. Similarly $u : \mathcal{N}(n_0) \to \mathbb{R}^k$, $u(n) = \left[u^1(n), \dots, u^k(n)\right]$ with components that are functions defined on the same set $\mathcal{N}(n_0)$, is the sequence denoted by $u(n)$. We will consider the linear system

$$\Delta x(n) = P(n)[x(n) - x(n - r)] \tag{1}$$

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and its perturbed system

$$\Delta y(n) = P(n)[y(n) - y(n-r)] + Q(n)y(n) + R(n)y(n-r) \tag{2}$$

or

$$\Delta y(n) = F(n, y(n)) + G(n, y(n-r)), \tag{3}$$

where x, y are k-dimensional vector functions, $P, Q, R : \mathcal{N}(n_0) \to M^k$, $F, G : \mathcal{N}(n_0) \times \mathbb{R}^k \to \mathbb{R}^k$ are for any $n \in \mathcal{N}(n_0)$ continuous as a functions of $y \in \mathbb{R}^k$, Δ is the forward difference operator i.e., $\Delta v_n = v(n+1) - v(n)$ for any function $v : \mathcal{N}(n_0) \to \mathbb{R}^k$ We assume throughout that r is a natural number. Systems (1) and (2) are special cases of (3).

By a solution of a difference equation we mean a real sequence $\{y(n)\}$, $n = 0, 1, \dots$ satisfying it, and throughout this paper we will usually refer to a solution $\{y(n)\}$, $n = 0, 1, \dots$, simply as a solution $y(n)$.

Here we need to introduce the concept of an initial function. An initial function ϕ of (3) is a function from $\langle n_0 - r, n_0\rangle$ to $\mathbb{R}^k$. A solution $y(n) = y(n, n_0, \phi)$ of (3) is a sequence satisfying (3) for $n \in \mathcal{N}(n_0)$ and $y(s) = \phi(s)$ for $s \in \langle n_0 - r, n_0\rangle$.

Remark. For any n_0 and any ϕ on $\langle n_0 - r, n_0\rangle$, the solutions $x(n, n_0, \phi)$ and $y(n, n_0, \phi)$ of (1) and (2), respectively, are defined as solutions on the entire interval $\mathcal{N}(n_0)$. All initial functions are assumed to be bounded. When we refer to a solution $y(n, n_0, \phi)$ of (3) (in particular of (1) or (2)), we shall mean a solution with initial function ϕ on $\langle n_0 - r, n_0\rangle$ with the norm $\|\phi\| = \max\limits_{\langle n_0 - r, n_0\rangle} |\phi(n)|$, where $|\cdot|$ is a suitable norm in $\mathbb{R}^k$

Definition. Suppose that $F(n, 0) \equiv 0 \equiv G(n, 0)$ for $n \in \mathcal{N}(n_0)$. The zero solution $y(n) \equiv 0$, of (3) is eventually uniformly stable if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ and $T = T(\varepsilon) > 0$ such that, $|y(n, n_0, \phi)| < \varepsilon$, $n \geq n_0 \geq T$, provided $\|\phi\| < \delta$ on $\langle n_0 - r, n_0\rangle$.

Definition. A solution $y(n, n_0, \phi)$ of (3) is said to be bounded in the future if it is defined as a solution on (n_0, ∞) and if there exists a constant $M > 0$ such that $|y(n, n_0, \phi)| \leq M$ for all $n \in \langle n_0 - r, \infty\rangle$.

We need the following lemmas regarding l^p-sequences for our discussion. For p in the interval $1 \leq p < \infty$, l^p is the space of sequences $u = \{y_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} |y_i|^p < \infty$ with the norm $|u|_p = \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p} < \infty$, where $|\cdot|$ is an $\mathbb{R}^k$–norm.

Lemma 2.1. *If $q \in l^p$ for some $p \geq 1$, then*

(i) *$v, w \in l^p$, where $v = \{v(n)\}$, $v(n) = \sum_{s=n-r}^{n} q(s)$, $w = \{w(n)\}$, $w(n) = \sum_{s=n-r}^{n} |q(s)|$,*

(ii) *$\sum_{s=n-r}^{n} |q(s)| \to 0$ as $n \to \infty$.*

Lemma 2.2. *If $q_1 \in l^p$, $q_2 \in l^p$ for some $p \geq 1$ and if the product $q = q_1 q_2$ is defined, then $q \in l^{p/2}$.*

LEMMA 2.3. *If* $\Delta x \in l^1$, *then* $x(n) \to \text{const}$ as $n \to \infty$.

We omit the proofs of these lemmas because of their simplicity.

Main results

Before presenting the main results of the paper we will establish three preparatory lemmas.

LEMMA 3.1. *If* $P \in l^2$ *and if any solution* $x(n) = x(n, n_0, \phi)$ *of* (1) *is bounded in the future, then* $x(n) \to \text{const}$ *as* $n \to \infty$.

Proof. We have

$$x(n) - x(n-r) = \sum_{s=n-r}^{n-1} \Delta x(s) = \sum_{s=n-r}^{n-1} P(s)[x(s) - x(s-r)].$$

Thus,

$$\Delta x(n) = P(n)[x(n) - x(n-r)] = P(n) \sum_{s=n-r}^{n-1} P(s)[x(s) - x(s-r)].$$

Since $P \in l^2$ and $x(n)$ is bounded in the future, it follows from Lemma 2.1 that $z \in l^2$, where $z = \left\{\sum_{s=n-r}^{n-1} P(s)[x(s) - x(s-r)]\right\}$. Hence, by Lemma 2.2 the sequence Δx is in l^1 and the conclusion follows from Lemma 2.3.

LEMMA 3.2. *If* $P \in l^2$, *then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for any* $n_0 \geq 0$, $\|\phi\| < \delta$ *on* $\langle n_0 - r, n_0\rangle$ *implies*

$$|x(n, n_0, \phi) - x(n-r, n_0, \phi)| < \varepsilon \ \text{ for } \ n_0 \leq n \leq n_0 + r$$

Proof. Let $x(n) = x(n, n_0, \phi)$. Then $x(n-r) = \phi(n-r)$ for $\langle n_0, n_0 + r\rangle$. From Lemma 2.1 $\sum_{s=n}^{n+r} |P(s)| \to 0$ as $n \to \infty$. Then there exists a constant $M_1 > 0$ such that $\sum_{s=n}^{n+r} |P(s)| \leq M_1$ for $n \in \mathcal{N}(n_0)$. Let $\varepsilon > 0$ be given and choose $\delta < \min\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{(1+M_1)e^{M_1}}\right)$. Then for $\|\phi\| < \delta$ on $\langle n_0 - r, n_0\rangle$ and $n \in \langle n_0, n_0 + r\rangle$ we have

$$\begin{aligned}
|x(n)| &\leq |x(n_0)| + \sum_{s=n_0}^{n-1} |P(s)| \left[|x(s) - x(s-r)|\right] \\
&\leq |\phi(n_0)| + \sum_{s=n_0}^{n-1} |P(s)||\phi(s-r)| + \sum_{s=n_0}^{n-1} |P(s)||x(s)| \\
&\leq |\phi(n_0)| + \sum_{s=n_0}^{n_0+r} |P(s)|\,|\phi(s-r)| + \sum_{s=n_0}^{n-1} |P(s)|\,|x(s)| \\
&\leq \delta(1+M) + \sum_{s=n_0}^{n-1} |P(s)||x(s)|.
\end{aligned}$$

By a discrete version of Gronwall inequality we obtain

$$|x(n)| \le \delta(1+M)e^M < \varepsilon/2$$

So $|x(n)-x(n-r)| < \varepsilon$ for $n \in \langle n_0, n_0+r\rangle$, since $\|\phi\| < \delta < \varepsilon/2$.

LEMMA 3.3. *If $P \in l^2$, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any $n_0 \ge T_1 \ge 0$, $\|\phi\| < \delta$ on $\langle n_0 - r, n_0\rangle$ implies $|x(n,n_0,\phi) - x(n-r,n_0,\phi)| < \varepsilon$ for all $n > n_0$ (T_1 is chosen such that $\sum_{s=n-r}^{n} |P(s)| < 1$ for all $n \ge T_1$).*

Proof. Let $\varepsilon > 0$ be given and δ chosen as in Lemma 3.2. Then

$$|x(n)-x(n-r)| < \varepsilon$$

for $n_0 \le n \le n_0 + r$, and any $n_0 \ge 0$, where $x(n)$ is a solution of (1). Let $n_0 \ge T_1$ and let there exist an $n > n_0 + r$ such that

$$|x(n)-x(n-r)| =: |u(n)| = \varepsilon.$$

Let $n^* > n_0 + r$ be the first such moment. Then $\varepsilon = |u(n^*)| > |u(n)|$ for $T_1 \le n_0 \le n < n^*$ and

$$|u(n^*)| = |x(n^*) - x(n^*-r)| = \Big|\sum_{s=n^*-r}^{n^*-1} \Delta x(s)\Big|$$

$$= \Big|\sum_{s=n^*-r}^{n^*-1} P(s)\,[x(s)-x(s-r)]\Big| = \Big|\sum_{s=n^*-r}^{n^*-1} P(s)u(s)\Big| = \sum_{s=n^*-r}^{n^*-1} |P(s)|\,|u(s)|.$$

Since $|u(n^*)| \ge |u(s)|$ for $n^*-r \le s \le n^*-1$, it follows that

$$|u(n^*)| \le |u(n^*)| \sum_{s=n^*-r}^{n^*-1} |P(s)| \tag{4}$$

From (4) we have $\sum_{s=n^*-r}^{n^*-1} |P(s)| \ge 1$, which contradicts the choice of T_1, since $n^* > n_0 \ge T_1$. So $|x(n)-x(n-r)| < \varepsilon$ for all $n \ge n_0$.

THEOREM 3.1. *If $P \in l^2$, then the zero solution of (1) is uniformly stable.*

Proof. We show that the zero solution is eventually uniformly stable and then we apply a continual dependence argument. Let $\eta > 0$ be given. We wish to find a $\delta_0 > 0$ and a $T \ge 0$ such that $n_1 \ge T$ and $\|\phi\| < \delta$ imply $|x(n,n_1,\phi)| < \eta$ for all $n \ge n_1$and any n$_1 \ge T$. Let $\varepsilon = 1$ and choose $\delta = \delta(\varepsilon) < 1$ according to Lemma 3.3. Let $\delta_0 = \min(\delta, \eta/3)$. Now, by Lemma 2.1 we have $u \in l^2 \Rightarrow w \in l^2$, where $u = \{|P(n)|\}$, $w = \big\{\sum_{s=n-r}^{n} |P(n)|\big\}$, hence (by Lemma 2.2) $v \in l^1$, $v = \big\{|P(n)| \sum_{s=n-r}^{n} |P(s)|\big\}$. Let $T \ge T_1 + r$ be chosen so that

$$\sum_{m-n_1}^{n-1} \Big(|P(m)| \sum_{s=m-r}^{m-1} |P(s)|\Big) < \frac{\eta}{3} \quad \text{and} \quad \sum_{s=n_1}^{n_1+r} |P(s)| < \frac{\eta}{3}$$

for all $n \geq n_1 \geq T \geq T_1 + r$ (T_1 was defined in Lemma 3.3). Thus, for $n_1 \geq T$ and $\|\phi\| < \delta_0$, we have, for $n_1 \leq n \leq n_1 + r$

$$|x(n, n_1, \phi)| \leq |x(n_1)| + \sum_{s=n_1}^{n-1} |P(s)| < \frac{2}{3}\eta,$$

where we used the fact that $|x(n) - x(n-r)| < 1$. If $n_1 + r \leq n$, then

$$\begin{aligned} |x(n)| &\leq |\phi(n_1)| + \sum_{s=n_1}^{n_1+r} |P(s)| + \sum_{s=n_1+r}^{n-1} |P(s)|\, |x(s) - x(s-r)| \\ &\leq |\phi(n_1)| + \sum_{s=n_1}^{n_1+r} |P(s)| + \sum_{s=n_1+r}^{n-1} |P(s)| \sum_{u=s-r}^{s-1} |P(u)|\, |x(u) - x(u-r)| \\ &\leq \frac{\eta}{3} + \frac{\eta}{3} + \sum_{s=n_1+r}^{n-1} \Big(|P(s)| \sum_{u=s-r}^{s-1} |P(u)| \Big) < \eta. \end{aligned}$$

In this case $|x(n, n_1, \phi)| < \eta$ for $n \geq n_1 \geq T$ and $\|\phi\| < \delta_0$. The zero solution of (1) is eventually uniformly stable. Since solutions depend continuously on initial conditions, we can find $\delta_1 \leq \delta_0$ such that $|x(n, n^*, \phi)| < \delta_0$, for $\|\phi\| < \delta_1$ and for any $n^* \leq n \leq T$. Hence $|x(n, n_1, \phi)| < \eta$ for all $n \geq n_1$, $\|\phi\| < \delta_1$, and any $n_1 \geq n_0$. We have uniform stability of the zero solution.

Now we consider the perturbed system

$$\Delta y(n) = P(n)\,[y(n) - y(n-r)] + f(n, y(n), y(n-r)).$$

Lemma 3.4. *Assume:* 1° $P \in l^2$*;* 2° *there exists a function* $\omega(n,u,v)$ *defined on* $\mathcal{N}(n_0) \times \mathbb{R}_+ \times \mathbb{R}_+$*, nondecreasing in* u, v *and such that* $\omega(n,a,b) \in l^2$ *for each* $a, b \in \mathbb{R}_+$ *and furthermore*

$$|f(n,u,v)| \leq \omega(n, |u|, |v|), \quad f(n,0,0) = 0, \quad n \in \mathcal{N}(n_0), \quad |u|, |v| < \infty,$$

3° *any solution* $y(n)$ *of (5) is bounded in the future. Then* $y(n) \to \text{const}$*, as* $n \to \infty$*.*

The proof is similar to the proof of Lemma 3.1

Remark. If $f(n, y(n), y(n-r)) \equiv Q(n)y(n) + R(n)y(n-r)$ and $Q \in l^2$, $R \in l^2$, then $y(n) \to \text{const}$, as $n \to \infty$, where $y(n)$ is a solution of (2).

Lemma 3.5. *Let the solution* $x(n)$ *of* (1) *be bounded in the future, and* $Q, R \in l^1$*. Then, the solution* $y(n)$ *of* (2) *is bounded in the future*

Proof. If $y(n)$ is a solution of (2), then we have

$$y(n) = x(n) + \sum_{s=0}^{n-1} \{Q(s)y(s) + R(s)y(s-r)\} \tag{6}$$

where $x(n)$ is a solution of (1). Denote by

$$v(n) = \max\{|y(n)| : |y(n-r)|\}. \tag{7}$$

Then from (6) and the condition 1° we obtain

$$v(n) \leq M + \sum_{s=0}^{n-1} \{|Q(s)| + |R(s)|\} v(s).$$

Now applying the discrete version of Gronwall inequality we have

$$v(n) \leq M \exp\Big(\sum_{s=0}^{n-1} \{|Q(s)| + |R(s)|\}\Big).$$

By (7) this gives the required result.

THEOREM 3.2. *If $Q, R \in l^1$, and the zero solution of* (1) *is uniformly stable, then the zero solution of* (2) *is uniformly stable.*

Proof. As the zero solution of (1) is uniformly stable (Theorem 3.1) and $Q, R \in l^1$ it follows from Lemma 3.5 that the zero solution of (2) is also uniformly stable.

THEOREM 3.3. *If the conditions* 1°–3° *of Lemma* 3.4 *hold, and if there exists a solution $z(n)$ of the difference equation $\Delta z(n) = \omega(n, z(n), z(n))$ with an initial value $z(0) > M_0 > 0$, $M_0 = \text{const}$, then any $y(n)$ solution of* (5), *with an initial condition $\|\phi\| < M_0$, satisfies $|y(n)| < z(n)$ for all $n \in \mathcal{N}(n_0)$.*

Proof. Any solution $y(n)$ of (5) is of the form

$$y(n) = x(n) + \sum_{s=0}^{n-1} f(s, y(s), y(s-r)).$$

Thus, from the condition 1° we obtain that

$$|y(n)| \leq M + \sum_{s=0}^{n-1} \omega(s, |y(s)|, |y(s-r)|).$$

Let $v(n) = \max\{|y(n)|, |y(n-r)|\}$; then

$$v(n) \leq M_0 + \sum_{s=0}^{n-1} \omega(s, v(s), v(s).$$

Hence by the assumption $z(0) > M_0$, where $z(n)$ is the solution of (8), we obtain [**4**]

$$|y(n)| < z(n) \text{ for all } n \in \mathcal{N}(n_0).$$

COROLLARY. *Let in Theorem* 3.3 $\omega(n, z(n), z(n)) = a(n)z(n)$, *and* $\sum_{n=0}^{\infty} a_n < \infty$. *Then the zero solution of* (5) *will be stable (uniformly stable), where the zero solution of* (1) *will be stable (uniformly stable).*

Proof. Consider the difference equation

$$\Delta z(n) = a(n)z(n), \quad z(0) = z_0.$$

Any solution of this equation is of the form

$$z(n) = z_0 \prod_{s=0}^{n-1} (1 + a(s)).$$

Taking a suitable $\delta > 0$ and for any $\varepsilon > 0$ we obtain that the zero solution of (5) will be stable (uniformly stable).

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