

**A NOTE ON STABILITY OF MINIMAL SURFACES  
IN  $n$ -DIMENSIONAL HYPERBOLIC SPACE  $H^n(c)$**

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**Abstract.** We improve a result of Barbosa-do Carmo about stability of minimal surfaces in  $n$ -dimensional hyperbolic space  $H^n(c)$ .

**1. Introduction.** In [1] Barbosa and do Carmo obtain the following well-known result

**THEOREM 1 [1].** *Let  $M$  be a minimal surface in an  $n$ -dimensional hyperbolic space  $H^n(c)$ . Assume that  $D$  is a simply connected compact domain with piecewise smooth boundary on  $M$ . Let  $A$  denote the second fundamental form of  $M$ . If*

$$(1) \quad \int_D (|c| + \frac{|A|^2}{2}) dv < \frac{4\pi}{3},$$

*then  $D$  is stable.*

In this note, we improve the Theorem above as follows

**THEOREM 2.** *Let  $M$  be a minimal surface in an  $n$ -dimensional hyperbolic space  $H^n(c)$ . Assume that  $D$  is a simply connected compact domain with piecewise smooth boundary on  $M$ . Let  $A$  denote the second fundamental form of  $M$ . If*

$$(2) \quad \int_D (\frac{|c|}{5} + \frac{|A|^2}{2}) dv < \frac{4\pi}{3},$$

*then  $D$  is stable.*

*Remark.* Obviously, our condition (2) is better than condition (1) of Barbosa-do Carmo's.

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**2. Preliminaries.** Let  $H^n(c)$  be an  $n$ -dimensional simply connected space of constant negative curvature  $c$ ; we also call it the hyperbolic space. Let  $M$  be a minimal surface in  $H^n(c)$ ; we denote by  $K$  the Gauss curvature of  $M$  with respect to the induced metric  $ds_M^2$ . Let  $A$  be the second fundamental form of  $M$ .

We need the following lemmas to prove Theorem 2.

smallskip LEMMA 1. *If  $M$  be a minimal surface in  $H^n(c)$ , then*

$$(3) \quad |\nabla(|A|^2)|^2 \leq 2|A|^2|\nabla A|^2.$$

*Proof.* Let  $M$  be a minimal surface  $H^n(c)$ . By an elementary observation one can see that at each point the dimension of the image of the second fundamental form  $A$  of  $M$  is at most 2. Thus we may choose  $e_3, \dots, e_n$  so that  $h_{ij}^\alpha = 0$  for all  $i, j$  and  $\alpha \geq 5$ , i.e., we may choose the basis  $e_1, e_2, \dots, e_n$  so that the component  $h_{ij}^\alpha$  of  $A$  satisfy

$$(4) \quad (h_{ij}^3) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h_{ij}^5) = \dots = (h_{ij}^n) = 0,$$

for some functions  $\lambda$  and  $\mu$ . Let  $|A|^2 = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2$  be the square length of the second fundamental form of  $M$  and  $K_N = \sum_{\alpha, \beta, i, j} R_{\alpha\beta ij}^2$  be the normal scalar curvature of  $M$ . By (4) and Ricci equation we easily check that  $|A|^2 = 2(\lambda^2 + \mu^2)$ ,  $K_N = 16\lambda^2\mu^2$ .

Noting  $\sum_k (h_{11k}^\alpha)^2 = \sum_k (h_{12k}^\alpha)^2$ ,  $3 \leq \alpha \leq n$ , by (4), we have

$$(5) \quad \begin{aligned} |\nabla(|A|^2)|^2 &= 4 \sum_k \left( \sum_{i, j, \alpha} h_{ij}^\alpha h_{ijk}^\alpha \right)^2 \\ &= 16 \sum_k (\lambda h_{11k}^3 + \mu h_{12k}^4)^2 \\ &\leq 16 \sum_k (\lambda^2 + \mu^2) [(h_{11k}^3)^2 + (h_{12k}^4)^2] \\ &= 8|A|^2 \sum_k [(h_{11k}^3)^2 + (h_{11k}^4)^2]. \end{aligned}$$

On the other hand, we have

$$(6) \quad \begin{aligned} |\nabla A|^2 &= 2 \sum_{i, k, \alpha} (h_{iik}^\alpha)^2 = 4 \sum_{k, \alpha} (h_{11k}^\alpha)^2 \\ &\geq 4 \sum_k [(h_{11k}^3)^2 + (h_{11k}^4)^2]. \end{aligned}$$

We get (3) from (5) and (6). *The proof of Lemma 1 is completed.*

LEMMA 2. *If  $M$  be a minimal surface in  $H^n(c)$ , then*

$$\begin{aligned} \frac{1}{2}\Delta(|A|^2) &= |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4 + 2(\lambda^2 - \mu^2)^2 \\ &\geq |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4. \end{aligned}$$

*Proof.* Denote the matrix  $(h_{ij}^\alpha)$  by  $H_\alpha$ ,  $3 \leq \alpha \leq n$ . By Gauss-Codazzi-Ricci equations it was shown in [4] that

$$\begin{aligned} \frac{1}{2}\Delta(|A|^2) &= \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + \sum_{\alpha,i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} + h_{li}^\alpha R_{lkjk}) \\ &\quad + \sum_{\alpha,\beta,i,j,k} h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk} \\ &= |\nabla A|^2 + \sum_{\alpha,\beta} \text{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta} (\text{tr}(H_\alpha H_\beta))^2 + 2c|A|^2. \end{aligned}$$

By (4), it is easy to check the following formulas

$$(8) \quad \sum_{\alpha,\beta} \text{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 = -16\lambda^2\mu^2, \quad \sum_{\alpha,\beta} (\text{tr}(H_\alpha H_\beta))^2 = 4(\lambda^4 + \mu^4).$$

Substituting (8) into (7), we get

$$\begin{aligned} \frac{1}{2}\Delta(|A|^2) &= |\nabla A|^2 + 2c|A|^2 - 8(\lambda^2 + \mu^2)^2 + 4(\lambda^4 + \mu^4) \\ &= |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4 + 2(\lambda^2 - \mu^2)^2. \end{aligned}$$

We completed the proof of Lemma 2.

The following proposition is crucial to prove our Theorem 2.

PROPOSITION 1. *Let  $M$  be a minimal surface in  $H^n(c)$ ,  $ds_M^2$  be the induced metric. Then the Gauss curvature  $\overline{K}$  of the conformal metric  $\overline{ds^2} = \sigma ds_M^2$  satisfies  $\overline{K} \leq 2$ , where*

$$\sigma = \frac{|c|}{5} + \frac{|A|^2}{2} > 0.$$

*Proof.* By Gauss equation  $2K = 2c - |A|^2$ ,

$$(9) \quad \sigma = \frac{|c|}{5} + \frac{|A|^2}{2} = \frac{4c}{5} - K.$$

Thus we can define a conformal metric  $\overline{ds^2} = \sigma ds_M^2$  on  $M$ . As it is wellknown, the Gauss curvature  $\overline{K}$  of  $\overline{ds^2}$  satisfies (for example, see [2])

$$(10) \quad -\sigma\overline{K} = -K + \frac{1}{2} \frac{\Delta\sigma}{\sigma} - \frac{|\nabla\sigma|^2}{2\sigma^2},$$

where  $\Delta$  is the Laplacian operator of the metric  $ds_M^2$ .

By (9) and (10), we get

$$(11) \quad -\sigma \bar{K} = \sigma - \frac{4c}{5} + \frac{1}{2} \frac{\Delta \sigma}{\sigma} - \frac{|\nabla \sigma|^2}{2\sigma^2}.$$

By use of Lemma 1, Lemma 2 and (9),

$$(12) \quad \begin{aligned} \frac{1}{2} \Delta \sigma &= \frac{1}{4} \Delta(|A|^2) \geq \frac{1}{2} |\nabla A|^2 + c|A|^2 - \frac{3}{4} |A|^4 \\ &\geq \frac{1}{2} \frac{|\nabla \sigma|^2}{\sigma} - 3\sigma^2 + \frac{4c\sigma}{5}. \end{aligned}$$

Combining (11) with (12), we obtain

$$\bar{K} \leq 2.$$

We completed the proof of Proposition 1.

**3. The Proof of Theorem 2.** By use of our Proposition 1, we can prove Theorem 2 in the same way as Barbosa and do Carmo did in [1] for Theorem 1. So we omit the proof of Theorem 2 here.

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