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ON MOORE-PENROSE INVERSE OF BLOCK MATRICES AND FULL-RANK FACTORIZATION

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Dedicated to Professor Petar Madić on the occasion of his 75th birthday

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Abstract. We develop a few methods for computing the Moore-Penrose inverse, based on full-rank factorizations which arise from different block decompositions of rectangular matrices. In this way, the paper is a continuation of the previous works given by Noble [**5**] and Tewarson [**10**]. We compare the obtained results with the known block representations of the Moore-Penrose inverse. Moreover, efficient block representations of the weighted Moore-Penrose inverse are introduced using the same principles.

1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices, and $\mathbb{C}_r^{m\times n}=\{X\in\mathbb{C}^{m\times n}:\operatorname{rank}(X)=r\}$. With $A^{|r}$ and $A_{|r}$ we denote the submatrix of A which contains the first r columns of A and the first r rows of A, respectively. Similarly, $A^{r|}$ and $A_{r|}$ denote the last r columns and the last r rows of A, respectively. Finally, $A_{r|}^{r|}$ denotes the submatrix of A generated by the first r columns and the last r rows of A. The identity matrix of the order k is denoted by I_k, and $\mathbb{O}$ denotes the zero matrix of a convenient size.

Penrose [**6**], [**7**] has shown the existence and uniqueness of a solution $X\in\mathbb{C}^{n\times m}$ of the following four equations:

$$(1)\ AXA=A,\quad (2)\ XAX=X,\quad (3)\ (AX)^*=AX,\quad (4)\ (XA)^*=XA,$$

for any $A\in\mathbb{C}^{m\times n}$. For a sequence $\mathcal{S}$ of elements from the set $\{1,2,3,4\}$, the set of matrices which satisfy the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$.

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We use the following useful expansion for the Moore-Penrose generalized inverse $A^\dagger$ of A, based on the full-rank factorization $A = PQ$ of A [**1**], [**2**]:

$$A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*.$$

The *weighted Moore-Penrose inverse* is investigated in [**3**] and [**8**]. The main results of these papers are:

PROPOSITION 1.1. [**8**] *Let given positive-definite matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$. For any matrix $A \in \mathbb{C}^{m\times n}$ there exists the unique solution $X = A^\dagger_{M\circ,\circ N} \in A\{1,2\}$ satisfying*

$$(5) \qquad (MAX)^* = MAX \qquad\qquad (6) \qquad (XAN)^* = XAN.$$

Similarly, we use the following notations:

$A^\dagger_{M\circ,N\circ}$ denotes the unique solution of the equations (1), (2), and

$$(7) \qquad (MAX)^* = MAX, \qquad\qquad (8) \qquad (NXA)^* = NXA;$$

$A^\dagger_{\circ M,N\circ}$ is the unique solution of the equations (1), (2) and

$$(9) \qquad (AXM)^* = AXM, \qquad\qquad (10) \qquad (NXA)^* = NXA;$$

$A^\dagger_{\circ M,\circ N}$ denotes the unique solution of the equations (1), (2) and

$$(11) \qquad (AXM)^* = AXM, \qquad\qquad (12) \qquad (XAN)^* = XAN.$$

PROPOSITION 1.2 *[**8**] The equation* (5) *is equivalent to* $(AXM^{-1})^* = AXM^{-1}$, *and* (6) *can be expressed in the form* $(N^{-1}XA)^* = N^{-1}XA$.

PROPOSITION 1.3 *[**8**] If $A = PQ$ is a full rank factorization of A, then:*

$$\begin{aligned} A^\dagger_{M\circ,\circ N} &= (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^* \\ &= NQ^*(P^*MANQ^*)^{-1}P^*M. \end{aligned}$$

Using these notations, the following fact can be easily verified.

PROPOSITION 1.4 (a) $\quad A^\dagger_{M\circ,\circ N} = A^\dagger_{\circ M^{-1},\circ N} = A^\dagger_{M\circ,N^{-1}\circ} = A^\dagger_{\circ M^{-1},N^{-1}\circ}$,

(b) $\quad A^\dagger_{M\circ,N\circ} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* = A_{M\circ,\circ N^{-1}}$,

(c) $\quad A^\dagger_{\circ M,N\circ} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^* = A_{M^{-1}\circ,\circ N^{-1}}$,

(d) $A^{\dagger}_{\circ M,\circ N} = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*.$

From the part (a) of Proposition 1.4 and Proposition 1.2, it is easy to conclude that each of the indices, used in notation of the weighted Moore-Penrose inverse, can be written in one of the following form:

$$\circ M, \circ N \qquad M\circ, \circ N \qquad \circ M, N\circ \qquad M\circ, N\circ$$

For the sake of clarity, we use the notation $\varphi(M,N)$ for an arbitrary of these indices. Following Proposition 1.3 and Proposition1.4, we conclude the following:

1. $\varphi(M,N) = \circ M, \circ N \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*;$
2. $\varphi(M,N) = M\circ, \circ N \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^*;$
3. $\varphi(M,N) = \circ M, N\circ \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*;$
4. $\varphi(M,N) = M\circ, N\circ \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*.$

The cases 1–4 can be written in the following way:

$$A^{\dagger}_{\varphi(M,N)} = (QN^{[-1]})^*(Q(QN^{[-1]})^*)^{-1}((M^{[-1]}P)^*P)^{-1}(M^{[-1]}P)^*$$

where $M^{[-1]}$ stands for one of the matrices M or M^{-1}, and $N^{[-1]}$ denotes N or N^{-1}, in view of one of the rules 1–4.

We restate the main block decompositions [**4**], [**11–13**]. For a given matrix $A \in \mathbb{C}_r^{m\times n}$ there exist the regular matrices R, G, the permutation matrices E, F and the unitary matrices U, V, such that:

(T_1) $\quad RAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1;$ $\qquad (T_2)$ $\quad RAG = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2;$

(T_3) $\quad RAF = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_3;$ $\qquad (T_4)$ $\quad EAG = \begin{bmatrix} I_r & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_4;$

(T_5) $\quad UAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1;$ $\qquad (T_6)$ $\quad RAV = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1;$

(T_7) $\quad UAV = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2;$ $\qquad (T_8)$ $\quad UAF = \begin{bmatrix} B & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_5;$

(T_9) $\quad EAV = \begin{bmatrix} B & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_6;$

(T_{10a}) $\quad EAF = \begin{bmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{bmatrix} = N_7;$

where the multipliers S, T satisfy $T=A_{11}^{-1}A_{12}, \quad S=A_{21}A_{11}^{-1}$;

$$(T_{10b}) \quad EAF = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix} = N_7.$$

(T_{11}) Transformation of similarity for square matrices:

$$RAR^{-1} = RAFF^*R^{-1} = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} F^*R^{-1} = \begin{bmatrix} T_1 & T_2 \\ \mathbb{O} & \mathbb{O} \end{bmatrix}.$$

For the sake of completeness and comparison with our results, we describe known block representations of the Moore-Penrose inverse.

In the begining, we restate the block representations of the Moore-Penrose inverse from [**11**], [**13**]. For $A \in \mathbb{C}_r^{m\times n}$, let

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad G = [\,G_1, \quad G_2\,], \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad V = [\,V_1, \quad V_2\,],$$

where R_1, U_1 are the first r rows of R and U, respectively, and G_1, V_1 denote the first r columns of G and V, respectively. Then the Moore-Penrose inverse can be represented in the following way, where the block representations (M_i) correspond to the block decompositions (T_i), $i \in \{1, \dots, 9\}$.

$$\begin{aligned} (M_1) \quad A^\dagger &= G\begin{bmatrix} I_r & -R_1R_2^\dagger \\ -G_2^\dagger G_1 & G_2^\dagger G_1 R_1 R_2^\dagger \end{bmatrix} R = G\begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} [\,I_r, \quad -R_1R_2^\dagger\,] R \\ &= (G_1 - G_2G_2^\dagger G_1)(R_1 - R_1R_2^\dagger R_2), \end{aligned}$$

$$\begin{aligned} (M_2) \quad A^\dagger &= G\begin{bmatrix} B^{-1} & -B^{-1}R_1R_2^\dagger \\ -G_2^\dagger G_1 B^{-1} & G_2^\dagger G_1 B^{-1} R_1 R_2^\dagger \end{bmatrix} R \\ &= G\begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} B^{-1} [\,I_r, \quad -R_1R_2^\dagger\,] R \\ &= (G_1 - G_2G_2^\dagger G_1)B^{-1}(R_1 - R_1R_2^\dagger R_2), \end{aligned}$$

$$\begin{aligned} (M_3) \quad A^\dagger &= F\begin{bmatrix} (I_r+KK^*)^{-1} & -(I_r+KK^*)^{-1}R_1R_2^\dagger \\ K^*(I_r+KK^*)^{-1} & -K^*(I_r+KK^*)^{-1}R_1R_2^\dagger \end{bmatrix} R \\ &= F\begin{bmatrix} I_r \\ K^* \end{bmatrix} (I_r+KK^*)^{-1} \begin{bmatrix} I_r, & -R_1R_2^\dagger \end{bmatrix} R, \end{aligned}$$

$$\begin{aligned} (M_4) \quad A^\dagger &= G\begin{bmatrix} (I_r+K^*K)^{-1} & (I_r+K^*K)^{-1}K^* \\ -G_2^\dagger G_1(I_r+K^*K)^{-1} & -G_2^\dagger G_1(I_r+K^*K)^{-1}K^* \end{bmatrix} R \\ &= G\begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} (I_r+K^*K)^{-1} [\,I_r, \quad K^*\,] E, \end{aligned}$$

$$(M_5)\quad A^\dagger = G\begin{bmatrix} I_r & \mathbb{O} \\ -G_2^\dagger G_1 & \mathbb{O}\end{bmatrix} U = G\begin{bmatrix} I_r \\ -G_2^\dagger G_1\end{bmatrix}[\,I_r, \quad \mathbb{O}\,]\,U$$
$$= (G_1 - G_2G_2^\dagger G_1)U_1,$$

$$(M_6)\quad A^\dagger = V\begin{bmatrix} I_r & -R_1R_2^\dagger \\ \mathbb{O} & \mathbb{O}\end{bmatrix} R = V\begin{bmatrix} I_r \\ \mathbb{O}\end{bmatrix}[\,I_r, \quad -R_1R_2^\dagger\,]\,R$$
$$= V_1(R_1 - R_1R_2^\dagger R_2),$$

$$(M_7)\quad A^\dagger = V\begin{bmatrix} B^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}\end{bmatrix} U = V\begin{bmatrix} I_r \\ \mathbb{O}\end{bmatrix}[\,B^{-1}, \quad \mathbb{O}\,]\,U = V_1B^{-1}U_1,$$

$$(M_8)\quad A^\dagger = F\begin{bmatrix} B^*(BB^*+KK^*)^{-1} & \mathbb{O} \\ K^*(BB^*+KK^*)^{-1} & \mathbb{O}\end{bmatrix} U$$
$$= F\begin{bmatrix} B^* \\ K^*\end{bmatrix}(BB^*+KK^*)^{-1}[\,I_r, \quad \mathbb{O}\,]\,U,$$

$$(M_9)\quad A^\dagger = V\begin{bmatrix} (B^*B+K^*K)^{-1}B^* & (B^*B+K^*K)^{-1}K^* \\ \mathbb{O} & \mathbb{O}\end{bmatrix} E$$
$$= V\begin{bmatrix} I_r \\ \mathbb{O}\end{bmatrix}(B^*B+K^*K)^{-1}[\,B^*, \quad K^*\,]\,E.$$

These results are obtained by solving the equations (1)–(4).

Block decomposition (T_{10a}) is investigated in [**5**], [**13**], but in two different ways. In [**13**], the Moore-Penrose inverse is represented by solving the corresponding set of matrix equations. The results in [**5**] are obtained using a full-rank factorization, implied by the block decomposition (T_{10a}). The corresponding representation of the Moore-Penrose inverse is:

$$(M_{10a})\quad A^\dagger = \begin{bmatrix} I_r \\ T^*\end{bmatrix}\left(A_{11}^*[\,I_r, \quad S^*\,]\,EAF\begin{bmatrix} I_r \\ T^*\end{bmatrix}\right)^{-1} A_{11}^*[\,I_r, \quad S^*\,]$$
$$= F\begin{bmatrix} I_r \\ T^*\end{bmatrix}(I_r+TT^*)^{-1}A_{11}^{-1}(I_r+S^*S)^{-1}[\,I_r, \quad S^*\,].$$

The following block representation of the Moore-Penrose inverse was obtained in [**10**] and [**13**] for the block decomposition (T_{10b}):

$$(M_{10b})\quad A^\dagger = \begin{bmatrix} I_r \\ (A_{11}^{-1}A_{12})^*\end{bmatrix}\left(I_r + A_{11}^{-1}A_{12}(A_{11}^{-1}A_{12})^*\right)^{-1}$$
$$\times A_{11}^{-1}\left(I_r + (A_{21}A_{11}^{-1})^*A_{21}A_{11}^{-1}\right)^{-1}[\,I_r, \quad (A_{21}A_{11}^{-1})^*\,].$$

Also, in [**14**] is given the following block representation of the Moore-Penrose inverse, based on the formula $A^\dagger = A^*TA^*, \quad T \in A^*AA^*\{1\}$: If $A \in \mathbb{C}_r^{m\times n}$ has the form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}$, $A_{11} \in \mathbb{C}_r^{r\times r}$, then

$$(M'_{10b})\quad A^\dagger = [\,A_{11},\quad A_{12}\,]^*\,K_{11}^*\begin{bmatrix}A_{11}\\ A_{21}\end{bmatrix}^*,$$

$$\text{where } K_{11} = \left([\,A_{11},\quad A_{12}\,]\,A^*\begin{bmatrix}A_{11}\\ A_{21}\end{bmatrix}\right)^{-1}.$$

Block decomposition (T_{11}) is investigated in [**9**], but only for square matrices and the group inverse.

The main idea of this paper is to continue the papers [**5**] and [**10**]. In other words, we use the following algorithm: from the presented block decompositions of matrices find the corresponding full-rank factorizatons $A = PQ$, and then apply described general representations for $A^\dagger$ and $A^\dagger_{\varphi(M,N)}$. Main advantages of the introduced block representations are their simply derivation, computation and possibility of natural generalization.

2. The Moore-Penrose inverse.

In the following theorem we derive a few representations of the Moore-Penrose inverse, using described block decompositions and full-rank factorizations.

THEOREM 2.1 *The Moore-Penrose inverse of a given matrix $A \in \mathbb{C}_r^{m\times n}$ can be represented as follows, where each block representation (G_i) is derived from the block decomposition (T_i), $i \in \{1,\dots,9,10a,10b,11\}$:*

$$(G_1)\quad A^\dagger = \left(G^{-1}{}_{|r}\right)^*\left(\left(R^{-1|r}\right)^* A\left(G^{-1}{}_{|r}\right)^*\right)^{-1}\left(R^{-1|r}\right)^*$$
$$= \left(G^{-1}{}_{|r}\right)^*\left((RR^*)^{-1}{}^{|r}_{|r}(G^*G)^{-1}{}^{|r}_{|r}\right)^{-1}\left(R^{-1|r}\right)^*,$$

$$(G_2)\quad A^\dagger = \left(G^{-1}{}_{|r}\right)^*\left(\left(R^{-1|r}B\right)^* A\left(G^{-1}{}_{|r}\right)^*\right)^{-1}\left(R^{-1|r}B\right)^*$$
$$= \left(G^{-1}{}_{|r}\right)^*\left(B^*(RR^*)^{-1}{}^{|r}_{|r}B(G^*G)^{-1}{}^{|r}_{|r}\right)^{-1}B^*\left(R^{-1|r}\right)^*,$$

$$(G_3)\quad A^\dagger = F\begin{bmatrix}I_r\\ K^*\end{bmatrix}\left(\left(R^{-1|r}\right)^* AF\begin{bmatrix}I_r\\ K^*\end{bmatrix}\right)^{-1}\left(R^{-1|r}\right)^*$$
$$= \left(F^{|r} + F^{n-r|}K^*\right)\left((RR^*)^{-1}{}^{|r}_{|r}(I_r + KK^*)\right)^{-1}\left(R^{-1|r}\right)^*,$$

$$(G_4)\quad A^\dagger = \left(G^{-1}{}_{|r}\right)^*\left([\,I_r,\quad K^*\,]\,EA\left(G^{-1}{}_{|r}\right)^*\right)^{-1}[\,I_r,\quad K^*\,]\,E$$
$$= \left(G^{-1}{}_{|r}\right)^*\left((I_r + K^*K)(G^*G)^{-1}{}^{|r}_{|r}\right)^{-1}\left(E_{|r} + K^*E_{n-r|}\right),$$

$$(G_5)\quad A^\dagger = \left(G^{-1}{}_{|r}\right)^*\left(U_{|r}A\left(G^{-1}{}_{|r}\right)^*\right)^{-1}U_{|r} = \left(G^{-1}{}_{|r}\right)^*\left((G^*G)^{-1}{}^{|r}_{|r}\right)^{-1}U_{|r},$$

$$(G_6)\quad A^{\dagger}=V^{|r}\left(\left(R^{-1^{|r}}\right)^{*}AV^{|r}\right)^{-1}\left(R^{-1^{|r}}\right)^{*}=V^{|r}\left((RR^{*})^{-1}{}^{|r}_{|r}\right)^{-1}\left(R^{-1^{|r}}\right)^{*},$$

$$(G_7)\quad A^{\dagger}=V^{|r}\left(B^{*}U_{|r}AV^{|r}\right)^{-1}B^{*}U_{|r}=V^{|r}B^{-1}U_{|r},$$

$$(G_8)\quad A^{\dagger}=F\begin{bmatrix}B^{*}\\K^{*}\end{bmatrix}\left(U_{|r}AF\begin{bmatrix}B^{*}\\K^{*}\end{bmatrix}\right)^{-1}U_{|r}=F\begin{bmatrix}B^{*}\\K^{*}\end{bmatrix}(BB^{*}+KK^{*})^{-1}U_{|r},$$

$$\begin{aligned}(G_9)\quad A^{\dagger}&=V^{|r}\left([\,B^{*},\quad K^{*}\,]\,EAV^{|r}\right)^{-1}[\,B^{*},\quad K^{*}\,]\,E\\&=V^{|r}\,(B^{*}B+K^{*}K)^{-1}\,[\,B^{*},\quad K^{*}\,]\,E,\end{aligned}$$

$$\begin{aligned}(G_{10a})\ A^{\dagger}&=F\begin{bmatrix}I_r\\T^{*}\end{bmatrix}\left(A_{11}^{*}\,[\,I_r,\quad S^{*}\,]\,EAF\begin{bmatrix}I_r\\T^{*}\end{bmatrix}\right)^{-1}A_{11}^{*}\,[\,I_r,\quad S^{*}\,]\,E\\&=\left(F^{|r}+F^{n-r|}T^{*}\right)((I_r+S^{*}S)A_{11}(I_r+TT^{*}))^{-1}\left(E_{|r}+S^{*}E_{n-r|}\right),\end{aligned}$$

$$\begin{aligned}(G_{10b})\ A^{\dagger}&=F\begin{bmatrix}A_{11}^{*}\\A_{12}^{*}\end{bmatrix}\left((A_{11}^{*})^{-1}[A_{11}^{*},\quad A_{21}^{*}]EAF\begin{bmatrix}A_{11}^{*}\\A_{12}^{*}\end{bmatrix}\right)^{-1}(A_{11}^{*})^{-1}[A_{11}^{*},\quad A_{21}^{*}]E\\&=F\begin{bmatrix}A_{11}^{*}\\A_{12}^{*}\end{bmatrix}(A_{11}A_{11}^{*}+A_{12}A_{12}^{*})^{-1}A_{11}(A_{11}^{*}A_{11}+A_{21}^{*}A_{21})^{-1}[A_{11}^{*},\quad A^{\ *}_{21}]E,\end{aligned}$$

$$\begin{aligned}(G_{11})\ A^{\dagger}&=R^{*}\begin{bmatrix}I_r\\(T_1^{-1}T_2)^{*}\end{bmatrix}\left(\left(R^{-1^{|r}}T_1\right)^{*}AR^{*}\begin{bmatrix}I_r\\(T_1^{-1}T_2)^{*}\end{bmatrix}\right)^{-1}\left(R^{-1^{|r}}T_1\right)^{*}\\&=R^{*}\begin{bmatrix}I_r\\(T_1^{-1}T_2)^{*}\end{bmatrix}\left(T_1^{*}(RR^{*})^{-1}{}^{|r}_{|r}[T_1,\quad T_2]RR^{*}\begin{bmatrix}I_r\\(T_1^{-1}T_2)^{*}\end{bmatrix}\right)^{-1}\left(R^{-1^{|r}}T_1\right)^{*}.\end{aligned}$$

Proof. (G_1) Starting from (T_1), we obtain

$$A=R^{-1}\begin{bmatrix}I_r&\mathbb{O}\\\mathbb{O}&\mathbb{O}\end{bmatrix}G^{-1}=R^{-1}\begin{bmatrix}I_r\\\mathbb{O}\end{bmatrix}[\,I_r,\quad\mathbb{O}\,]\,G^{-1},$$

which implies

$$P=R^{-1}\begin{bmatrix}I_r\\\mathbb{O}\end{bmatrix}=R^{-1^{|r}},\qquad Q=[\,I_r,\quad\mathbb{O}\,]\,G^{-1}=G^{-1}{}_{|r}.$$

Now, we get

$$\begin{aligned}A^{\dagger}&=Q^{*}\,(P^{*}AQ^{*})^{-1}\,P^{*}=\left(G^{-1}{}_{|r}\right)^{*}\left(\left(R^{-1^{|r}}\right)^{*}A\left(G^{-1}{}_{|r}\right)^{*}\right)^{-1}\left(R^{-1^{|r}}\right)^{*}\\&=\left(G^{-1}{}_{|r}\right)^{*}\left((R^{*})^{-1}{}_{|r}R^{-1^{|r}}G^{-1}{}_{|r}(G^{*})^{-1^{|r}}\right)^{-1}\left(R^{-1^{|r}}\right)^{*}\\&=\left(G^{-1}{}_{|r}\right)^{*}\left((RR^{*})^{-1}{}^{|r}_{|r}(G^{*}G)^{-1}{}^{|r}_{|r}\right)^{-1}\left(R^{-1^{|r}}\right)^{*}.\end{aligned}$$

The other block representations of the Moore-Penrose inverse can be developed in a similar way.

(G_5) The block decomposition (T_5) implies

$$P = U^* \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = U^{*|r}, \qquad Q = [\, I_r, \quad \mathbb{O}\,]\, G^{-1} = G^{-1}{}_{|r},$$

which means

$$\begin{aligned} A^\dagger &= \left(G^{-1}{}_{|r}\right)^* \left(U_{|r} A \left(G^{-1}{}_{|r}\right)^*\right)^{-1} U_{|r} \\ &= \left(G^{-1}{}_{|r}\right)^* \left(U_{|r} U^{*|r} G^{-1}{}_{|r} (G^*)^{-1|r}\right)^{-1} U_{|r} = \left(G^{-1}{}_{|r}\right)^* \left((G^* G)^{-1}{}_{|r}^{|r}\right)^{-1} U_{|r}. \end{aligned}$$

(G_7) It is easy to see that (T_7) implies

$$P = U^* \begin{bmatrix} B \\ \mathbb{O} \end{bmatrix} = U^{*|r} B, \qquad Q = [\, I_r, \quad \mathbb{O}\,]\, V^* = V^*{}_{|r}.$$

Now,

$$A^\dagger = V^{|r} \left(B^* U_{|r} U^{*|r} B V^*{}_{|r} V^{|r}\right)^{-1} B^* U_{|r} = V^{|r} (B^* B)^{-1} B^* U_{|r} = V^{|r} B^{-1} U_{|r}.$$

(G_{10a}) From (T_{10a}) we obtain

$$A = E^* \begin{bmatrix} A_{11} & A_{11} T \\ S A_{11} & S A_{11} T \end{bmatrix} F^* = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} \, [\, I_r, \quad T\,]\, F^*,$$

which implies, for example, the following full rank factorization of A:

$$P = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}, \quad Q = [\, I_r, \quad T\,]\, F^*.$$

Consequently, the Moore-Penrose inverse of A is

$$A^\dagger = F \begin{bmatrix} I_r, \\ T^* \end{bmatrix} \left(A_{11}^* \, [\, I_r, \quad S^*\,]\, E A F \begin{bmatrix} I_r, \\ T^* \end{bmatrix} \right)^{-1} A_{11}^* \, [\, I_r, \quad S^*\,]\, E.$$

This part of the proof can be completed using

$$EAF = \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} \, [\, I_r, \quad T\,]$$

and

$$F \begin{bmatrix} I_r \\ T^* \end{bmatrix} = F^{|r} + F^{n-r|} T^*, \quad [\, I_r, \quad S^*\,]\, E = E_{|r} + S^* E_{n-r|}.$$

(G_{10b}) Follows from

$$P = E^* \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} A_{11}^{-1}, \quad Q = [\, A_{11}, \quad A_{12} \,] F^*. \qquad \square$$

Remark 2.1. (i) A convenient method for finding the matrices S, T and A_{11}^{-1}, required in (T_{10a}), was introduced in [**5**], and it was based on the following extended Gauss-Jordan transformation:

$$\begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & \mathbb{O} \end{bmatrix} \longrightarrow \begin{bmatrix} I & T & A_{11}^{-1} \\ \mathbb{O} & \mathbb{O} & -S \end{bmatrix}.$$

(ii) In [**10**] it was used the following full-rank factorization of the matrix A, represented in the form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$:

$$P = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \qquad Q = [\, I_r, \quad A_{11}^{-1}A_{12} \,].$$

3. The weighted Moore-Penrose inverse.

Using the general representation of the weighted Moore-Penropse inverse presented in Proposition 1.3: $A^\dagger_{M\circ,\circ N} = NQ^*(P^*MANQ^*)^{-1}P^*M$, and the algorithm of Theorem 2.1, we obtain the following block representation of the weighted Moore-Penrose inverse $A^\dagger_{M\circ,\circ N}$.

THEOREM 3.1 *The weighted Moore-Penrose inverse $A^\dagger_{M\circ,\circ N}$ of $A \in \mathbb{C}_r^{m\times n}$ possesses the following block representations (Z_i), which correspond to the block decompositions (T_i), $i \in \{1, \dots, 9, 10a, 10b, 11\}$:*

(Z_1) $N\left(G^{-1}{}_{|r}\right)^* \left(\left(R^{-1^{|r}}\right)^* MAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} \left(R^{-1^{|r}}\right)^* M,$

(Z_2) $N\left(G^{-1}{}_{|r}\right)^* \left(\left(R^{-1^{|r}}B\right)^* MAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} \left(R^{-1^{|r}}B\right)^* M,$

(Z_3) $NF\begin{bmatrix} I_r \\ K^* \end{bmatrix} \left(\left(R^{-1^{|r}}\right)^* MANF\begin{bmatrix} I_r \\ K^* \end{bmatrix}\right)^{-1} \left(R^{-1^{|r}}\right)^* M,$

(Z_4) $N\left(G^{-1}{}_{|r}\right)^* \left([\, I_r, \quad K^* \,] EMAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} [\, I_r, \quad K^* \,] EM,$

(Z_5) $N\left(G^{-1}{}_{|r}\right)^* \left(U_{|r}MAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} U_{|r}M,$

$$(Z_6) \quad NV^{|r}\left(\left(R^{-1^{|r}}\right)^* MANV^{|r}\right)^{-1}\left(R^{-1^{|r}}\right)^* M,$$

$$(Z_7) \quad NV^{|r}\left(B^*U_{|r}MANV^{|r}\right)^{-1} B^*U_{|r}M,$$

$$(Z_8) \quad NF\begin{bmatrix} B^* \\ K^* \end{bmatrix}\left(U_{|r}MANF\begin{bmatrix} B^* \\ K^* \end{bmatrix}\right)^{-1} U_{|r}M,$$

$$(Z_9) \quad NV^{|r}\left([\,B^*, \quad K^*\,]\,EMANV^{|r}\right)^{-1}[\,B^*, \quad K^*\,]\,EM,$$

$$(Z_{10a}) \; NF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\left(A_{11}^*[\,I_r, \quad S^*\,]\,EMANF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1} A_{11}^*[\,I_r, \quad S^*\,]\,EM,$$

$$(Z_{10b}) \; NF\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\left((A_{11}^*)^{-1}[A_{11}^*, \quad A_{21}^*]EMANF\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\right)^{-1}(A_{11}^*)^{-1}[A_{11}^*, \quad A_{21}^*]EM,$$

$$(Z_{11}) \quad NR^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\left(\left(R^{-1^{|r}}T_1\right)^* MANR^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\right)^{-1}\left(R^{-1^{|r}}T_1\right)^* M.$$

The following representations can be obtained from the main properties of the weighted Moore-Penrose inverse and Theorem 3.1.

Corollary 3.1. *The weighted Moore-Penrose inverse* $A^{\dagger}_{\varphi(M,N)}$ *of* $A \in \mathbb{C}_r^{m\times n}$ *can be represented as follows:*

$$(W_1) \; N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left(\left(R^{-1^{|r}}\right)^* M^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1}\left(R^{-1^{|r}}\right)^* M^{[-1]},$$

$$(W_2) \; N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left(\left(R^{-1^{|r}}B\right)^* M^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1}\left(R^{-1^{|r}}B\right)^* M^{[-1]},$$

$$(W_3) \; N^{[-1]}F\begin{bmatrix} I_r \\ K^* \end{bmatrix}\left(\left(R^{-1^{|r}}\right)^* M^{[-1]}AN^{[-1]}F\begin{bmatrix} I_r \\ K^* \end{bmatrix}\right)^{-1}\left(R^{-1^{|r}}\right)^* M^{[-1]},$$

$$(W_4) \; N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left([I_r, \quad K^*]EM^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1}[I_r, \quad K^*]\,EM^{[-1]},$$

$$(W_5) \; N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left(U_{|r}M^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1} U_{|r}M^{[-1]},$$

$$(W_6) \; N^{[-1]}V^{|r}\left(\left(R^{-1^{|r}}\right)^* M^{[-1]}AN^{[-1]}V^{|r}\right)^{-1}\left(R^{-1^{|r}}\right)^* M^{[-1]},$$

$$(W_7) \; N^{[-1]}V^{|r}\left(B^*U_{|r}M^{[-1]}AN^{[-1]}V^{|r}\right)^{-1} B^*U_{|r}M^{[-1]},$$

(W_8) $N^{[-1]}F\begin{bmatrix}B^*\\K^*\end{bmatrix}\left(U_{|r}M^{[-1]}AN^{[-1]}F\begin{bmatrix}B^*\\K^*\end{bmatrix}\right)^{-1}U_{|r}M^{[-1]}$,

(W_9) $N^{[-1]}V^{|r}\left([\,B^*,\quad K^*\,]\,EM^{[-1]}AN^{[-1]}V^{|r}\right)^{-1}[\,B^*,\quad K^*\,]\,EM^{[-1]}$,

(W_{10a}) $N^{[-1]}F\begin{bmatrix}I_r\\T^*\end{bmatrix}\left(A_{11}^*[I_r,\quad S^*]EM^{[-1]}AN^{[-1]}F\begin{bmatrix}I_r\\T^*\end{bmatrix}\right)^{-1}A_{11}^*[I_r,\quad S^*]EM^{[-1]}$,

$$(W_{10b})\ N^{[-1]}F\begin{bmatrix}A_{11}^*\\A_{12}^*\end{bmatrix}\left((A_{11}^*)^{-1}[A_{11}^*,\quad A_{21}^*]EM^{[-1]}AN^{[-1]}F\begin{bmatrix}A_{11}^*\\A_{12}^*\end{bmatrix}\right)^{-1}$$
$$\times\,(A_{11}^*)^{-1}\,[\,A_{11}^*,\quad A_{21}^*\,]\,EM^{[-1]},$$

(W_{11}) $N^{[-1]}R^*\begin{bmatrix}I_r\\(T_1^{-1}T_2)^*\end{bmatrix}\left(\left(R^{-1^{|r}}T_1\right)^*M^{[-1]}AN^{[-1]}R^*\begin{bmatrix}I_r\\(T_1^{-1}T_2)^*\end{bmatrix}\right)^{-1}\left(R^{-1^{|r}}T_1\right)^*M^{[-1]}$.

4. Examples.

Example 4.1. Consider $A=\begin{bmatrix}-1&0&1&2\\-1&1&0&-1\\0&-1&1&3\\0&1&-1&-3\\1&-1&0&1\\1&0&-1&-2\end{bmatrix}$. Using the Gauss-Jordan transformation, we get the following reduced row-echelon form of the matrix A:

$$RAF=R_A=\begin{bmatrix}I_r&K\\\mathbb{O}&\mathbb{O}\end{bmatrix}=\begin{bmatrix}1&0&-1&-2\\0&1&-1&-3\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

The matrix R_A is obtained using the permutation matrix $F=I_4$, and the following regular matrix:

$$R=\begin{bmatrix}-1&0&0&0&0&0\\-1&1&0&0&0&0\\-1&1&1&0&0&0\\1&-1&0&1&0&0\\0&1&0&0&1&0\\1&0&0&0&0&1\end{bmatrix}.$$

Using the method (G_3) we obtain

$$R^{-1^{|r}}=\begin{bmatrix}-1&0\\-1&1\\0&-1\\0&1\\1&-1\\1&0\end{bmatrix},\quad F\begin{bmatrix}I_r\\K^*\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\-1&-1\\-2&-3\end{bmatrix},$$

and the following Moore-Penrose inverse of A:

$$A^{\dagger}=\begin{bmatrix} -\frac{5}{34} & -\frac{3}{17} & \frac{1}{34} & -\frac{1}{34} & \frac{3}{17} & \frac{5}{34} \\ \frac{4}{51} & \frac{13}{102} & -\frac{5}{102} & \frac{5}{102} & -\frac{13}{102} & -\frac{4}{51} \\ \frac{7}{102} & \frac{5}{102} & \frac{1}{51} & -\frac{1}{51} & -\frac{5}{102} & -\frac{7}{102} \\ \frac{1}{17} & -\frac{1}{34} & \frac{3}{34} & -\frac{3}{34} & \frac{1}{34} & -\frac{1}{17} \end{bmatrix}.$$

Example 4.2. For the matrix A used in Example 3.1 we obtain (see [**5**])

$$A_{11}^{-1}=\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \quad S=\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad T=\begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}.$$

Using (G_{10a}) we obtain the same Moore-Penrose inverse of A.

Example 4.3. For the matrix $A=\begin{bmatrix} 4 & -1 & 1 & 2 \\ -2 & 2 & 0 & -1 \\ 6 & -3 & 1 & 3 \\ -10 & 4 & -2 & -5 \end{bmatrix}$ we obtain

$$R=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}, \quad T_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2=\begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}, \quad F=I_4.$$

Then, one can verify the following:

$$\left(R^{-1|_r}T_1\right)^*=\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \quad R^*\begin{bmatrix} I_r \\ \left(T_1^{-1}T_2\right)^* \end{bmatrix}=\begin{bmatrix} -4 & -6 \\ 1 & 3 \\ -1 & -1 \\ -2 & -3 \end{bmatrix}.$$

Finally, using (G_{11}), we get

$$A^{\dagger}=\begin{bmatrix} \frac{8}{81} & \frac{10}{81} & -\frac{2}{81} & -\frac{2}{27} \\ \frac{47}{162} & \frac{79}{162} & -\frac{16}{81} & -\frac{5}{54} \\ \frac{7}{54} & \frac{11}{54} & -\frac{2}{27} & -\frac{1}{18} \\ \frac{4}{81} & \frac{5}{81} & -\frac{1}{81} & -\frac{1}{27} \end{bmatrix}.$$

Example 4.4. Consider $A = \begin{bmatrix} 1 & -5 & 1 & 4 \\ -2 & 7 & 0 & 1 \\ 0 & -3 & 2 & 9 \end{bmatrix}$ and positive definite matrices

$$M = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}, \quad N = \begin{bmatrix} 4 & 0 & -1 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 2 & 5 & -2 \\ 0 & 1 & -2 & 6 \end{bmatrix}.$$

Block decomposition (T_1) can be obtained by applying transformation (T_3) two times:

$$R_1AF_1 = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_3, \qquad R_2N_3^TF_2 = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1.$$

Then, the regular matrices R, G can be computed as follows:

$$N_1 = N_1^T = F_2^TN_3R_2^T = F_2^TR_1AF_1R_2^T \Rightarrow R = F_2^TR_1, \quad G = F_1R_2^T.$$

For given matrix the following can be obtained:

$$N_3 = \begin{bmatrix} 1 & 0 & -\frac{7}{3} & -11 \\ 0 & 1 & -\frac{2}{3} & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} -\frac{7}{3} & -\frac{5}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{3} & 0 \\ -2 & -1 & 1 \end{bmatrix}, \quad F_1 = I_4,$$

$$N_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{7}{3} & \frac{2}{3} & 1 & 0 \\ 11 & 3 & 0 & 1 \end{bmatrix}, \quad F_2 = I_3.$$

From $R = R_1$, $G = R_2^T$, we get

$$R^{-1^{|2}} = \begin{bmatrix} 1 & -5 \\ -2 & 7 \\ 0 & -3 \end{bmatrix}, \quad G^{-1}{}_{|2} = \begin{bmatrix} 1 & 0 & -\frac{7}{3} & -11 \\ 0 & 1 & -\frac{2}{3} & -3 \end{bmatrix}.$$

Using formula (Z_1), we obtain the following representation for the weighted Moore-Penrose inverse of A:

$$A^{\dagger}_{M\circ,\circ N} = \begin{bmatrix} -\frac{8841}{207506} & -\frac{13865}{207506} & -\frac{13865}{207506} \\ \frac{23355}{207506} & \frac{38035}{207506} & -\frac{14947}{207506} \\ \frac{2149}{103753} & \frac{11585}{103753} & -\frac{7035}{103753} \\ \frac{42301}{207506} & \frac{25265}{207506} & \frac{3465}{207506} \end{bmatrix}.$$

Example 4.5. Similarly, block decomposition (T_1) of the matrix A, considered in Example 4.1 can be obtained by transformation (T_3) two times, by means of the following matrices:

$$R_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_1 = I_4,$$

$$R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \quad F_2 = I_6.$$

From $R = R_1$, $G = R_2^T$, we get

$$R^{-1^{|2}} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad G^{-1}{}_{|2} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -3 \end{bmatrix}.$$

Using formula (G_1), we obtain the Moore-Penrose inverse $A^\dagger$, as in Example 4.1.

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