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# BANDS OF NIL-EXTENSIONS OF RIGHT SIMPLE SEMIGROUPS

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**Abstract**. We prove a general theorem when the semigroup is a band of nil-extensions of right simple semigroups. We also describe normal bands of right simple semigroups, normal bands of right Archimedean semigroups and normal bands of nil-extensions of right simple semigroups.

Nil-extensions of some classes of semigroups were considered by many authors, for example by L.N. Ševrin, J.L. Galbiati, M.L. Veronesi, S. Bogdanović and M. Ćirić. All of these results are presented in a survey paper [**2**].

Let $N$ be the set of all positive integers. A semigroup $S$ is *right Archimedean* (or *r-Archimedean*) if, for every $a, b \in S$, there exists $n \in N$ such that $a^n \in bS$. A semigroup $S$ is *right simple* if $a \in bS$ for every $a, b \in S$. A semigroup $S$ is *right $\pi$-regular semigroup* if for all $a \in S$ there exists $n \in N$ such that $a^n \in a^{n+1}S$. A semigroup $S$ with the zero 0 is *nil* if for every $a \in S$ there exists $n \in N$ such that $a^n = 0$. An ideal extension $S$ of $T$ is a *nil-extension* if $S/T$ is a nil semigroup. A semigroup $B$ is a *band* if $x^2 = x$ for each $x \in B$.

A semigroup $S$ is a *band $Y$ of semigroups $S_\alpha$* if $S = \bigcup_{\alpha \in Y} S_\alpha$, $Y$ is a band, $S_\alpha \cap S_\beta = \emptyset$ for $\alpha, \beta \in Y$ with $\alpha \neq \beta$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. A congruence $\rho$ on $S$ is called *band congruence* if $S/\rho$ is a band.

For undefined notions and notation we refer to [**1**], [**3**] and [**4**].

THEOREM 1. [**6**] *A semigroup $S$ is a band of $r$-Archimedean semigroups if and only if*

$$(1)\qquad (\forall a \in S)(\forall x, y \in S^1)(\exists i, j \in N)(xay)^i \in xa^2yS,\ (xa^2y)^j \in xayS. \quad \square$$

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In [**6**] is proved that if (1) holds, then the relation $\rho$ defined on $S$ by

$$a\rho b \Longleftrightarrow (\forall x,y \in S^1)(\exists i,j \in N)(xay)^i \in xbyS,\ (xby)^j \in xayS \tag{2}$$

is a band congruence on $S$.

THEOREM 2. [**1**] *A semigroup $S$ is a nil-extension of a right simple semigroup if and only if $S$ is $r$-Archimedean and right $\pi$-regular.* $\square$

THEOREM 3. *The following conditions on a semigroup $S$ are equivalent:*

(i) *$S$ is a band of nil-extensions of right simple semigroups,*

(ii) $(\forall a \in S)(\forall x,y \in S^1)(\exists m,n \in N)(\forall p,q \in N)$

$$(xay)^m \in (xa^2y)^pS,\ (xa^2y)^n \in (xay)^qS,$$

(iii) $(\forall a \in S)(\forall x,y \in S^1)(\exists m,n \in N)$

$$(xay)^m \in (xa^2y)^mS,\ (xa^2y)^n \in (xay)^nS.$$

*Proof.* (i)$\Longrightarrow$(ii). Let $S$ be a band $Y$ of nil-extensions $S_\alpha$ of right simple semigroups $K_\alpha$. Then $S^1$ is a band $Y^1$ of nil-extensions $S_\alpha$ of right simple semigroups $K_\alpha$, $\alpha \in Y^1$ and $S_1 = 1 = K_1$. If $x \in S_\alpha$, $a \in S_\beta$, $y \in S_\gamma$ then $xay, xa^2y \in S_{\alpha\beta\gamma}$. Now there exist $m,n \in N$ such that $(xay)^m, (xa^2y)^n \in K_{\alpha\beta\gamma}$. Also, $(xa^2y)^{np} \in K_{\alpha\beta\gamma}$ for every $p \in N$. Since $K_{\alpha\beta\gamma}$ is a right simple semigroup it follows that

$$(xay)^m \in (xa^2y)^{np}K_{\alpha\beta\gamma} \subseteq (xa^2y)^pS.$$

Similarly, $(xa^2y)^n \in (xay)^qS$ for every $q \in N$, and so (ii) holds.

(ii)$\Longrightarrow$(iii). This follows trivially.

(iii)$\Longrightarrow$(i). From (iii) we have

$$(\forall a \in S)(\forall x,y \in S^1)(\exists m,n \in N)\ (xay)^m \in xa^2yS,\ (xa^2y)^n \in xayS,$$

and by Theorem 1 we conclude that semigroup $S$ is a band of $r$-Archimedean semigroups $S_\alpha$, $\alpha \in Y$. Also, from (iii) for $x = a = y \in S_\alpha$, it follows that $a^{3m} \in a^{4m}S$ and so there exists $u \in S_\beta$ such that $a^{3m} = a^{4m}u$. From above we have $\alpha = \alpha\beta$. Now $a^{2m-1}u^2 \in S_\alpha$ and

$$a^{3m} = a^{4m}u = a^m a^{3m}u = a^m a^{4m}uu = a^{3m+1}a^{2m-1}u^2 \in a^{3m+1}S_\alpha.$$

Hence, $S_\alpha$ is a right $\pi$-regular $r$-Archimedean semigroup and by Theorem 2 it follows that $S_\alpha$ is a nil-extension of a right simple semigroup, whence semigroup $S$ is a band of nil-extensions of right simple semigroups. $\square$

Recall that a band $B$ is a *normal band* if $efge = egfe$ for every $e, f, g \in B$.

THEOREM 4. *A semigroup $S$ is a normal band of $r$-Archimedean semigroups if and only if*

$$(\forall u, v, w, t \in S)(\exists n \in N)\ (uvwt)^n \in uwvtS. \tag{3}$$

*Proof.* Let $S$ be a normal band $Y$ of $r$-Archimedean semigroups $S_\alpha$. If $u \in S_\alpha,\ v \in S_\beta,\ w \in S_\gamma,\ t \in S_\delta$ then $uvwt \in S_{\alpha\beta\gamma\delta}$. Since $Y$ is a normal band we have $uwvtvuwt \in S_{\alpha\beta\gamma\delta}$ and, since $S_{\alpha\beta\gamma\delta}$ is a right Archimedean semigroup, there exists $n \in N$ such that

$$(uvwt)^n \in uwvtvuwtS_{\alpha\beta\gamma\delta} \subseteq uwvtS$$

and so (3) holds.

Conversely, let statement (3) hold in $S$ and let $a \in S,\ x, y \in S^1$. Then, by (3), for $u = xa,\ v = ayxa^2,\ w = yxa,\ t = ay$, there exists $n \in N$ such that

$$(xa^2y)^{3n} = (xa \cdot ayxa^2 \cdot yxa \cdot ay)^n \in xa \cdot yxa \cdot ayxa^2 \cdot ayS \subseteq xayS.$$

Also, for $u = xa,\ v = yxayx,\ w = ay,\ t = xay$, there exists $m \in N$ such that

$$(xay)^{4m} = (xa \cdot yxayx \cdot ay \cdot xay)^m \in xa \cdot ay \cdot yxayx \cdot xayS \in xa^2yS.$$

By Theorem 1 we have that $S$ is a band $Y$ of $r$-Archimedean semigroups $S_\alpha$.

We shall prove that the congruence $\rho$ defined by (2) is a normal band congruence on $S$. Let $a, b, c \in S$ and $x, y \in S^1$. For $u = xa,\ v = b,\ w = c,\ t = ay$ by (3) there exists $n \in N$ such that $(xabcay)^n \in xacbayS$. Similarly, $(xacbay)^m \in xabcayS$ for some $m \in N$. Hence $abca\rho acba$ and $\rho$ is a normal band congruence on $S$. It follows that $S$ is a normal band of $r$-Archimedean semigroups. $\square$

THEOREM 5. *The following conditions on a semigroup $S$ are equivalent:*

(i) *$S$ is a normal band of nil-extensions of right simple semigroups,*

(ii) $(\forall u, v, w, t \in S)(\exists n \in N)(\forall k \in N)\ (uvwt)^n \in (uwvtvuwt)^k S,$

(iii) $(\forall u, v, w, t \in S)(\exists n \in N)\ (uvwt)^n \in (uwvtvuwt)^n S.$

*Proof.* (i)$\Longrightarrow$(ii). Let $S$ be a normal band $Y$ of nil-extensions $S_\alpha$ of right simple semigroups $K_\alpha$. If $u \in S_\alpha,\ v \in S_\beta,\ w \in S_\gamma,\ t \in S_\delta$, then $uvwt \in S_{\alpha\beta\gamma\delta}$. Since $\alpha\beta\gamma\delta = \alpha\gamma\beta\delta\beta\alpha\gamma\delta$ we have that $uwvtvuwt \in S_{\alpha\beta\gamma\delta}$. Now, there exist $n, m \in N$ such that $(uvwt)^n, (uwvtvuwt)^m \in K_{\alpha\beta\gamma\delta}$ and, also $(uwvtvuwt)^{mk} \in K_{\alpha\beta\gamma\delta}$ for every $k \in N$. Since $K_{\alpha\beta\gamma\delta}$ is a right simple semigroup, we conclude that

$$(uvwt)^n \in (uwvtvuwt)^{mk} K_{\alpha\beta\gamma\delta} \subseteq (uwvtvuwt)^k S.$$

(ii)$\Longrightarrow$(iii). This follows trivially.

(iii)$\Longrightarrow$(i). From (iii) we have

$$(\forall u, v, w, t \in S)(\exists n \in N)\ (uvwt)^n \in (uwvtvuwt)^n S \subseteq uwvtS$$

and by Theorem 4 $S$ is a normal band $Y$ of $r$-Archimedean semigroups $S_\alpha$. If $a \in S_\alpha$, then from (iii) for $a = u = v = w = t$ there exists $n \in N$ such that $u^{4n} \in u^{8n}S$ and $u^{4n} = u^{8n}s$ for some $s \in S$. If $s \in S_\beta$, then it follows that $\alpha = \alpha\beta$. Now $u^{4n-1}s \in S_\alpha$ and so

$$u^{4n} = u^{8n}s = u^{4n+1}u^{4n-1}s \in u^{4n+1}S_\alpha.$$

Hence $S_\alpha$ is right $\pi$-regular and by Theorem 2 we conclude that $S_\alpha$ is a nil-extension of right simple semigroup and $S$ is a normal band of nil-extensions of right simple semigroups. $\square$

Theorem 6. *A semigroup $S$ is a normal band of right simple semigroups if and only if*

$$(\forall u, v, w, t \in S)\ uvwt \in uwvtS,\ u \in u^2S. \tag{4}$$

*Proof.* Let $S$ be a normal band $Y$ of right simple semigroups $S_\alpha$. If $u \in S_\alpha$, $v \in S_\beta$, $w \in S_\gamma$, $t \in S_\delta$, then $uvwt$, $uwvtvuwt \in S_{\alpha\beta\gamma\delta}$. Since $S_{\alpha\beta\gamma\delta}$ is a right simple semigroup we conclude that

$$uvwt \in uwvtvuwtS \subseteq uwvtS.$$

Since $u, u^2 \in S_\alpha$, we have $u \in u^2S$.

Conversely, let (4) hold. We define on $S$ a relation $\sigma$ by

$$a\sigma b \iff a \in bS,\ b \in aS.$$

Clearly $\sigma \subseteq \mathcal{R}$ where $\mathcal{R}$ is the Green $\mathcal{R}$-relation. Conversely, let $a\mathcal{R}b$. If $a \neq b$, then $a \in bS$, $b \in aS$ and so $a\sigma b$. If $a = b$, then by (4) we have $a \in a^2S \subseteq aS$, and so $a\sigma a$. Hence $\mathcal{R} \subseteq \sigma$ and so $\sigma = \mathcal{R}$.

Let $a\mathcal{R}b$, $c \in S$; then $ac \in bSc$. Hence $ac = btc$ for some $t \in S$. Now by (4) we have $ac = btc \in btc^2S \subseteq bctcS \subseteq bcS$. Similarly, $bc \in acS$ and so $ac\mathcal{R}bc$. Since $\mathcal{R}$ is a left congruence on an arbitrary semigroup, we conclude that $\mathcal{R}$ is a congruence relation on $S$.

Let $a \in S$; then $a^2 \in aS$. By (4) we have $a \in a^2S$ and so $a\mathcal{R}a^2$. Hence $\mathcal{R}$ is a band congruence. If $a, b, c \in S$, then by (4) it follows that $abca\mathcal{R}acba$ and so $\mathcal{R}$ is a normal band congruence on S. Now, $S = \bigcup_{\alpha \in Y} S_\alpha$, $Y$ is a normal band and $S_\alpha$ are $\mathcal{R}$-classes.

Let $a\mathcal{R}b$; then $a\mathcal{R}b^2$, whence $a = b^2x$ for some $x \in S$. If $a, b \in S_\alpha$ and $x \in S_\beta$, then $\alpha = \alpha\beta$. By (4) from $a \in a^2S$ we have $a = a^2y$ for some $y \in S$. If $y \in S_\gamma$ then $\alpha = \alpha\gamma$. Now from above we have

$$a = a^2y = aay = b^2xay = bbxay \in bS_{\alpha\beta\alpha\gamma} = bS_\alpha.$$

Hence $S_\alpha$ is a right simple semigroup and so the semigroup $S$ is a normal band of right simple semigroups. $\square$

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