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RESTRICTED CONVERGENCE AND p-MAX STABLE LAWS

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Abstract. Nondegenerate limit laws for maxima of iid random variables with power normalization are called p-max stable laws. We prove that if maxima of iid random variables with power normalization converge weakly on a bounded interval, they converge for every $x \in R$.

The class of p-max stable laws, obtained as limiting for maxima of iid random variables under power normalization, was introduced by Pancheva (1985). Namely, let $\{X_n,\ n \geq 1\}$ be an iid sequence of random variables with common distribution $F(x)$. Put $M_n = \max_{1 \leq i \leq n} X_i$. Suppose that for each $n \in N$ there exist $\alpha_n > 0$, $\beta_n > 0$, such that under power normalization $T_n = \alpha_n |x|^{\beta_n} \operatorname{sgn}(x)$, we have

$$P\{T_n^{-1}(M_n) \leq x\} = F^n(T_n(x)) = F^n(\alpha_n |x|^{\beta_n} \operatorname{sgn}(x)) \to G(x) \tag{1}$$

weakly as $n \to \infty$, where G is assumed nondegenerate. It has been shown by Pancheva that then G is of the power type of one of the following six distributions (We say that two distribution functions V_1 and V_2 are of the same power type (p-type) if for some $A > 0$ and $B > 0$ $V_1(x) = V_2(A|x|^B \operatorname{sgn}(x)))$ for all $x \in R$):

$$\begin{aligned}
H_{1,\alpha}(x) &= \exp(-(\log x)^{-\alpha}), \quad x \geq 1, \text{ for some } \alpha > 0\\
H_{2,\alpha}(x) &= \exp(-(-\log x)^{\alpha}), \quad 0 \leq x \leq 1, \text{ for some } \alpha > 0\\
H_{3,\alpha}(x) &= \exp(-(-\log(-x))^{-\alpha}), \quad -1 \leq x \leq 0, \text{ for some } \alpha > 0\\
H_{4,\alpha}(x) &= \exp(-(\log(-x))^{\alpha}), \quad x \leq -1, \text{ for some } \alpha > 0\\
\Phi_1(x) &= \exp(-x^{-1}), \quad x \geq 0\\
\Psi_1(x) &= \exp(x), \quad x \leq 0.
\end{aligned} \tag{2}$$

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The class of p-max stable laws contains all laws of the p-type (2).

When in (1) normalization is linear, $T_n(x) = a_n x + b_n$, $a_n > 0$, $b_n \in R$, then we obtain as limiting the well known class of extreme value distributions (or $l-$max stable laws) which consists of the following three types

$$\Phi_\alpha(x) = \exp(-x^{-\alpha}), \quad x \geq 0, \text{ for some } \alpha > 0$$
$$\Psi_\alpha(x) = \exp(-(-x)^\alpha), \quad x < 0, \text{ for some } \alpha > 0$$
$$\Lambda(x) = \exp(-e^{-x}), \quad x \in R.$$

Various properties of the class of p-max stable laws, parallel to those of l-max stable laws, were investigated by Mohan and Ravi (1989), (1992a), (1992b), and Ravi (1991), (1992). The fact shown by Mohan and Ravi (1992a) which makes p-max stable laws interesting is the following:

Let us first introduce the notion of the domain of attraction for p-max stable laws: We say that F belongs to the domain of attraction of G under power normalization (and denote that by $F \in D_p(G)$) if there exist constants $\alpha_n > 0$, $\beta_n > 0$, $n \geq 1$ such that (1) holds. The notion of the domain of attraction for l-max stable laws is the usual one: we say that F belongs to the domain of attraction of G ($F \in D_l(G)$) if there exist constants $a_n > 0$ and $b_n \in R$ such that $P\{M_n \leq a_n x + b_n\} = F^n(a_n x + b_n) \to G(x)$ weakly as $n \to \infty$.

Mohan and Ravi compared domains of attraction of l-max stable laws with those of p-max stable laws and showed that every probability distribution attracted to an l-max stable law is also attracted to some p-max stable law and that strict inclusion $D_l(G) \subset D_p(G)$ holds.

On initiative of V.M. Zolotarev the theory of sums of independent random variables was considered from the new point of view. He gave a conjecture concerning Central Limit Theorem with necessary and sufficient conditions of quite another type than the usual ones. The conjecture was that the restricted convergence of sums of iid random variables on the half line $(-\infty, \tau) \subset R$ continues to R and the limit df is the normal df. Various results of that kind can be found in the monograph of Rossberg, Jesiak and Siegel (1985) and in Rossberg (1994).

Theorem of that type concerned with the restricted convergence of extreme values under linear normalization has been proved by Gnedenko and Senusi-Bereksi (1983).

Our aim is to prove the following theorem about the restricted convergence of maxima of iid random variables under power normalization, namely to prove that from the convergence of maxima with power normalization on a bounded interval to some of the p-max stable laws - follows the convergence on the whole line.

THEOREM. *If for suitably chosen constants $a_n > 0$, $b_n > 0$, $n \geq 1$*

$$F^n(\alpha_n |x|^{\beta_n} \operatorname{sgn}(x)) \to G(x), n \to \infty, \tag{3}$$

for all x from the interval $J \subset R$, where G is strictly monotone and continuous, $G(x) \neq 0$, $G(x) \neq 1$ for $x \in J$, then (3) *holds for all $x \in R$ and the limit distribution is of the p-type* (2).

Proof. The proof consists of two parts. First, we establish limit behaviour of constants α_n and β_n. Then we use that fact in proving that the convergence holds for all $x \in$R. From (1) we get that for any $t > 0$

$$F^{[nt]}(\alpha_{[nt]}|x|^{\beta_{[nt]}} \operatorname{sgn}(x)) \to G(x) \tag{4}$$

as $n \to \infty$, and also that

$$F^{[nt]}(\alpha_n|x|^{\beta_n} \operatorname{sgn}(x)) = (F^n(\alpha_n|x|^{\beta_n} \operatorname{sgn}(x)))^{[nt]/n} \to G^t(x) \tag{5}$$

where $[t]$ is, as usual, greatest integer less than or equal to t.

Define $F^{\leftarrow}(y) = \inf\{s : F(s) \geq y\}$. According to the Theorem 25.7 of Billingsley (1979), see also Resnick (1987), relations (4) and (5) can be inverted so that for $J = [c, d]$, $y \in (G(c), G(d))$

$$\left(\frac{F^{[nt]\leftarrow}(y)\operatorname{sgn}(y)}{\alpha_{[nt]}}\right)^{\beta_{[nt]}^{-1}} \underset{n\to\infty}{\longrightarrow} G^{\leftarrow}(y) \quad y \in (G(c), G(d)) \tag{6}$$

and

$$\left(\frac{F^{[nt]\leftarrow}(y)\operatorname{sgn}(y)}{\alpha_n}\right)^{\beta_n^{-1}} \underset{n\to\infty}{\longrightarrow} G^{t\leftarrow}(y), \quad y \in (G^t(c), G^t(d)). \tag{7}$$

There exists a subinterval J_0 of J such that for t sufficiently close to 1, $t \in [1-s, 1+s]$, all points of continuity of $G^{t\leftarrow}$ for $y \in J_0$ are also points of continuity of $G^{\leftarrow}$.

Let $y_1, y_2, y_1 < y_2$ be continuity points of both $G^{t\leftarrow}$ and $G^{\leftarrow}$. Then from (6) and (7) we have

$$\left(\frac{F^{[nt]\leftarrow}(y_i)\operatorname{sgn}(y_i)}{\alpha_{[nt]}}\right)^{\beta_{[nt]}^{-1}} \underset{n\to\infty}{\longrightarrow} G^{\leftarrow}(y_i), \quad i = 1, 2 \tag{8}$$

$$\left(\frac{F^{[nt]\leftarrow}(y_i)\operatorname{sgn}(y_i)}{\alpha_n}\right)^{\beta_n^{-1}} \underset{n\to\infty}{\longrightarrow} G^{t\leftarrow}(y_i), \quad i = 1, 2. \tag{9}$$

Divide two relations in (8) by each other and do the same in (9) to obtain:

$$\left(\frac{F^{[nt]\leftarrow}(y_1)\operatorname{sgn}(y_1)}{F^{[nt]\leftarrow}(y_2)\operatorname{sgn}(y_2)}\right)^{\beta_{[nt]}^{-1}} \underset{n\to\infty}{\longrightarrow} \frac{G^{\leftarrow}(y_1)}{G^{\leftarrow}(y_2)} \tag{10}$$

$$\left(\frac{F^{[nt]\leftarrow}(y_1)\operatorname{sgn}(y_1)}{F^{[nt]\leftarrow}(y_2)\operatorname{sgn}(y_2)}\right)^{\beta_n^{-1}} \underset{n\to\infty}{\longrightarrow} \frac{G^{t\leftarrow}(y_1)}{G^{t\leftarrow}(y_2)}. \tag{11}$$

Taking logarithms of both sides of (10) and (11) and dividing we get

$$\frac{\beta_n}{\beta_{[nt]}} \underset{n\to\infty}{\longrightarrow} \frac{\log G^{\leftarrow}(y_1) - \log G^{\leftarrow}(y_2)}{\log G^{t\leftarrow}(y_1) - \log G^{t\leftarrow}(y_2)} := \beta(t) \tag{12}$$

for $t \in I = [1-s, 1+s]$. In this case all conditions of Lemma 1.5 of Seneta (1976) are fulfilled, according to which the relation (12) holds for all $t > 0$ and $\beta(t) = t^\theta$, $\theta \in R$.

If we divide (6) by (7) for $y = y_1$ we get

$$\left(\frac{\alpha_n}{\alpha_{[nt]}}\right)^{\beta_{[nt]}^{-1}} \underset{n\to\infty}{\longrightarrow} \frac{G^{\leftarrow}(y_1)}{(G^{t\leftarrow}(y_1))^{\beta(t)}} := \alpha(t).$$

Logarithming both sides we get

$$\frac{\log \alpha_n - \log \alpha_{[nt]}}{\beta_{[nt]}} \underset{n\to\infty}{\longrightarrow} \log \alpha(t).$$

From Theorem 2.10 of Seneta (1976) it follows that $\log \alpha(t)$ is of the form $\log \alpha(t) = c \log t$ when $\theta = 0$ and $\log \alpha(t) = c(t^\theta - 1)$ when $\theta \neq 0$. Therefore $\alpha(t) = t^c$ when $\theta = 0$ and $\alpha(t) = \exp(c(t^\theta - 1))$ when $\theta \neq 0$. We shall write α in the case when $\theta \neq 0$ in the form used by Mohan and Ravi (1992b), namely, put $\exp(-c) = d$, then $\exp(c(t^\theta - 1)) = d^{1-t^\theta}$, so

$$\beta(t) = t^\theta,\ \ \theta \in R, \quad \alpha(t) = d^{1-t^\theta},\ \ \theta \neq 0, \quad \alpha(t) = t^c,\ \ \theta = 0. \tag{13}$$

According to the Helley's selection theorem, it is possible to select a convergent subsequence from the sequence $F^n(\alpha_n |x|^{\beta_n} \operatorname{sgn}(x))$, which converges to a nondecreasing function, say $H(x)$. So

$$H(x) = \lim_{k\to\infty} F^{n_k}(\alpha_{n_k} |x|^{\beta_{n_k}} \operatorname{sgn}(x)). \tag{14}$$

From the convergence of the p-types theorem (see Mohan and Ravi 1989 and 1992b) we can replace constants α_{n_k} and β_{n_k} in (14) by equivalent ones a_{n_k} and b_{n_k}, satisfying

$$\frac{\beta_{n_k}}{b_{n_k}} \to 1, \quad \left(\frac{\alpha_{n_k}}{a_{n_k}}\right)^{b_{n_k}^{-1}} \to 1, \quad k \to \infty,$$

without changing the limit in (14). From (13) it follows that

$$\lim_{n\to\infty} \frac{\beta_n t^{-\theta}}{\beta_{[nt]}} = 1$$

$$\lim_{n\to\infty} \left(\frac{\alpha_n t^{-c\beta_{[nt]}}}{\alpha_{[nt]}}\right)^{\beta_{[nt]}^{-1}} = 1 \quad \text{when } \theta = 0,$$

$$\lim_{n\to\infty} \left(\frac{\alpha_n d^{-(1-t^\theta)\beta_{[nt]}}}{\alpha_{[nt]}}\right)^{\beta_{[nt]}^{-1}} = 1 \quad \text{when } \theta \neq 0.$$

So $\beta_n t^{-\theta}$ are equivalent to $\beta_{[nt]}$, $\alpha_n t^{-c\beta_{[nt]}}$ are equivalent to $\alpha_{[nt]}$ when $\theta = 0$ and $\alpha_n d^{-(1-t^\theta)\beta_{[nt]}}$ are equivalent to $\alpha_{[nt]}$ when $\theta \neq 0$.

Suppose $\theta \neq 0$. Choose arbitrary point $x_0 \in J$, $x_0 \neq 0, -d, d$. Point x_0 belongs to one of the intervals $I_1 = (d, +\infty)$, $I_2 = (0, d)$, $I_3 = (-d, 0)$, $I_4 = (-\infty, -d)$. For any other point x from the same interval I_k there exists $t_x > 0$,

$$t_x = \left(\frac{\log|x/d|}{\log|x_0/d|}\right)^{\theta^{-1}}$$

so that

$$|x| = d^{1-t_x^\theta}|x_0|^{t_x^\theta}. \tag{15}$$

Note that x and x_0 are of the same sign. Then we have from (14)

$$\begin{aligned}
H(x) &= \lim_{k\to\infty} F^{[n_k t_x]}(\alpha_{[n_k t_x]}|x|^{\beta_{[n_k t_x]}}\operatorname{sgn}(x)) \\
&= \lim_{k\to\infty} F^{[n_k t_x]}(\alpha_{n_k} d^{-(1-t_x^\theta)\beta_{n_k} t_x^{-\theta}}(d^{1-t_x^\theta}|x_0|^{t_x^\theta})^{\beta_{n_k} t_x^{-\theta}}\operatorname{sgn}(x_0)) \\
&= \lim_{k\to\infty} F^{[n_k t_x]}(\alpha_{n_k} d^{-(1-t_x^\theta)\beta_{n_k} t_x^{-\theta}}(d^{(1-t_x^\theta)\beta_{n_k} t_x^{-\theta}}|x_0|^{\beta_{n_k}}\operatorname{sgn}(x_0)) \\
&= \lim_{k\to\infty} F^{[n_k t_x]}(\alpha_{n_k}|x_0|^{\beta_{n_k}}\operatorname{sgn}(x_0)) = G^{t_x}(x_0) \\
&= \exp\left(\log G(x_0)\left(\frac{\log|x/d|}{\log|x_0/d|}\right)^{\theta^{-1}}\right).
\end{aligned} \tag{16}$$

When $x \in I_1 = (d, +\infty)$, put

$$\frac{\log G(x_0)}{(\log|x_0/d|)^{\theta^{-1}}} = -g, \quad g > 0. \tag{17}$$

Then

$$\begin{aligned}
H(x) &= \exp(-g(\log|x/d|)^{\theta^{-1}}) = \exp(-(g^\theta \log|x/d|)^{\theta^{-1}}) \\
&= \exp(-(\log(d^{-g^\theta} x^{g^\theta}))^{\theta^{-1}}),
\end{aligned}$$

which is the law of the p-type $H_{1,1/\theta}$ for $\theta < 0$. Here $g = -\log H(de)$. From (17) it follows that

$$G(x_0) = \exp(-(\log(d^{-g^\theta} x_0^{g^\theta}))^{\theta^{-1}}),$$

and starting from another point different from x_0 from the interval J, we would obtain the same function H which coincides on J with G.

The remaining cases when $x \in I_k$, $k = 2, 3, 4$ are analogous. Starting from (16), if $x \in I_2$, put

$$\frac{-\log G(x_0)}{(-\log|x_0/d|)^{\theta^{-1}}} = g, \quad g > 0;$$

then
$$H(x)=\exp(-(-\log(d^{-g^\theta}x^{g^\theta}))^{\theta^{-1}}),\theta>0,$$
where $g=-\log H(d/e)$ and H is of the p-type $H_{2,1/\theta}$. If $x\in I_3$, put
$$\frac{-\log G(x_0)}{(-\log|x_0/d|)^{\theta^{-1}}}=g,\quad g>0;$$
then
$$H(x)=\exp(-(-\log(d^{-g^\theta}(-x)^{g^\theta}))^{\theta^{-1}}),\quad \theta<0$$
where $g=-\log H(-d/e)$ and H is of the p-type $H_{3,1/\theta}$. If $x\in I_4$, put
$$\frac{\log G(x_0)}{(\log|x_0/d|)^{\theta^{-1}}}=-g,\quad g>0;$$
then
$$H(x)=\exp(-(\log(d^{-g^\theta}(-x)^{g^\theta}))^{\theta^{-1}}),\theta>0,$$
where $g=-\log H(-de)$ and H is of the p-type $H_{4,1/\theta}$.

When $\theta=0$, fix arbitrary $0\neq x_0\in J$. Then for any x of the same sign as x_0 there exists $t_x>0$ ($t_x=|x/x_0|^{1/c}$) such that x can be written as $|x|=|x_0|t_x^c$. As in the previous case we select a convergent subsequence from the sequence $F^n(\alpha_n|x|^{\beta_n}\operatorname{sgn}(x))$ and replace constants α_n and β_n by equivalent ones. We have:
$$\begin{aligned}H(x)&=\lim_{k\to\infty}F^{[n_kt_x]}(\alpha_{[n_kt_x]}|x|^{\beta_{[n_kt_x]}}\operatorname{sgn}(x))\\&=\lim_{k\to\infty}F^{[n_kt_x]}(\alpha_{n_k}t_x^{-c\beta_{n_k}}|x_0t^c|^{\beta_{n_k}}\operatorname{sgn}(x_0))\\&=\lim_{k\to\infty}F^{[n_kt_x]}(\alpha_{n_k}|x_0|^{\beta_{n_k}}\operatorname{sgn}(x_0))=G^{t_x}(x_0)\\&=\exp(\log\mathrm{G}(x_0)|x/x_0|^{1/c}).\end{aligned}\tag{18}$$
When $x>0$ put
$$\frac{-\log G(x_0)}{|x_0|^{1/c}}=g$$
Then (18) is equal to
$$H(x)=\exp(-gx^{c^{-1}}),\quad c<0,$$
$g=-\log H(1)$ and H is of the p-type Φ. When $x<0$ put $\dfrac{-\log G(x_0)}{|x_0|^{1/c}}=g$. Then $H(x)=\exp(-g(-x)^{c^{-1}})$, $c>0$, $g=-\log H(-1)$ and H is of the p-type Ψ.

Since H depends on the value $G(x_0)$, $x_0\in J$, every other convergent subsequence of $F^n(\alpha_n|x|^{\beta n}\operatorname{sgn}(x))$, would have the same limit (14).

Hence, we proved that weak convergence of maxima with power normalization to one of the p-max stable laws (2) on a restricted interval J implies the convergence on I, where I is the interval on which that law takes values different from 0 and from 1. Because of the continuity, the convergence holds also in the left and right end points. The proof is completed.

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