PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 61 (75), 1997, 90–96

ON A TYPE OF SEMI-SYMMETRIC METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

U.C. De and **S.C. Biswas**

Communicated by Mileva Prvanović

Abstract. The properties of Riemannian manifolds admitting a semi-symmetric metric connection were studied by many authors ([**1**], [**2**], [**3**], [**4**], [**5**], [**6**]). In [**4**] an expression of the curvature tensor of a manifold was obtained under assumption that the manifold admits a semi-symmetric metric connection with vanishing curvature tensor and recurrent torsion tensor. Also in [**7**] Prvanović and Pušić obtained an expression for curvature tensor of a Riemannian manifold, locally decomposable Riemannian space and the Kähler space which admits a semi-symmetric metric connection $\tilde\nabla$ with vanishing curvature tensor and torsion tensor T^h_{1m} satisfying $\tilde\nabla_k\tilde\nabla_j T^h_{1m} - \tilde\nabla_j\tilde\nabla_k T^h_{1m} = 0$.

We study a type of semi-symmetric metric connection $\tilde\nabla$ satisfying $\tilde R(X,Y)T = 0$ and $\omega(\tilde R(X,Y)Z) = 0$, where T is the torsion tensor of the semi-symmetric connection, $\tilde R$ is the curvature tensor corresponding to $\tilde\nabla$ and ω is the associated 1-form of T.

0. Introduction. Let (M^n, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. A linear connection $\tilde\nabla$ on (M^n, g) is said to be a semi-symmetric metric connection if the torsion tensor T of the connection $\tilde\nabla$ and the metric tensor g of the manifold satisfy the following conditions:

$$T(X,Y) = \omega(Y)X - \omega(X)Y \tag{0.1}$$

for any vector fields X, Y where ω is a 1-form associated with the torsion tensor of the connection $\tilde\nabla$ and

$$(\tilde\nabla_Z g)(X,Y) = 0$$

Then we have [**1**] for any vector fields X, Y, Z

$$\tilde\nabla_X Y = \nabla_X Y + \omega(Y)X - g(X,Y)\rho$$

AMS Subject Classification (199?): Primary 53C05.

where

$$g(X,\rho)=\omega(X), \tag{0.4}$$

the 1-form ω and the vector field ρ are usually called 1-form and vector field associated with torsion tensor T and

$$(\tilde{\nabla}_X\omega)(Y)=(\nabla_X\omega)(Y)-\omega(X)\omega(Y)+\omega(\rho)g(X,Y)$$

Also, we have [**1**]

$$\tilde{R}(X,Y)Z=R(X,Y)Z-\alpha(Y,Z)X+\alpha(X,Z)Y-g(Y,Z)AX+g(X,Z)AY \tag{0.6}$$

where

$$\alpha(Y,Z)=g(AY,Z)=(\nabla_Y\omega)(Z)-\omega(Y)\omega(Z)+\frac{1}{2}\omega(\rho)g(Y,Z), \tag{0.7}$$

$\tilde{R}$ and R are respective curvature tensor for the connections $\tilde{\nabla}$ and ∇, A being a $(1-1)$ tensor field.

Now, let us suppose that the connection (1) satisfies the following conditions:

$$\tilde{R}(X,Y)T=0 \tag{0.8}$$

$$\omega(\tilde{R}(X,Y)Z)=0 \tag{0.9}$$

where $\tilde{R}(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y.

1. Expression for the curvature tensor of the semi-symmetric metric connection. The condition (0.8) gives

$$\begin{aligned}\tilde{R}(X,Y)T(U,V)&-T(\tilde{R}(X,Y)U,V)\\&-T(U,\tilde{R}(X,Y)V)-(\tilde{\nabla}_{T(X,Y)}T)(U,V)=0.\end{aligned} \tag{1.1}$$

Now

$$\begin{aligned}(\tilde{\nabla}_{T(X,Y)}T)(U,V)&=(\tilde{\nabla}_{\omega(Y)X-\omega(X)Y}T)(U,V)\\&=\omega(Y)(\tilde{\nabla}_XT)(U,V)-\omega(X)(\tilde{\nabla}_YT)(U,V)\\&=\omega(Y)[(\nabla_X\omega)(V)U-(\nabla_X\omega)(U)V-\omega(\rho)\{g(X,U)V-g(X,V)U\}]\\&\quad-\omega(X)[(\nabla_Y\omega)(V)U-(\nabla_Y\omega)(U)V-\omega(\rho)\{g(Y,U)V-g(Y,V)U\}]\end{aligned} \tag{1.2}$$

From (1.1) and (1.2) we get

$$\begin{aligned}&\omega(\tilde{R}(X,Y)U)V-\omega(\tilde{R}(X,Y)V)U\\&-\omega(Y)[(\nabla_X\omega)(V)U-(\nabla_X\omega)(U)V]+\omega(X)[(\nabla_Y\omega)(V)U-(\nabla_Y\omega)(U)V]\\&+\omega(\rho)[\omega(Y)\{g(X,U)V-g(X,V)U\}-\omega(X)\{g(Y,U)V-g(Y,V)U\}]=0\end{aligned} \tag{1.3}$$

Now using the condition (0.9) it follows from (1.3)

$$\begin{aligned}(1.4)\quad &\omega(Y)[(\nabla_X\omega)(V)U-(\nabla_X\omega)(U)V]-\omega(X)[(\nabla_Y\omega)(V)U-(\nabla_Y\omega)(U)V]\\ &-\omega(\rho)[\omega(Y)\{g(X,U)V-g(X,V)U\}-\omega(X)\{g(Y,U)V-g(Y,V)U\}]=0\end{aligned}$$

Contracting U in (1.4) we obtain

$$(1.5)\quad \omega(X)(\nabla_Y\omega)(V)-\omega(Y)(\nabla_X\omega)(V)+\omega(\rho)[\omega(X)g(Y,V)-\omega(Y)g(X,V)]=0$$

Putting $Y=\rho$ in (1.5) we get

$$(1.6)\qquad (\nabla_X\omega)(Z)=\frac{\omega(X)}{\omega(\rho)}(\nabla_\rho\omega)(Z)-g(X,Z)\omega(\rho)+\omega(X)\omega(Z)$$

where we take $V=Z$. From (0.7) and (1.6) we get

$$\alpha(X,Z)=\frac{\omega(X)}{\omega(\rho)}(\nabla_\rho\omega)(Z)-\frac{1}{2}\omega(\rho)g(X,Z)$$

Now putting the value of $\alpha(X,Z)$ in (0.6) we obtain

$$(1.8)\quad \begin{aligned}{}'\tilde{R}(X,Y,Z,U)={}&{}'R(X,Y,Z,U)-\frac{1}{\omega(\rho)}[\omega(Y)g(X,U)(\nabla_\rho\omega)(Z)\\ &-\omega(X)g(Y,U)(\nabla_\rho\omega)(Z)+\omega(X)g(Y,Z)(\nabla_\rho\omega)(U)\\ &-\omega(Y)g(X,Z)(\nabla_\rho\omega)(U)]+\omega(\rho)[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)]\end{aligned}$$

where ${}'\tilde{R}(X,Y,Z,U)=g(\tilde{R}(X,Y)Z,U)$. Thus we can state:

THEOREM 1. *Let a Riemannian manifold admits a semi-symmetric metric connection* (0.1) *satisfying* (0.8) *and* (0.9). *Then the curvature tensor of the semi-symmetric metric connection has the form* (1.8). *If, in particular,* $\tilde{R}=0$, *then from* (0.6) *we get*

$$\begin{aligned}&{}'R(X,Y,Z,U)\\ &\quad=\alpha(Y,Z)g(X,U)-\alpha(X,Z)g(Y,U)+g(Y,Z)\alpha(X,U)-g(X,Z)\alpha(Y,U)\end{aligned}$$

Now putting $X=U=e_i$ in the above expression where $\{e_i\}$ is an orthonormal basis of the tangent space at any point and taking summation over $i\le i\le n$, we get

$$\begin{aligned}S(Y,Z)&=(n-1)\alpha(Y,Z)+\sum_i\alpha(e_i,e_i)g(Y,Z)-\alpha(Y,Z)\\ &=(n-2)\alpha(Y,Z)+\sum_i\alpha(e_i,e_i)g(Y,Z).\end{aligned}$$

Since S is symmetric, we get $\alpha(Y,Z) = \alpha(Z,Y)$. Hence from (0.7) we get $(\nabla_Y\omega)(Z) = (\nabla_Z\omega)(Y)$. Therefore $(\nabla_\rho\omega)(Y) = (\nabla_Y\omega)(\rho)$. From (1.6) we have

$$(\nabla_X\omega)(\rho) = \beta\omega(X) \tag{1.10}$$

where $\beta = (\nabla_\rho\omega)(\rho)/\omega(\rho)$. Now taking $\tilde{R} = 0$ and using (1.10) in (1.8) we get

$$\begin{aligned}{}'R(X,Y,Z,U) = \nu[&g(X,U)\omega(Y)\omega(Z) - g(Y,U)\omega(X)\omega(Z)\\ &+ g(Y,Z)\omega(X)\omega(U) - g(X,Z)w(Y)\omega(U)]\\ &+ \omega(\rho)[g(X,Z)g(Y,U) - g(Y,Z)g(X,U)]\end{aligned} \tag{1.11}$$

where

$$\nu = \frac{(\nabla_\rho\omega)(\rho)}{\omega(\rho)\omega(\rho)}$$

The expression (1.11) has been obtained by Prvanović and Pušić in [**7**].

According to Smaranda [**8**] a Riemannian manifold whose curvature tensor $'R$ is of the form (1.11) is said to be of almost constant curvature. In view of this we can state the following:

THEOREM 2. *If a Riemannian manifold admits a semi-symmetric metric connection* (0.1) *whose curvature tensor vanishes and satisfies the condition* (0.8)*, then the manifold is of almost constant curvature.*

Remarks. The conditions (0.8) and (0.9) of our paper are weaker than the conditions of [**4**] and also of [**7**], since it is known that in a Riemannian manifold $(\tilde{\nabla}_X T)(Y,Z) = B(X)T(Y,Z)$ where B is a 1-form, implies $\tilde{R}(X,Y).T = 0$ and $\tilde{R} = 0$ implies $\omega(\tilde{R}(X,Y)Z) = 0$, but the converse is not necessarily true in general. From (1.8) it can be easily seen that $'\tilde{R}$ satisfies the properties

$$'\tilde{R}(X,Y,Z,U) = -'\tilde{R}(Y,X,Z,U) \text{ and } '\tilde{R}(X,Y,Z,U) = -'\tilde{R}(X,Y,U,Z)$$

Also we get

$$'\tilde{R}(X,Y,Z,U) = {}'\tilde{R}(Z,U,X,Y), \tag{1.12}$$

$$'\tilde{R}(X,Y,Z,U) + {}'\tilde{R}(Y,Z,X,U) + {}'\tilde{R}(Z,X,Y,U) = 0 \tag{1.13}$$

if and only if

$$\omega(Y)(\nabla_\rho\omega)(Z) = \omega(Z)(\nabla_\rho\omega)(Y) \tag{1.14}$$

2. Symmetry condition of the Ricci tensor of $\tilde{\nabla}$. In this section necessary and sufficient conditions for the symmetry of the Ricci tensor of the semi-symmetric metric connection are obtained by proving the following:

THEOREM 3. *A necessary and sufficient condition for the Ricci tensor of the semi-symmetric metric connection $\tilde{\nabla}$ to be symmetric is that the* (0.4)*-curvature tensor* $'\tilde{R}$ *of the connection* $\tilde{\nabla}$ *satisfies either of the following two conditions:*
(i) $'\tilde{R}(X,Y,Z,U) = {}'\tilde{R}(Z,U,X,Y)$
(ii) $'\tilde{R}(X,Y,Z,U) + {}'\tilde{R}(Y,Z,X,U) + {}'\tilde{R}(Z,X,Y,U) = 0$

Proof. Let S and $\tilde{S}$ denote the Ricci tensors of the Levi-Civita connection and the semi-symmetric connection respectively. Putting $X = U = e_i$ in (1.8) we get

$$\tilde{S}(Y,Z) = S(Y,Z) - a(n-2)\omega(Y)(\nabla_\rho\omega)(Z) - ag(Y,Z)(\nabla_\rho\omega)(\rho) + (n-1)\omega(\rho)g(Y,Z) \tag{2.1}$$

where $a = 1/\omega(\rho)$. From (2.1) it follows that $\tilde{S}(Y,Z) = \tilde{S}(Z,Y)$ if and only if $\omega(Y)(\nabla_\rho\omega)(Z) = \omega(z)(\nabla_\rho\omega)(Y)$. But from (1.12), (1.13) and (1.14) we see that (1.12) and (1.13) hold if and only if (1.14) holds. Hence $\tilde{S}$ is symmetric if and only if either of the two conditions (1.12) and (1.13) hold. This completes the proof.

Using the above theorem we now prove the following:

THEOREM 4. *If a Riemannian manifold* (M^n, g) *admits a semi-symmetric metric connection* $\tilde{\nabla}$ *satisfying* (0.8) *and* (0.9) *whose curvature tensor is recurrent with associated 1-form* C *and symmetric Ricci tensor, then either* $C(\rho) = 2\omega(\rho)$ *or* $\tilde{R}(X,Y)Z = 0$.

Proof. Since $\omega(\tilde{R}(X,Y)Z) = 0$, we get

$$\tilde{R}(X,Y,Z,\rho) = 0 \tag{2.2}$$

Also since $\tilde{S}$ is symmetric, we get from Theorem 3

$$'\tilde{R}(X,Y,Z,U) = {}'\tilde{R}(Z,U,X,Y) \tag{2.3}$$

Putting $U = \rho$ in (2.3) and using (2.2) we get $'\tilde{R}(Z,\rho,X,Y) = 0$, that is

$$\tilde{R}(Z,\rho)X = 0 \tag{2.4}$$

Applying the Bianchi Second identity for the curvature tensor $\tilde{R}$ of the connection $\tilde{\nabla}$ we obtain

$$\begin{aligned} \tilde{R}(T(U,X),Y)Z + \tilde{R}(T,(X,Y),U)Z + \tilde{R}(T(Y,U),X)Z \\ + (\tilde{\nabla}_U\tilde{R})(X,Y)Z + (\tilde{\nabla}_X\tilde{R})(Y,U)Z + (\tilde{\nabla}_Y\tilde{R})(U,X)Z = 0 \end{aligned} \tag{2.5}$$

Since the curvature tensor is recurrent with associated 1-form C, then

$$(\tilde{\nabla}_X\tilde{R})(Y,Z)U = C(X)\tilde{R}(Y,Z)U \tag{2.6}$$

Now using (0.1) and (2.6) in (2.5), we find that
(2.7)
$$[C(U)-2\omega(U)]\tilde{R}(X,Y)Z+[C(Y)-2\omega(Y)]\tilde{R}(U,X)Z+[C(X)-2\omega(X)]\tilde{R}(Y,U)Z=0$$

for the vector fields X, Y, Z and U.

Putting $U=\rho$ in (2.7) and using (2.4) we find

$$[C(\rho)-2\omega(\rho)]\tilde{R}(X,Y)Z=0$$

Thus either $C(\rho)=2\omega(\rho)$ or $\tilde{R}(X,Y)Z=0$. This completes the proof of the theorem.

If, in particular, the 1-form $C=0$, then it follows from (2.7) that $\tilde{R}(X,Y)Z=0$ or $\omega(\rho)=0$. If $\omega(\rho)=0$, then from (0.4) it follows that $\rho=0$, since g is positive definite. But $\rho=0$ would mean that $\tilde{\nabla}=\nabla$ and hence $\tilde{\nabla}$ would not be semi-symmetric. Hence $\tilde{R}(X,Y)Z=0$. But it is known [**1**] that if a Riemannian manifold (M^n,g) $(n>3)$ admits a semi-symmetric metric connection whose curvature tensor vanishes, then the manifold is conformally flat. Hence we can state the following corollary.

COROLLARY. *If a Riemannian manifold* (M^n,g) $(n>3)$ *admits a semi-symmetric metric connection satisfying* (0.8) *and* (0.9) *whose curvature tensor is covariant constant and Ricci tensor is symmetric, then the manifold is conformally flat.*

3. Existence of a torse-forming vector field. In this section we consider a Riemannian manifold (M^n,g) $(n>3)$ that admits a semi-symmetric metric connection $\tilde{\nabla}$ whose Ricci tensor is symmetric and satisfies the conditions (0.8) and (0.9). It is shown that if a Riemannian manifold admits such a connection, then the manifold admits a torse-forming vector field [**9**].

If the connection (0.1) satisfies the conditions (0.8) and (0.9), then we get from (1.6)

$$(\nabla_X\omega)(Y)=\frac{\omega(X)}{\omega(\rho)}(\nabla_\rho\omega)(Y)-g(X,Y)\omega(\rho)+\omega(X)\omega(Y) \tag{3.1}$$

Since $\tilde{S}$ is symmetric we get from Theorem 3

$$\omega(Y)(\nabla_\rho\omega)(X)=\omega(X)(\nabla_\rho\omega)(Y) \tag{3.2}$$

Putting $Y=\rho$ in (3.2) we get

$$(\nabla_\rho\omega)(X)=\beta\omega(X) \tag{3.3}$$

where $\beta=(\nabla_\rho\omega)(\rho)/\omega(\rho)$. Using (3.3) in (3.1) we obtain

$$(\nabla_X\omega)(Y)=(\nu+1)\omega(X)\omega(Y)-g(X,Y)\omega(\rho)$$

Hence $(\nabla_X \omega)(Y) = f\omega(X)\omega(Y) + hg(X,Y)$ where f and h are scalars. Thus we get the following:

THEOREM 5. *If a Riemannian manifold admits a semi-symmetric metric connection $\tilde{\nabla}$ with symmetric Ricci tensor and satisfies the conditions (8) and (9), then the manifold always admits a torse-forming vector field.*

REFERENCES

[1] K. Yano, *On semi-symmetric connection*, Rev. Roumaine Math. Pure Appl. **15** (1970), 1479–1586.

[2] K. Yano and T. Imai, *On semi-symmetric metric F-connection*, Tensor **29** (1975), 134–138.

[3] O.C. Andonie, *Sur Une connexion semi-symtrique qui laisse invariant la tenseur de Bochner*, Ann. Fac. Sci. Kuishawa Zaire **1** (1976), 247–253.

[4] M.C. Chaki and A. Konar, *On a type of semi-symmetric connection on a Riemannian manifold*, J. Pure Math. **1** (1981), 77–80.

[5] M. Prvanović, *Product semi-symmetric connection of the locally decompossible Riemannian space*, Bull. Acad. Serbe Sci. Art. Cl. Sci. Math. Natur. **64** (1979), 17–27.

[6] U.C. De, *On a type of semi-symmetric metric connection on a Riemannian manifold*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Math. **38** (1991), 105–108.

[7] M. Prvanović and N. Pušić, *On manifolds admitting some semi-symmetric metric connection*, Indian J. Math. **37** (1995), 37–67.

[8] D. Smaranda, *Pseudo Riemannian recurrent manifolds with almost constant curvature*, The XVIII Int. Conf. on Geometry and Topology (Oradea, 1987) 4–7, Preprint 88–2 Univ. "Babes Balyai" Cluj-Napoca, 1988. 175–180.

[9] J.A. Schouten, *Ricci Calculus*, Springer-Verlag, Berlin, 1954.

Department of Mathematics
University of Kalyani
Kalyani 741235
West Bengal, India

(Received 17 09 1996)