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SEMI-FREDHOLM OPERATORS AND PERTURBATIONS

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Abstract. It is well known that the set of semi-Fredholm operators is an open semigroup in the set of all bounded linear operators on Banach spaces [**3**]. Perturbations theorems for semi-Fredholm operators are of great interest (see e.g. [**3**], [**4**], [**6**], [**9**], [**13**], [**14**], [**15**] and [**20**]). The main results is a general perturbation theorem for semi-Fredholm operators. Then as a corollary we get some well known results of [**6**] and [**7**].

1. Introduction and preliminaries

In this paper X and Y are complex Banach spaces, $B(X,Y)$ ($K(X,Y)$) the set of all bounded (compact) linear operators from X into Y. We shall write $B(X)$ $(K(X))$ instead of $B(X,X)$ $(K(X,X))$.

An operator $T \in B(X,Y)$ is in $\Phi_+(X,Y)$ $(\Phi_-(X,Y))$ if the range $R(T)$ is closed in Y and the dimension $\alpha(T)$ of the null space $N(T)$ of T is finite (the codimension $\beta(T)$ of $R(T)$ in Y is finite). Operators in $\Phi_+(X,Y) \cup \Phi_-(X,Y)$ are called semi-Fredholm operators. For such operators the index is defined by $i(T) = \alpha(T) - \beta(T)$. We set $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$. The operators in $\Phi(X,Y)$ are called Fredholm operators. We shall write $\Phi_+(X)$ (resp. $\Phi_-(X)$, $\Phi(X)$) instead of $\Phi_+(X,X)$ (resp. $\Phi_-(X,X)$, $\Phi(X,X)$).

Since index is locally constant (see [**3**, Theorems (4.2.1), (4.2.2), (4.4.1)]) we have

Lemma 1. *Let A, $B \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$ and f be a continuous map from $[0,1]$ into $B(X,Y)$ such that $f(0) = A$, $f(1) = B$ and $f([0,1]) \subset \Phi_+(X,Y) \cup \Phi_-(X,Y)$; then $i(A) = i(B)$.*

Let U denote the closed unit ball of X. Let $T \in B(X,Y)$ and

$$m(T) = \inf\{\|Tx\| : \|x\| = 1\}$$

AMS Subject Classification (1991): Primary 47A10, 47A53, 47A55 be the *minimum modulus* of T, and let

$$n(T) = \sup\{\epsilon \geq 0 \,:\, \epsilon U \subset TU\}$$

be the *surjection modulus* of T.

Obviously $m(T) > 0$ if and only if there is a number $c > 0$ such that $c\|x\| \leq \|Tx\|, x \in X$, and in this case we say that operator T is a bounded below. It is well known that $m(T) > 0$ if and only if the null space of T is zero and the range of T is closed, and $n(T) > 0$ if and only if T is surjective.

Further, for T, $S \in B(X,Y)$ we have

$$m(T+S) \leq m(T) + \|S\|$$

and analogously

$$n(T+S) \leq n(T) + \|S\|.$$

It is well known that if an operator T is bounded below (surjective) and the norm of a perturbation S is smaller than $m(T)$ $(n(T))$, then $T+S$ is bounded below (surjective). Namely,

$$\begin{aligned} m(T) = m(T+S-S) &\leq m(T+S) + \|S\| < m(T+S) + m(T) \\ &\Rightarrow m(T+S) > 0. \end{aligned}$$

Obviously a bounded below operator is Φ_+ and a surjective operator is Φ_-.

An operator $T \in B(X,Y)$ is *strictly singular* ($T \in S(X,Y)$) if, for every infinite dimensional (closed) subspace M of X, the restriction of T to M, $T|_M$, is not a homeomorphism, i.e., $m(T|_M) = 0$. An operator $T \in B(X,Y)$ is *strictly cosingular* ($T \in CS(X,Y)$) if, for every infinite codimensional closed subspace V of Y the composition $Q_V T$ is not surjective, where Q_V is the quotient map from Y onto Y/V, i.e., $n(Q_V T) = 0$. It is well known that $K(X,Y) \subset S(X,Y)$ and $K(X,Y) \subset CS(X,Y)$.

Let S be a subset of a Banach space A. The perturbation class associated with S is denoted $P(S)$ and $P(S) = \{a \in A \,:\, a + s \in S \text{ for all } s \in S\}$. The perturbation class associated with $\Phi_+(X,Y)$ (resp. $\Phi_+(X)$, $\Phi_-(X,Y)$, $\Phi_-(X)$) is denoted by $P(\Phi_+(X,Y))$ (resp. $P(\Phi_+(X))$, $P(\Phi_-(X,Y))$, $P(\Phi_-(X))$).

For $T \in B(X,Y)$, we set (see [**18**], [**19**])

$$m_e(T) = \operatorname{dist}(T,\, B(X,Y) \backslash \Phi_+(X,Y))$$

for the *essential minimum modulus* and

$$n_e(T) = \operatorname{dist}(T,\, B(X,Y) \backslash \Phi_-(X,Y))$$

for the *essential surjection modulus*.

For $T \in B(X)$, the quantities

$$s_+(T) = \sup\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow \lambda I - T \in \Phi_+(X)\}$$
$$s_-(T) = \sup\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow \lambda I - T \in \Phi_-(X)\}$$

are *semi-Fredholm radii* of the operator T (see [**18**], [**19**]).

We shall use π to denote the natural homomorphism of $B(X)$ onto the Calkin algebra $C(X) = B(X)/K(X)$. $C(X)$ is itself a Banach algebra in the quotient algebra norm

$$\|\pi(T)\| = \inf\{\|T + K\| \,:\, K \in K(X)\}.$$

Let $r_e(T)$ denote spectral radius of the element $\pi(T)$ in $C(X)$, $T \in B(X)$, i.e., $r_e(T) = \lim_{n\to\infty} (\|\pi(T^n)\|)^{\frac{1}{n}}$ and it is called *essential spectral radius* of T. Recall that $r_e(T) = \sup\{|\lambda| \,:\, \lambda I - T \notin \Phi(X)\}$ (see [**3**]). An operator $T \in B(X)$ is *Riesz* operator if and only if $r_e(T) = 0$ [**3**, Theorem 3.3.1]. Let $R(X)$ denote the set of Riesz operators in $B(X)$.

2. Results

If $f : B(X,Y) \mapsto [0,\infty)$, set $N(f) = \{T \in B(X,Y) \,:\, f(T) = 0\}$. The main result in this paper is the following perturbation theorem.

THEOREM 1. *Let f be a seminorm on $B(X,Y)$, and $h : B(X,Y) \mapsto [0,\infty)$ a function such that for $A,\ B \in B(X,Y)$*

(1) $$h(A) > 0 \iff A \in \Phi_+(X,Y),$$

(2) $$h(A + B) \leq h(A) + f(B),$$

(3) $$K(X,Y) \subset N(f) \text{ and } f(A) \leq \|A\|;$$

then:

(*a*) *$h(A + C) = h(A)$ for all $C \in N(f)$;*

(*b*) *If $f(B) < h(A)$, then $A,\ A + B \in \Phi_+(X,Y)$ and $i(A) = i(A + B)$;*

(*c*) *$N(f)$ is closed subspace of $B(X,Y)$ and $N(f) \subset P(\Phi_+(X,Y))$;*

(*d*) *If $\|B\| < h(A)$, then $A,\ A + B \in \Phi_+(X,Y)$ and $i(A + B) = i(A)$;*

(*e*) *$m_e(A) \geq h(A)$.*

For $A \in B(X)$ we have

(*f*) *$s_+(A) \geq h(A)$;*

(*g*) *$s_+(A) \geq \overline{\lim}_{n\to\infty} (h(A^n))^{\frac{1}{n}}$;*

(*h*) *If $f(A) < h(I)$, then $I - A \in \Phi(X)$ and $i(I - A) = 0$;*

(*i*) *If $f(A^n) < h(I)$ for some $n > 1$, then $I - A \in \Phi(X)$ and $i(I - A) = 0$;*

(*j*) *$r_e(A) = \lim_{n\to\infty} (f(A^n))^{\frac{1}{n}}$;*

(*k*) *If* $AB - BA \in P(\Phi_+(X))$ *and* $r_e(B) < \overline{\lim}_{n\to\infty}(h(A^n))^{\frac{1}{n}}$, *then* $A,\ A + B \in \Phi_+(X)$ *and* $i(A + B) = i(A)$.

Proof: (a) Let $C \in N(f)$. By (2) we have

$$h(A + C) \le h(A) + f(C) = h(A),$$
$$h(A) = h(A + C + (-C)) \le h(A + C) + f(-C) = h(A + C)$$

and hence $h(A) = h(A + C)$.

(b) Let $f(B) < h(A)$ and $\lambda \in [0, 1]$. By (2) we have

$$h(A) = h(A + \lambda B + (-\lambda B)) \le h(A + \lambda B) + f(-\lambda B) = h(A + \lambda B) + \lambda f(B) < \\ < h(A + \lambda B) + h(A),$$

and hence $h(A + \lambda B) > 0$. Further, by (1) it follows that $A + \lambda B \in \Phi_+(X, Y)$ and hence $A,\ A + B \in \Phi_+(X, Y)$. Now by Lemma 1 we have $i(A + B) = i(A)$.

(c) Let $A,\ B \in N(f)$ and $\lambda,\ \mu \in \mathbb{C}$. Since f is a seminorm on $B(X, Y)$ it follows that

$$0 \le f(\lambda A + \mu B) \le f(\lambda A) + f(\mu B) = |\lambda| f(A) + |\mu| f(B) = 0 \Longrightarrow \\ f f(\lambda A + \mu B) = 0 \Longrightarrow \lambda A + \mu B \in N(f).$$

So $N(f)$ is a subspace of $B(X, Y)$.

Let $A_n \in N(f)$, $n \in \mathbb{N}$ and $A \in B(X, Y)$ such that $\|A_n - A\| \to 0$ when $n \to \infty$. Then

$$0 \le f(A) = f(A - A_n + A_n) \le f(A - A_n) + f(A_n) = f(A - A_n) \le \|A_n - A\|.$$

It follows that $f(A) = 0$, so $A \in N(f)$. Hence $N(f)$ is closed.

Let $A \in \Phi_+(X, Y)$ and $B \in N(f)$. By (1) it follows that $f(B) = 0 < h(A)$. Now by (b) we have $A + B \in \Phi_+(X, Y)$. Hence $B \in P(\Phi_+(X, Y))$, and (c) is proved.

(d) Let $\|B\| < h(A)$. By (3) $f(B) \le \|B\|$ and this implies $f(B) < h(A)$. Now by (b) we get $A,\ A + B \in \Phi_+(X, Y)$ and $i(A + B) = i(A)$.

(e) Since $m_e(A) = \max\{\epsilon \ge 0 : \|B\| < \epsilon \Rightarrow A + B \in \Phi_+(X, Y)\}$, (d) implies (e).

(f) Obviously $s_+(A) \ge m_e(A)$ and hence (f) follows from (e).

(g) It is known that $s_+(A^n) = [s_+(A)]^n$, $n \in \mathbb{N}$. Hence by (f) we have $s_+(A) = (s_+(A^n))^{\frac{1}{n}} \ge (h(A^n))^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. It implies (g).

(h) Let $f(A) < h(I)$. Now (b) implies $I - A \in \Phi_+(X)$ and $i(I - A) = i(I) = 0$. Hence $I - A \in \Phi(X)$.

(i) Let $f(A^n) < h(I)$ for some $n > 1$, and let $\lambda \in [0,1]$. Then $f((\lambda A)^n) = \lambda^n f(A^n) \le f(A^n) < h(I)$ and by (h) it follows that $I - (\lambda A)^n \in \Phi(X)$. Since

$$\begin{aligned} I - (\lambda A)^n &= (I - \lambda A)(I + \lambda A + \ \dots\ + \lambda^{n-1}A^{n-1}) \\ &= (I + \lambda A + \ \dots\ + \lambda^{n-1}A^{n-1})(I - \lambda A) \end{aligned}$$

by [**3**, Corollary 1.3.6] we have $I - \lambda A \in \Phi(X)$. Hence $I - A \in \Phi(X)$. Further, by Lemma 1 we get $i(I - A) = i(I) = 0$.

(j) Let $\lambda \in \mathbb{C}$ and $|\lambda| > (h(I))^{-\frac{1}{n}}(f(A^n))^{\frac{1}{n}}$ for some $n \in \mathbb{N}$. Then $h(I) > f((A/\lambda)^n)$ and by (i) it follows $I - A/\lambda \in \Phi(X)$, i.e., $\lambda I - A \in \Phi(X)$. Hence $r_e(A) \le (h(I))^{-\frac{1}{n}}(f(A^n))^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. This implies

$$r_e(A) \le \lim_{n\to\infty} (h(I))^{-\frac{1}{n}} \varliminf_{n\to\infty} (f(A^n))^{\frac{1}{n}} = \varliminf_{n\to\infty} (f(A^n))^{\frac{1}{n}} .$$

From (3) it follows that for all $T \in B(X)$ and $K \in K(X)$

$$\begin{aligned} &f(T + K) \le f(T) + f(K) = f(T), \\ &f(T) = f(T + K + (-K)) \le f(T + K) + f(-K) = f(T + K), \end{aligned}$$

so that $f(T) = f(T + K) \le \|T + K\|$. Thus

$$f(T) \le \inf\{\|T + K\| \,:\, K \in K(X)\} = \|\pi(T)\|.$$

Hence

$$r_e(A) \le \varliminf_{n\to\infty} (f(A^n))^{\frac{1}{n}} \le \varlimsup_{n\to\infty} (f(A^n))^{\frac{1}{n}} \le \lim_{n\to\infty} (\|\pi(A^n)\|)^{\frac{1}{n}} = r_e(A),$$

and we get (j). (k) Let $AB - BA \in P(\Phi_+(X))$ and $r_e(B) < \varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}}$. Let ϵ be such that $r_e(B) < \epsilon < \varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}}$. By (j) we have $\lim_{n\to\infty} (f(B^n))^{\frac{1}{n}} < \epsilon < \varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}}$. Hence there exists $n \in \mathbb{N}$ such that $(f(B^n))^{\frac{1}{n}} < \epsilon < (h(A^n))^{\frac{1}{n}}$, i.e., $f(B^n) < h(A^n)$. From (b) it follows $A^n - B^n \in \Phi_+(X)$. Since $P(\Phi_+(X))$ is a two sided ideal of $B(X)$ (see [**3**, Lemma 5.5.5]), from $AB - BA \in P(\Phi_+(X))$ we get $A^n - B^n = C(A - B) + P$, where $C = A^{n-1} + BA^{n-2} + \cdots + B^{n-1}$ and $P \in P(\Phi_+(X))$. Thus, $C(A - B) \in \Phi_+(X)$, and by [**3**, Corollary 1.3.4] we get $A - B \in \Phi_+(X)$. Let us remark that from our proof, it follows that $A + \lambda B \in \Phi_+(X)$ for $0 \le \lambda \le 1$. Now by Lemma 1, we have $i(A + B) = i(A)$. $\square$

Remark 1. Let us remark that we can get (g) as a consequence of (k). If $\varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}} = 0$, then the inequality (g) obviously holds. Suppose that $\varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}} > 0$. For $\lambda \in \mathbb{C}$, let $|\lambda| < \varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}}$ and $B = \lambda I$. Then we have $r_e(B) = |\lambda| < \varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}}$ and $AB = BA$. By (k) we have $\lambda I - A \in \Phi_+(X)$. Therefore $s_+(A) \ge \varlimsup_{n\to\infty} (h(A^n))^{\frac{1}{n}}$.

The next theorem is a dual part of Theorem 1. We omit the proof.

THEOREM 1'. *Let f be a seminorm on $B(X,Y)$, and $h : B(X,Y) \mapsto [0,\infty)$ a function such that for $A,\ B \in B(X,Y)$*

$$h(A) > 0 \iff A \in \Phi_-(X,Y), \tag{1}$$

$$h(A+B) \le h(A) + f(B), \tag{2}$$

$$K(X,Y) \subset N(f) \text{ and } f(A) \le \|A\|, \tag{3}$$

then:

(a) $h(A+C) = h(A)$ *for all* $C \in N(f)$*;*

(b) *If* $f(B) < h(A)$*, then* $A,\ A+B \in \Phi_-(X,Y)$ *and* $i(A) = i(A+B)$*;*

(c) $N(f)$ *is closed subspace of* $B(X,Y)$ *and* $N(f) \subset P(\Phi_-(X,Y))$*;*

(d) *If* $\|B\| < h(A)$*, then* $A,\ A+B \in \Phi_-(X,Y)$ *and* $i(A+B) = i(A)$*;*

(e) $n_e(A) \ge h(A)$.

For $A \in B(X)$ *we have*

(f) $s_-(A) \ge h(A)$*;*

(g) $s_-(A) \ge \overline{\lim\limits_{n\to\infty}}\,(h(A^n))^{\frac{1}{n}}$*;*

(h) *If* $f(A) < h(I)$*, then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$*;*

(i) *If* $f(A^n) < h(I)$ *for some* $n > 1$*, then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$*;*

(j) $r_e(A) = \lim\limits_{n\to\infty}(f(A^n))^{\frac{1}{n}}$*;*

(k) *If* $AB - BA \in P(\Phi_-(X))$ *and* $r_e(B) < \overline{\lim}_{n\to\infty}(h(A^n))^{\frac{1}{n}}$*, then* $A,\ A+B \in \Phi_-(X)$ *and* $i(A+B) = i(A)$.

Set

$$\Phi_+^-(X,Y) = \{T \in \Phi_+(X,Y)\ :\ i(T) \le 0\},$$
$$\Phi_-^+(X,Y) = \{T \in \Phi_-(X,Y)\ :\ i(T) \ge 0\}.$$

We shall write $\Phi_+^-(X)$ $(\Phi_-^+(X))$ instead of $\Phi_+^-(X,X)$ $(\Phi_-^+(X,X))$

For $A \in B(X,Y)$, set

$$m_{\Phi_+^-}(A) = \operatorname{dist}(A, B(X,Y)\backslash\Phi_+^-(X,Y)),$$
$$n_{\Phi_-^+}(A) = \operatorname{dist}(A, B(X,Y)\backslash\Phi_-^+(X,Y)),$$

and

$$s_+^-(A) = \sup\{\epsilon \ge 0\ :\ |\lambda| < \epsilon \Longrightarrow \lambda I - A \in \Phi_+^-(X)\},$$
$$s_-^+(A) = \sup\{\epsilon \ge 0\ :\ |\lambda| < \epsilon \Longrightarrow \lambda I - A \in \Phi_-^+(X)\}.$$

Let us remark that $m_e(A) \geq m_{\Phi_+^-}(A)$ ($n_e(A) \geq n_{\Phi_-^+}(A)$) and if $m_{\Phi_+^-}(A) > 0$ ($n_{\Phi_-^+}(A) > 0$), then $m_e(A) = m_{\Phi_+^-}(A)$ ($n_e(A) = n_{\Phi_-^+}(A)$) (because index is locally constant). Also $s_+(A) \geq s_+^-(A)$ ($s_-(A) \geq s_-^+(A)$) and if $s_+^-(A) > 0$ ($s_-^+(A) > 0$), then $s_+(A) = s_+^-(A)$ ($s_-(A) = s_-^+(A)$).

Let us remark that $\Phi_+^-(X,Y)$ ($\Phi_-^+(X,Y)$) is an open subset of $\Phi_+(X,Y)$ ($\Phi_-(X,Y)$) and that $\Phi_+(X,Y)$ ($\Phi_-(X,Y)$) does not contain any boundary point of $\Phi_+^-(X,Y)$ ($\Phi_-^+(X,Y)$) (because index is locally constant). By [**3**, Lemma 5.5.4] it follows that $P(\Phi_+(X,Y)) \subset P(\Phi_+^-(X,Y))$ ($P(\Phi_-(X,Y)) \subset P(\Phi_-^+(X,Y))$). Rakočević proved in [**10**] that $P(\Phi_+(X)) = P(\Phi_+^-(X))$ ($P(\Phi_-(X)) = P(\Phi_-^+(X))$). We set the following question: does the equality $P(\Phi_+(X,Y)) = P(\Phi_+^-(X,Y))$ ($P(\Phi_-(X,Y)) = P(\Phi_-^+(X,Y))$) hold?

Analogously as Theorem 1 the following two theorems can be proved.

THEOREM 2. *Let f be a seminorm on $B(X,Y)$, and $h : B(X,Y) \mapsto [0,\infty)$ a function such that for $A,\ B \in B(X,Y)$*

(1) $$h(A) > 0 \iff A \in \Phi_+^-(X,Y),$$

(2) $$h(A+B) \leq h(A) + f(B),$$

(3) $$K(X,Y) \subset N(f) \text{ and } f(A) \leq \|A\|,$$

then:

(a) $h(A+C) = h(A)$ *for all* $C \in N(f)$*;*

(b) *If* $f(B) < h(A)$*, then* $A,\ A+B \in \Phi_+(X,Y)$ *and* $i(A) = i(A+B) \leq 0$*;*

(c) $N(f)$ *is closed subspace of* $B(X,Y)$ *and* $N(f) \subset P(\Phi_+^-(X,Y))$*;*

(d) *If* $\|B\| < h(A)$*, then* $A,\ A+B \in \Phi_+(X,Y)$ *and* $i(A+B) = i(A) \leq 0$*;*

(e) $m_{\Phi_+^-}(A) \geq h(A)$.

For $A \in B(X)$ *we have*

(f) $s_+^-(A) \geq h(A)$*;*

(g) $s_+^-(A) \geq \overline{\lim\limits_{n\to\infty}} (h(A^n))^{\frac{1}{n}}$*;*

(h) *If* $f(A) < h(I)$*, then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$*;*

(i) *If* $f(A^n) < h(I)$ *for some* $n > 1$*, then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$*;*

(j) $r_e(A) = \lim\limits_{n\to\infty} (f(A^n))^{\frac{1}{n}}$*;*

(k) *If* $AB - BA \in P(\Phi_+(X))$ *and* $r_e(B) < \overline{\lim}_{n\to\infty} (h(A^n))^{\frac{1}{n}}$*,*
then $A,\ A+B \in \Phi_+(X)$ *and* $i(A+B) = i(A) \leq 0$.

THEOREM 2′. *Let f be a seminorm on $B(X,Y)$, and $h : B(X,Y) \mapsto [0,\infty)$ a function such that for $A,\ B \in B(X,Y)$*

(1) $$h(A) > 0 \iff A \in \Phi_-^+(X,Y),$$

(2) $$h(A+B) \leq h(A) + f(B),$$

(3) $$K(X,Y) \subset N(f) \text{ and } f(A) \leq \|A\|,$$

then:

(a) $h(A+C)=h(A)$ *for all* $C\in N(f)$;

(b) *If* $f(B)<h(A)$, *then* $A,\ A+B\in\Phi_-(X,Y)$ *and* $i(A)=i(A+B)\geq 0$;

(c) $N(f)$ *is closed subspace of* $B(X,Y)$ *and* $N(f)\subset P(\Phi_+^-(X,Y))$;

(d) *If* $\|B\|<h(A)$, *then* $A,\ A+B\in\Phi_-(X,Y)$ *and* $i(A+B)=i(A)\geq 0$;

(e) $n_{\Phi_-^+}(A)\geq h(A)$.

For $A\in B(X)$ *we have*

(f) $s_-^+(A)\geq h(A)$;

(g) $s_-^+(A)\geq \overline{\lim}_{n\to\infty}(h(A^n))^{\frac{1}{n}}$;

(h) *If* $f(A)<h(I)$, *then* $I-A\in\Phi(X)$ *and* $i(I-A)=0$;

(i) *If* $f(A^n)<h(I)$ *for some* $n>1$, *then* $I-A\in\Phi(X)$ *and* $i(I-A)=0$;

(j) $r_e(A)=\lim_{n\to\infty}(f(A^n))^{\frac{1}{n}}$;

(k) *If* $AB-BA\in P(\Phi_-(X))$ *and* $r_e(B)<\overline{\lim}_{n\to\infty}(h(A^n))^{\frac{1}{n}}$, *then* $A,\ A+B\in\Phi_-(X)$ *and* $i(A+B)=i(A)\geq 0$.

Now we shall list several examples of known functions, which satisfy the conditions of Theorem 1, Theorem 1', Theorem 2 or Theorem 2'.

Examples. 1. For $A\in B(X,Y)$ set

$$\|A\|_C=\inf\{\|A+K\|\,:\,K\in K(X,Y)\},$$
$$m_C(A)=\sup\{m(A+K)\,:\,K\in K(X,Y)\}\quad (\text{ see } [\mathbf{8}])$$
$$n_C(A)=\sup\{n(A+K)\,:\,K\in K(X,Y)\}.$$

The functions $\|\cdot\|_C$ and m_C $(\|\cdot\|_C$ and $n_C)$ satisfy the conditions of Theorem 2 (Theorem 2') (see [**17**]).

2. The functions $\|\cdot\|_C$ and m_e $(\|\cdot\|_C$ and $n_e)$ satisfy the conditions of Theorem 1 (Theorem 1') (see [**19**, Proposition 1]).

3. If Ω is a nonempty subset of X, then the Hausdorff measure of noncompactness of Ω, is denoted by $q(\Omega)$, and $q(\Omega)=\inf\{\epsilon>0\,:\,\Omega$ has a finite ϵ-net in $X\}$. For $A\in B(X,Y)$ the Hausdorff measure of noncompactness of A, denoted by $\|A\|_q$, is defined by

$$\|A\|_q=\inf\{k\geq 0\,:\,q_Y(A\Omega)\leq kq_X(\Omega),\,\Omega\subset X\text{ is bounded.}\}$$

It is easy to see that

$$\|A\|_q=\sup\{q_Y(A\Omega)\,:\,\Omega\subset X,\,q_X(\Omega)=1\}.$$

Set (see [**7**])

$$m_q(A)=\inf\{q_Y(A\Omega)\,:\,\Omega\subset X,\,q_X(\Omega)=1\}.$$

The functions $\|\cdot\|_q$ and m_q satisfy the conditions of Theorem 1 (see [**7**, Theorem 4.10], [**1**, p. 73] or [**11**, Posledica 2.12.12]). Fainstein [**4**] proved that

$$\|A\|_q = \inf\{\|Q_N A\| \,:\, N \text{ finite-dimensional subspace of } Y\},$$

where Q_N is the quotient map from Y into Y/N.

Set (see [**4**] and [**20**])

$$n_q(A) = \sup\{n(Q_N A) \,:\, N \text{ finite-dimensional subspace of } Y\}.$$

The functions $\|\cdot\|_q$ and n_q satisfy the conditions of Theorem 1' (see [**20**, Theorem 4.1]).

Let us remark that Theorem 4 in [**6**] follows from Theorem 1 (Theorem 1').

4. For $A \in B(X,Y)$ set

$$\|A\|_\mu = \inf\{\|A|_L\| \,:\, L \text{ subspace of } X,\ \operatorname{codim} L < \infty\},$$

and

$$m_\mu(A) = \sup\{m(A|_L) \,:\, L \text{ subspace of } X,\ \operatorname{codim} L < \infty\}.$$

We conclude that the functions $\|\cdot\|_\mu$ and m_μ satisfy the conditions of Theorem 1 (see [**7**] and [**13**, Lemma 2.13]). Hence Theorem 6.1 in [**7**] follows from Theorem 1.

5. Let $l_\infty(X)$ be the Banach space obtained from the space of all bounded sequences $x = (x_n)$ in X by imposing term-by-term linear combination and the supremum norm $\|x\| = \sup_n \|x_n\|$. Let $m(X)$ stand for the closed subspace

$$\{(x_n) \in l_\infty(X) \,:\, \{x_n : n \in \mathbf{N}\} \text{ relatively compact in } X\}$$

of $l_\infty(X)$. Let X^+ denote the quotient $l_\infty/m(X)$. Then $A \in B(X,Y)$ induces an operator $A^+ : X^+ \mapsto Y^+$, $(x_n) + m(X) \mapsto (Ax_n) + m(Y)$, $(x_n) \in l_\infty(X)$. The function $A \mapsto \|A^+\|$ is a measure of noncompactness, i.e., it is a seminorm on $B(X,Y)$ such that $\|A^+\| = 0 \Longleftrightarrow A \in K(X,Y)$ (see [**1**] and [**2**]).

The functions $A \mapsto \|A^+\|$ and $A \mapsto m(A^+)$ ($A \mapsto \|A^+\|$ and $A \mapsto n(A^+)$) satisfy the conditions of Theorem 1 (Theorem 1') (see [**2**, Theorem 2] and [**5**, Theorem 3.4]).

6. For $A \in B(X,Y)$ set

$$G_M(A) = \inf_{N \subset M} \|A|_N\|,\quad G(A) = G_X(A),\ \Delta_M(A) = \sup_{N \subset M} G_N(A),\quad \Delta(A) = \Delta_X(A),$$

where M, N denotes infinite dimensional subspaces of X

We conclude that the function Δ and G satisfy the conditions of Theorem 1 (see [**13**]).

Weis [**16**] introduced for $A \in B(X,Y)$ the following functions

$$K_V(A) = \inf_{W \supset V} \|Q_W A\|\,, \quad K(A) = K_{\{0\}}(A)\,,$$
$$\nabla_V(A) = \sup_{W \supset V} K_W(A)\,, \quad \nabla(A) = \nabla_{\{0\}}(A)\,,$$

where $V,\ W$ denote closed infinite codimensional subspaces of Y (we use the notations from [**20**]). It is not difficult to show that the functions ∇ and K satisfy the conditions of Theorem 1'.

Schechter [**13**] proved that $\Delta(A) \le \|A\|_\mu$, and similarly it can be proved that $\nabla(A) \le \|A\|_q,\ A \in B(X,Y)$. Therefore, the functions $\|\cdot\|_\mu$ and G ($\|\cdot\|_q$ and K) satisfy the conditions of Theorem 1 (Theorem 1').

7. For $A \in B(X,Y)$ set (see [**9**] and [**10**])

$$t_M(A) = \inf_{N \subset M} \|A|_N\|_q\,, \qquad t(A) = t_X(A),$$
$$g_M(A) = \sup_{N \subset M} t_N(A)\,, \qquad g(A) = g_X(A),$$

where $M,\ N$ denote infinite dimensional subspaces of X.

We conclude that the functions g and t satisfy the conditions of Theorem 1.

Remark 2. From the proof of Theorem 1 it is clear that if we replace the condition (2) of Theorem 1 ((2) of Theorem 1') by a weaker condition:

$(2')$ If $f(B) < h(A)$, then $A + B \in \Phi_+(X,Y)$

($(2')$ If $f(B) < h(A)$, then $A + B \in \Phi_-(X,Y)$),

then we can prove the assertions (c)–(k) of Theorem 1 (Theorem 1'). Zemánek [**20**] considered the following functions

$$u(A) = \sup\{m(A|_W) : W \text{ is closed subspace of } X \text{ with } \dim W = \infty\},$$
$$v(A) = \sup\{n(Q_V A) : V \text{ is closed subspace of } Y \text{ with } \operatorname{codim} V = \infty\}.$$

From the definition of strictly singular and strictly cosingular operators it is obvious that $u(A) = 0$ if and only if $A \in S(X,Y)$, and $v(A) = 0$ if and only if $A \in CS(X,Y)$. Zemánek denoted the quantities m_μ and n_q with B and M, respectively and proved: If $T,\ S \in B(X,Y)$ and $v(S) < M(T)$, then $T + S$ is a Φ_-–operator, and if $u(S) < B(T)$, then $T + S$ is a Φ_+–operator. Now it is clear that the functions u and B (v and M) satisfy the conditions (1), $(2')$ and (3) of Theorem 1 (Theorem 1').

The quantities $m_C,\ m_q,\ m_\mu,\ m(\cdot^+),\ m_e,\ G,\ t,\ \Delta',\ g'$ may be considered as substitutes for the minimum modulus of an operator and $n_C,\ n_q,\ n(\cdot^+),\ n_e,\ K,\ \nabla'$ as substitutes for the surjection modulus. Also we can say that measures of noncompactness $\|\cdot\|_C,\ \|\cdot\|_q,\ \|\cdot\|_\mu,\ \|\cdot^+\|$ generalize norm. Further, the quantities

Δ, g, u and ∇, v generalize measures of noncompactness in the same way as strictly singular and strictly cosingular operators generalize compact operators.

Let us introduce the following functions for $T \in B(X,Y)$:

$$\|T\|_{P\Phi_+} = \inf\{\|T-B\| \,:\, B \in P(\Phi_+(X,Y))\}\,,$$
$$\|T\|_{P\Phi_-} = \inf\{\|T-B\| \,:\, B \in P(\Phi_-(X,Y))\}\,.$$

Clearly $\|\cdot\|_{P\Phi_+}$ ($\|\cdot\|_{P\Phi_-}$) is seminorm on $B(X,Y)$ with property $\|T\|_{P\Phi_+} \le \|T\|$ ($\|T\|_{P\Phi_-} \le \|T\|$), $T \in B(X,Y)$. Since $P(\Phi_+(X,Y))$ ($P(\Phi_-(X,Y))$) is a closed set [**3**, Lemma 5.5.3] the function $\|\cdot\|_{P\Phi_+}$ ($\|\cdot\|_{P\Phi_-}$) disappears on $P(\Phi_+(X,Y))$ ($P(\Phi_-(X,Y))$). Since $K(X,Y) \subset P(\Phi_+(X,Y))$ ($P(\Phi_-(X,Y))$) [**3**, Corollary 1.3.7] we conclude that the functions $\|\cdot\|_{P\Phi_+}$ ($\|\cdot\|_{P\Phi_-}$) satisfy the condition (3) of Theorem 1 (Theorem 1').

LEMMA 2. *Let $T \in B(X,Y)$. Then*

(a) $$m_e(T) = m_e(T+A), \quad \text{for } A \in P(\Phi_+(X,Y)),$$
(b) $$n_e(T) = n_e(T+A), \quad \text{for } A \in P(\Phi_-(X,Y)).$$

Proof. (a) Let $A \in P(\Phi_+(X,Y))$. Since $P(\Phi_+(X,Y))$ is a linear subspace of $B(X,Y)$ (see [**3**, Lemma 5.5.3]) it follows that $-A \in P(\Phi_+(X,Y))$. It implies that $B \in \Phi_+(X,Y)$ if and only if $B + A \in \Phi_+(X,Y)$, i.e., $B \in B(X,Y)\backslash\Phi_+(X,Y)$ if and only if $B \in -A + B(X,Y)\backslash\Phi_+(X,Y)$. Hence

$$\begin{aligned} m_e(T) &= \inf\{\|T-B\| \,:\, B \in B(X,Y)\backslash\Phi_+(X,Y)\} \\ &= \inf\{\|T-(-A+C)\| \,:\, C \in B(X,Y)\backslash\Phi_+(X,Y)\} \\ &= \inf\{\|(T+A)-C\| \,:\, C \in B(X,Y)\backslash\Phi_+(X,Y)\} \\ &= m_e(T+A). \end{aligned}$$

(b) can be proved analogously. □

LEMMA 3. *Let $T,\ S \in B(X,Y)$. Then*

(a) $$m_e(T+S) \le m_e(T) + \|S\|_{P\Phi_+},$$
(b) $$n_e(T+S) \le n_e(T) + \|S\|_{P\Phi_-}.$$

Proof. Recall that

(4) $$m_e(A+B) \le m_e(A) + \|B\|, \quad A,\, B \in B(X,Y).$$

For each $A \in P(\Phi_+(X,Y))$, by Lemma 2 (a) and (4) we have

$$m_e(T+S) = m_e(T+S+A) \le m_e(T) + \|S+A\|,$$

hence

$$m_e(T+S) \le m_e(T) + \inf\{\|S+A\| : A \in P(\Phi_+(X,Y))\} = m_e(T) + \|S\|_{P\Phi_+}.$$

(b) can be proved analogously. □

We conclude that the functions $\|\cdot\|_{P\Phi_+}$ and m_e ($\|\cdot\|_{P\Phi_-}$ and n_e) satisfy the conditions of Theorem 1 (Theorem 1').

Let us introduce the following functions for $A \in B(X,Y)$:

$$\|A\|_S = \inf\{\|A+C\| : C \in S(X,Y)\},$$
$$\|A\|_{CS} = \inf\{\|A+C\| : C \in CS(X,Y)\},$$

and

$$m_S(A) = \sup\{m(A+C) : C \in S(X,Y)\},$$
$$n_{CS}(A) = \sup\{n(A+C) : C \in CS(X,Y)\}.$$

It is clear that

$$\text{(5)} \qquad \begin{aligned} m_S(A+P) &= m(A) \quad \text{for } P \in S(X,Y), \\ n_{CS}(A+P) &= n_{CS}(A) \quad \text{for } P \in CS(X,Y). \end{aligned}$$

Lemma 4. *Let* $A,\ B \in B(X,Y)$. *Then*

(a) $$m_S(A+B) \le m_S(A) + \|B\|_S,$$

(b) $$n_{CS}(A+B) \le n_{CS}(A) + \|B\|_{CS}.$$

Proof. For each $C \in S(X,Y)$ we have

$$m(T+S+C) \le m(T+C) + \|S\|.$$

It implies that

$$\sup\{m(T+S+C) : C \in S(X,Y)\} \le \sup\{m(T+C) : C \in S(X,Y)\} + \|S\|,$$

i.e.,

$$\text{(6)} \qquad m_S(A+B) \le m_S(A) + \|B\|.$$

Now as in the proof of Lemma 3, (a) follows from (5) and (6).

(b) can be proved analogously. □

Lemma 5. *For* $A \in B(X,Y)$

(a) $$m_S(A) > 0 \Longleftrightarrow A \in \Phi_+^-(X,Y),$$

(b) $$n_{CS}(A) > 0 \Longleftrightarrow A \in \Phi_-^+(X,Y).$$

Proof. (a) ($\Longrightarrow$) Let $m_S(A) > 0$. This implies that there is $C \in S(X,Y)$ such that $m(A+C) > 0$. Hence $A + C \in \Phi_+(X,Y)$ and $i(A+C) \leq 0$. Since $S(X,Y) \subset P(\Phi_+(X,Y))$, then $\lambda C \in P(\Phi_+(X,Y))$ for $\lambda \in [0,1]$ and we get $A + \lambda C \in \Phi_+(X,Y)$. It implies that $A \in \Phi_+(X,Y)$, and from Lemma 1 it follows that $i(A) = i(A+C) \leq 0$. Thus $A \in \Phi_+^-(X,Y)$.

($\Longleftarrow$) Assume $A \in \Phi_+^-(X,Y)$. Obviously $m_S(A) \geq m_C(A)$. Since (see [**17**])

$$m_C(A) > 0 \Longleftrightarrow A \in \Phi_+^-(X,Y),$$

it follows that $m_S(A) > 0$.

(b) can be proved analogously. ☐

Now we see that the functions $\|\cdot\|_S$ and m_S ($\|\cdot\|_{CS}$ and n_{CS}) satisfy the conditions of Theorem 2 (Theorem 2').

Let us introduce the following functions for $T \in B(X,Y)$:

$$m_{P\Phi_+}(T) = \sup\{m(T+C) \,:\, C \in P(\Phi_+(X,Y))\},$$
$$n_{P\Phi_-}(T) = \sup\{n(T+C) \,:\, C \in P(\Phi_-(X,Y))\}.$$

Similarly as above we get

$$m_{P\Phi_+}(T) > 0 \Longleftrightarrow T \in \Phi_+^-(X,Y),$$
$$n_{P\Phi_-}(T) > 0 \Longleftrightarrow T \in \Phi_-^+(X,Y).$$

and

$$m_{P\Phi_+}(T+S) \leq m_{P\Phi_+}(T) + \|S\|_{P\Phi_+},$$
$$n_{P\Phi_-}(T+S) \leq n_{P\Phi_-}(T) + \|S\|_{P\Phi_-} \quad T,\ S \in B(X,Y).$$

Thus, the functions $\|\cdot\|_{P\Phi_+}$ and $m_{P\Phi_+}$ ($\|\cdot\|_{P\Phi_-}$ and $n_{P\Phi_-}$) satisfy the conditions of Theorem 2 (Theorem 2').

Since the sets $\Phi_+^-(X,Y)$ and $\Phi_-^+(X,Y)$ are open, for $A \in B(X,Y)$ we have

$$m_{\Phi_+^-}(A) > 0 \Longleftrightarrow A \in \Phi_+^-(X,Y),$$
$$n_{\Phi_-^+}(A) > 0 \Longleftrightarrow A \in \Phi_-^+(X,Y).$$

Set

$$\|A\|_{P\Phi_+^-} = \inf\{\|A+C\| \,:\, C \in P(\Phi_+^-(X,Y))\},$$
$$\|A\|_{P\Phi_-^+} = \inf\{\|A+C\| \,:\, C \in P(\Phi_-^+(X,Y))\}.$$

Using the same metod as in Lemma 2 and Lemma 3, we conclude

$$m_{\Phi_+^-}(A+B) \le m_{\Phi_+^-}(A) + \|B\|_{P\Phi_+^-},$$
$$n_{\Phi_-^+}(A+B) \le n_{\Phi_-^+}(A) + \|B\|_{P\Phi_-^+}.$$

Now we see that the functions $\|\cdot\|_{P\Phi_+^-}$ and $m_{\Phi_+^-}$ ($\|\cdot\|_{P\Phi_+^-}$ and $n_{\Phi_-^+}$) satisfy the conditions of Theorem 2 (Theorem 2').

For $A \in B(X)$ recall that

$$\begin{aligned} s_+(A) &= \lim_{n\to\infty}(m_e(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(m_q(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(m_\mu(A^n))^{\frac{1}{n}} \\ &= \lim_{n\to\infty}(m((A^n)^+))^{\frac{1}{n}} = \lim_{n\to\infty}(G(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(t(A^n))^{\frac{1}{n}} \end{aligned} \tag{7}$$

and

$$\begin{aligned} s_-(A) &= \lim_{n\to\infty}(n_e(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(n_q(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(n((A^n)^+))^{\frac{1}{n}} \\ &= \lim_{n\to\infty}(K(A^n))^{\frac{1}{n}} \end{aligned} \tag{8}$$

(see [**19**], [**4**], [**15**], [**20**], [**19**]). Set (see [**20**])

$$m_\infty(A) = \sup\{m(A+F) \,:\, \dim R(F) < \infty\},$$
$$n_\infty(A) = \sup\{n(A+F) \,:\, \dim R(F) < \infty\}.$$

From the inequalities

$$m_\infty(A) \le m_C(A) \le m_S(A) \le m_{P\Phi_+}(A) \le m_{\Phi_+^-}(A),$$
$$n_\infty(A) \le n_C(A) \le n_{CS}(A) \le n_{P\Phi_-}(A) \le n_{\Phi_-^+}(A),$$

Theorem 2 (g), Theorem 2' (g) and by [**20**, Theorem 8.3] we get

$$\begin{aligned} s_+^-(A) &= \lim_{n\to\infty}(m_\infty(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(m_C(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(m_S(A^n))^{\frac{1}{n}} \\ &= \lim_{n\to\infty}(m_{P\Phi_+}(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(m_{\Phi_+^-}(A^n))^{\frac{1}{n}}, \end{aligned}$$

and

$$\begin{aligned} s_-^+(A) &= \lim_{n\to\infty}(n_\infty(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(n_C(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(n_{CS}(A^n))^{\frac{1}{n}} \\ &= \lim_{n\to\infty}(n_{P\Phi_-}(A^n))^{\frac{1}{n}} = \lim_{n\to\infty}(n_{\Phi_-^+}(A^n))^{\frac{1}{n}}. \end{aligned}$$

By Theorem 1(k), Theorem 1'(k), (7) and (8) we get:

Corollary 1. *Let* $A,\ B \in B(X)$.

(a) *If* $AB - BA \in P(\Phi_+(X))$ *and* $r_e(B) < s_+(A)$, *then* $A,\ A + B \in \Phi_+(X)$ *and* $i(A + B) = i(A)$.

(b) *If* $AB - BA \in P(\Phi_-(X))$ *and* $r_e(B) < s_-(A)$, *then* $A,\ A + B \in \Phi_-(X)$ *and* $i(A + B) = i(A)$.

Corollary 2. *Let* $A \in B(X)$ *and* $B \in R(X)$.

(a) *If* $A \in \Phi_+(X)$ *and* $AB - BA \in P(\Phi_+(X))$, *then* $A + B \in \Phi_+(X)$ *and* $i(A) = i(A + B)$.

(b) *If* $A \in \Phi_-(X)$ *and* $AB - BA \in P(\Phi_-(X))$, *then* $A + B \in \Phi_-(X)$ *and* $i(A) = i(A + B)$.

Proof. From Corollary 1. □

We are grtefull to the referee for pointing out that Zemaánek's result [**21**, Theorem 4] is related to our results.

Theorem 3. (Zemánek) *Let* $\omega(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X) \cup \Phi_-(X)\}$. *There exists a non-negative function* $\chi(\cdot)$ *defined on all bounded linear operators on* X *and having the following properties:*

(1) $|\chi(T) - \chi(S)| \leq \|T - S\|$ *for all operators* $T,\ S$;

(2) $\chi(T + C) = \chi(T)$ *for every* T *and every compact operator* C;

(3) $\omega(T) = \{\lambda \in \mathbb{C} : \chi(T - \lambda) = 0\}$;

(4) *for every point* λ_0 *in* $\mathbb{C}$ *we have* $\operatorname{dist}(\lambda_0, \omega(T)) = \lim_{n\to\infty}[\chi((T - \lambda_0)^n)]^{1/n}$.

Let us recall that Zemánek noted that the each of the four functions

$$\begin{aligned}
\chi_1(T) &= \max\{G(T), K(T)\},\\
\chi_2(T) &= \max\{B(T), M(T)\},\\
\chi_3(T) &= \max\{m_\infty(T), n_\infty(T)\},\\
\chi_4(T) &= \max\{m_e(T), n_e(T)\},\\
\chi_5(T) &= \max\{m(T^+), n(T^+)\},
\end{aligned}$$

satisfies Theorem 3. Let us remark that the following functions also satisfy this theorem:

$$\begin{aligned}
\chi_6(T) &= \max\{m_C(T), n_C(T)\},\\
\chi_7(T) &= \max\{m_S(T), n_S(T)\},\\
\chi_8(T) &= \max\{m_{P\Phi_+}(T), n_{P\Phi_-}(T)\},\\
\chi_9(T) &= \max\{m_{\Phi_+^-}(T), n_{\Phi_-^+}(T)\}.
\end{aligned}$$

3. Abstract case

Now, we show that some of the results above can be put in an abstract form, i.e., in general Banach algebra. Let $\mathcal{A}$ be a complex Banach algebra with identy 1, $\mathcal{K}$ two sided proper closed ideal, π the canonical homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{K}$, and G the group of invertibles in $\mathcal{A}/\mathcal{K}$. We write Φ to denote the semigroup $\pi^{-1}(G)$ and $P(\Phi)$ to denote the perturbation class associated with Φ. An (abstract) index consist of a homomorphism i of the semigroup Φ into the additive group $\mathbb{Z}$ of integers such that

(a) $i(x) = 0$ for all invertibile elements x in $\mathcal{A}$

(b) $i(1 + k) = 0$ for all k in $\mathcal{K}$.

It follows from the above definition that $i(x + k) = i(x)$, $(x \in \Phi,\ k \in \mathcal{K})$ and that if $x \in \Phi$, then there exists $\epsilon > 0$ such that for each $y \in \mathcal{A}$ with $\|x - y\| < \epsilon$ we have $y \in \Phi$ and $i(y) = i(x)$ (see [**2**]).

For $x \in \mathcal{A}$ define:

$$\|x\|_{P\Phi} = \inf\{\|x + y\| \,:\, y \in P(\Phi)\},$$
$$m_\Phi(x) = \operatorname{dist}(x, \mathcal{A}\backslash\Phi).$$

Let $r_e(x)$ be the spectral radius of the element $\pi(x)$ in the algebra $\mathcal{A}/\mathcal{K}$, i.e., $r_e(x) = \sup\{|\lambda| \,:\, \lambda - x \notin \Phi\}$.

Set $r_\Phi(x) = \inf\{|\lambda| \,:\, \lambda - x \notin \Phi\}$. It is easy to see that $r_\Phi(x) = \sup\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow \lambda - x \in \Phi\}$.

Now using the same method as above we conclude that

THEOREM 4. *Let $x,\ y \in \mathcal{A}$, then*

(a) $m_\Phi(x) = m_\Phi(x + z)$ *for* $z \in P(\Phi)$*;*

(b) $m_\Phi(x + y) \leq m_\Phi(x) + \|y\|_{P\Phi}$*;*

(c) *If* $\|y\|_{P\Phi} < m_\Phi(x)$*, then* $x,\ y \in \Phi$ *and* $i(x + y) = i(x)$*;*

(d) $r_\Phi(x) \geq m_\Phi(x)$*;*

(e) $r_\Phi(x) \geq \overline{\lim}_{n\to\infty}(m_\Phi(x^n))^{\frac{1}{n}}$*;*

(f) *If* $\|x\|_{P\Phi} < m_\Phi(1)$*, then* $1 - x \in \Phi$ *and* $i(1 - x) = 0$*;*

(g) *If* $\|x^n\|_{P\Phi} < m_\Phi(1)$ *for some* $n \in \mathbf{N}$*, then* $1 - x \in \Phi$ *and* $i(1 - x) = 0$*;*

(h) $r_e(x) = \lim\limits_{n\to\infty}(\|x^n\|_{P\Phi})^{\frac{1}{n}}$ *for* $x \in \mathcal{A}$*;*

(i) *If* $xy - yx \in P(\Phi)$ *and* $r_e(y) < \overline{\lim}_{n\to\infty}(m_\Phi(x^n))^{\frac{1}{n}}$*,*
then $x,\ x + y \in \Phi$ *and* $i(x + y) = i(x)$*.*

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