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$\mathcal{O}$–REGULARLY VARYING FUNCTIONS AND SOME ASYMPTOTIC RELATIONS

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Abstract. We prove that in the class of measurable positive functions defined on the interval $I_a = [\,a, +\infty)$ $(a > 0)$, the class of functions which preserve the strong asymptotic equivalence on the set of functions $\{x : I_a \mapsto \mathbb{R}^+,\, x(t) \to +\infty, t \to +\infty\}$, is a class of $\mathcal{O}$–regularly varying functions with continuous index function. We also prove a representation theorem for functions from this class, and a morphism-theorem for some asymptotic relations.

1. Introduction and auxiliary results

Throughout this paper, we shall use the following denotations:

$$I = (0, +\infty), \quad I_a = [a, +\infty) \quad (a > 0), \quad F\colon I_a \mapsto I,$$
$$K_F(t) = \varlimsup_{x \to +\infty} \frac{F(tx)}{F(x)} \qquad (t \in I).$$

Consider arbitrary functions x, y, F which are positive on the interval I_a, F is measurable on I_a, and $x(t), y(t) \to +\infty$ as $t \to +\infty$. Consider the relation

$$x(t) \sim y(t) \quad (t \to +\infty) \implies F(x(t)) \sim F(y(t)) \quad (t \to +\infty). \tag{1}$$

The class of all functions F satisfying relation (1) is denoted in this paper by CRV. This abbreviation is motivated by Theorem 2, where we proved that all functions $F \in CRV$ have *continuous* index functions.

Definition 1. A function $F\colon I_a \to I$ belongs to the class ORV if it is measurable and $K_F(t) < +\infty$ for every $t \in I$.

LEMMA 1. *For a positive and measurable function $F\colon I_a \mapsto I$, the next relations are mutually equivalent:*

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(a) *F satisfies relation* (1);

(b) *If* (a_n) *and* (b_n) *are arbitrary sequences tending to* $+\infty$, *then* $a_n \sim b_n$ *as* $n \to \infty$ *implies* $F(a_n) \sim F(b_n)$ *as* $n \to \infty$;

(c) *It holds true*

$$\lim_{\substack{\lambda\to 1\\ x\to+\infty}} \frac{F(\lambda x)}{F(x)} = 1.$$

Proof. (a) $\implies$ (b). Taking $x(t) = a_n$ and $y(t) = b_n$ for $n \le t < n+1$, we have

$$\lim_{n\to\infty} \frac{F(a_n)}{F(b_n)} = \lim_{t\to+\infty} \frac{F(a_{[t]})}{F(b_{[t]})} = 1.$$

(b) $\implies$ (c). Take any sequence $x_n > 0$ tending to $+\infty$, and any sequence $\lambda_n > 0$ tending to 1, as $n \to \infty$. Putting $a_n = \lambda_n x_n$, $b_n = x_n$ and using (b), we find

$$\lim_{n\to+\infty} \frac{F(\lambda_n x_n)}{F(x_n)} = 1,$$

whence we get (c).

(c) $\implies$ (a). By relation (c), we conclude that for every $\epsilon > 0$, there exist some $\Delta(\epsilon)$ and $\delta(\epsilon) > 0$ such that $|F(\lambda x)/F(x) - 1| < \epsilon$ for $x \ge \Delta(\epsilon)$ and $|1-\lambda| \le \delta(\epsilon)$. By assumptions from (a), we have $|x(t)/y(t) - 1| \le \delta(\epsilon)$ if $t \ge t_1$, and $y(t) \ge \Delta(\epsilon)$ if $t \ge t_0$. Taking $t_2 = \max\{t_0, t_1\}$, we finally find $|F(x(t))/F(y(t)) - 1| < \epsilon$ for $t \ge t_2$. $\square$

Examples. I. If F is an arbitrary function from the class RV with index $\rho \in \mathbb{R}$, then $F(x) = x^\rho L(x)$ for some function $L \in SRV$ and all $x \in I_a$. By Lemma 1(c) and the Uniform convergence theorem for slowly varying functions [3], we obtain $F \in CRV$.

II. The function $F(x) = 2 + \sin(\log x)$ $(x \in I_a)$ belongs to the class ORV, but does not belong to the class RV. Nevertheless we have $\lim_{\substack{\lambda\to 1\\ x\to+\infty}} \big(F(\lambda x)/F(x)\big) = 1$. The similar is true for the function $F(x) = \exp\{\sin(2\log x)\}$ $(x \in I_a)$.

III. If ϵ is an arbitrary bounded, measurable function on the interval I_B, then the function $F(x) = \exp\big\{\int_B^x \frac{\epsilon(t)}{t}\,dt\big\}$ $(x \in I_B)$ belongs to the class ORV, and we have the inequality $\lambda^a \le F(\lambda x)/F(x) \le \lambda^b$ $(x \in I_B, \lambda \ge 1)$, whenever $a \le \epsilon(t) \le b$ $(t \in I_B)$. For any value $\lambda < 1$, we have the reverse inequalities. Hence, we have $\lim_{\substack{\lambda\to 1\\ x\to+\infty}} \big(F(\lambda x)/F(x)\big) = 1$, i.e. $F \in CRV$.

Remark. By a straightforward calculation, it can be easily checked that

(a) if $F, G \in CRV$, then $F \cdot G \in CRV$;

(b) if $F \in CRV$, then $1/F \in CRV$;

(c) if $F, G \in CRV$, $F \circ G$ is a measurable function, and $G(x) \to +\infty$ as $x \to +\infty$, then $F \circ G \in CRV$.

Hence we conclude the class CRV is a multiplicative group, and the class $CRV^\infty = \{F \in CRV | \lim_{x\to+\infty} F(x) = +\infty\}$ is a groupoid with respect to the composition of functions.

IV. Let F be any function from the Matuszewska class [3] . Then for some $c \in \mathbb{R}$, and some measurable, bounded functions μ and ϵ on I, such that $\lim_{x\to\infty} \mu(x) = 0$, we have the representation

$$F(x) = \exp\Big\{c + \mu(x) + \int_1^x \frac{\epsilon(t)}{t}\,dt\Big\}. \tag{2}$$

From this representation, we conclude that $F \in CRV$.

V. The function $F(x) = 2 + \sin x$ $(x \in I)$ belongs to the class ORV, but the limit $\lim_{\substack{\lambda\to1\\ x\to+\infty}} \big(F(\lambda x)/F(x)\big)$ does not exist. Hence $F \notin CRV$.

Lemma 2. *Every function F from the class CRV defined on I_a, belongs to the class ORV.*

Proof. If $F \in CRV$ and $\epsilon > 0$, then there are an $x_0 \in I_a$ and $\delta > 0$ such that $1 - \epsilon \le F(\lambda x)/F(x) \le 1 + \epsilon$ whenever $x \ge x_0, |1 - \lambda| \le \delta$. Hence $F \in ORV$. $\square$

Since the class CRV is a proper subclass of the class ORV, and any function $F \in ORV$ has the representation

$$F(x) = \exp\Big\{\mu(x) + \int_B^x \frac{\epsilon(t)}{t}\,dt\Big\} \qquad (x \in I_B), \tag{3}$$

for some measurable, bounded functions μ and ϵ on I_B, by the remark above and Example III, we have

$$F \in CRV \iff \lim_{\substack{\lambda\to1\\ x\to+\infty}} \big(\mu(\lambda x) - \mu(x)\big) = 0. \tag{4}$$

Lemma 3. *For a continuous function $h\colon I_{\log A} \to \mathbb{R}$, the next conditions are equivalent*:

(a) *h is uniformly continuous on $I_{\log A}$*;

(b) $\lim_{\substack{\lambda\to0\\ x\to+\infty}} \big(h(\lambda + x) - h(x)\big) = 0.$

Proof. (a) $\implies$ (b). If h is uniformly continuous on $I_{\log A}$ and $\epsilon > 0$, then there is a $\delta > 0$ such that $|h(x') - h(x'')| < \epsilon$ for arbitrary $x', x'' \in I_{\log A}$ such that $|x' - x''| \le \delta$. Thus, we have $|h(\lambda + x) - h(x)| < \epsilon$ whenever $x, \lambda + x \in I_{\log A}$, and $|\lambda| \le \delta$. Hence, for sufficiently large $x \in I_{\log A}$ and $|\lambda| \le \delta$ we have $|h(\lambda + x) - h(x)| < \epsilon$, i.e. we have (b).

(b) $\implies$ (a). From (b) it follows that for any $\epsilon > 0$ there are x_0, λ_0 such that $|h(\lambda + x) - h(x)| < \epsilon$ whenever $x \ge x_0$ and $|\lambda| \le \lambda_0$. Hence, h is a uniformly continuous function on I_{x_0}. If $x_0 > \log A$, then h is uniformly continuous on the interval $[\log A, x_0]$, so h is uniformly continuous on $[\log A, +\infty)$. $\square$

Corollary 1. *For any function F, positive and continuous on the interval I_A, the next conditions are mutually equivalent*:

(a) $F \in CRV$;

(b) $F(x) = \exp\{h(\log x)\}$ $(x \geq A)$, *where the function* $h(x) = \log F(e^x)$ $(x \in I_{\log A})$ *is uniformly continuous on* $I_{\log A}$.

Lemma 4. *If $h : I_{\log A} \mapsto \mathbb{R}$ is a measurable and bounded function on $I_{\log A}$, then the next conditions are mutually equivalent*:

(a) $h(x) = \mu(x) + r(x)$ $(x \in I_{\log A})$, *where μ is a measurable, bounded function such that $\mu(x) \to 0$ as $x \to +\infty$, and r is bounded and uniformly continuous on* $I_{\log A}$;

(b)
$$\lim_{\substack{\lambda \to 0 \\ x \to +\infty}} \big(h(\lambda + x) - h(x)\big) = 0.$$

Proof. (a) $\Longrightarrow$ (b) is trivial.

(b) $\Longrightarrow$ (a). For any $\delta > 0$, denote $\omega_\delta(x) = \sup\{|h(x') - h(x)| : x, x' \in I_{\log A}\}$. Then for every $\epsilon > 0$, there is an $M = M(\epsilon)$ and a $\delta_0 = \delta_0(\epsilon) > 0$ such that $\omega_\delta(x) < \epsilon$ for $x \geq M, \delta \leq \delta_0$. Next for $t \geq B \geq \log A$, there is a function $\varphi_t(x) \in C(\mathbb{R})$ such that $\operatorname{supp} \varphi_t(x) \subseteq (t - 1/t, t + 1/t)$, $\varphi_t \geq 0$ on $\mathbb{R}$, $\int_{\mathbb{R}} \varphi_t(x)dx = 1$ and $F(t,x) = \varphi_t(x)$ is a continuous function with respect to the variables (t,x). Then $f(t) = \int_{I_B} \varphi_t(x)h(x)dx = \int_{I_B} F(t,x)h(x)dx$ is a continuous function on $t \in I_B$. Now $\nu(t) = h(t) - f(t)$ is a measurable function on I_B, and we have

$$|\nu(t)| = |h(t) - f(t)| = \Big| \int_{I_B} \varphi_t(x)(h(x) - h(t))\, dx \Big| \leq$$
$$= \int_{I_B \cap (t - 1/t, t+1/t)} \varphi_t(x)|h(x) - h(t)|\, dx \leq \omega_{2/t}(t).$$

Hence we obtain $\nu(t) \to 0$ as $t \to +\infty$, and consequently $h(t) = f(t) + \nu(t)$ $(t \in I_B)$. Since $\nu(t) \to 0$ as $t \to +\infty$, we find ν and consequently f satisfy condition (b). Denoting $B = \log A$, $f(x) = r(x)$ and $\nu(x) = \mu(x)$ for $x \in I_{\log A}$, we see that both r and μ are bounded on $I_{\log A}$. By Lemma 3, we immediately get condition (a). $\square$

Remark. By a straightforward calculation, one can check the function

$$\varphi_t(x) = F(t,x) = \begin{cases} 0, & x \in (-\infty, t - 1/2t\,] \cup [\,t + 1/2t, +\infty) \\ 2\,t(1 - 2t\,|x - t|), & |x - t| < 1/2t \end{cases}$$

defined for $t \geq B$, satisfies all the conditions from Lemma 4.

2. Representation and characterization theorems for the class CRV

Theorem 1 (Representation theorem). *A function F belongs to the class CRV if and only if we have*

$$F(x) = \exp\Big\{\tilde{\mu}(x) + r(\log x) + \int_B^x \frac{\epsilon(t)}{t}\, dt\Big\} \qquad (x \in I_B), \tag{5}$$

where ϵ is a measurable, bounded function on some interval I_B ($B \in I$), $\tilde{\mu}$ is a measurable, bounded function on the same interval I_B such that $\tilde{\mu}(x) \to 0$ as $x \to +\infty$, and r is a uniformly continuous, bounded function on the interval $I_{\log B}$.

Proof. By relation (4), a function $F \in CRV$ if and only if

$$\lim_{\substack{\lambda\to 1\\ x\to+\infty}} \Big(\mu_1(\log\lambda + \log x) - \mu_1(\log x)\Big) = 0,$$

where $\mu_1(\log x) = \mu(x)$ ($x \in I_B$), thus $\mu_1(x) = \mu(e^x)$ ($x \in I_{\log B}$). Using the substitution $\log\lambda = k$, $\log x = t$, the limit above can be written in the form

$$\lim_{\substack{k\to 0\\ t\to+\infty}} \big(\mu_1(k+t) - \mu_1(t)\big) = 0.$$

By Lemma 4, the previous condition is satisfied if and only if the measurable and bounded function $\mu_1(x) = \mu(e^x)$ can be written as $\mu_1(x) = \overline{\mu}(x) + r(x)$ on $I_{\log B}$, where $\overline{\mu}$ and r are bounded, $\overline{\mu}$ is measurable, $\overline{\mu}(x) \to 0$ as $x \to +\infty$, and r is uniformly continuous on $I_{\log B}$. Here we can obviously use $\mu(x) = \mu_1(\log x) = \overline{\mu}(\log x) + r(\log x)$ for $x \in I_B$ and $\overline{\mu}(\log x) = \tilde{\mu}(x)$ for $x \in I_B$. $\Box$

THEOREM 2 (Characterization theorem). *A function F belongs to the class CRV if and only if the index function $K_F(\lambda)$ is continuous in $\lambda \in I$.*

Proof. Assume $F \in CRV$. Then for every $\epsilon > 0$, there are a $\delta > 0$ and $\Delta > 0$ so that

$$1 - \epsilon < \frac{F(\lambda x)}{F(x)} < 1 + \epsilon,$$

if $x \geq \Delta$ and $|1-\lambda| \leq \delta$. Hence we get $1 - \epsilon < K_F(\lambda) < 1 + \epsilon$ if $|1-\lambda| \leq \delta$, thus $\lim_{\lambda\to 1} K_F(\lambda) = 1$. By a result from [2], it follows the function $K_F(\lambda)$ is continuous on I.

Next assume $F \in ORV$ and the function $K_F(\lambda)$ is continuous on I. By a result from [2], we then have

$$\lim_{u\to+\infty} \sup_{\lambda\in T} \Big(\sup_{x\geq u} \frac{F(\lambda x)}{F(x)} - K_F(\lambda)\Big) = 0,$$

for any compact $T \subseteq I$. Hence, for any $\epsilon > 0$, there are a $u_0 > 0$ and a $p > 1$ such that $F(\lambda x)/F(x) \leq K_F(\lambda) + \epsilon$ whenever $x \geq u_0$ and $1/p \leq \lambda \leq p$. Consequently, $\overline{\lim}_{\lambda\to 1, x\to+\infty} \big(F(\lambda x)/F(x)\big) \leq 1 + \epsilon$. If we put $y = \lambda x$ and $\mu\lambda = 1$, we obtain $F(y)/F(\mu y) \leq K_F\big(1/\mu\big) + \epsilon$ whenever $1/p \leq \mu \leq p$ and $y \geq p u_0$. Consequently, we obtain $\overline{\lim}_{\mu\to 1, y\to+\infty} \big(F(y)/F(\mu y)\big) \leq 1 + \epsilon$. Hence we get

$$\frac{1}{1+\epsilon} \leq \underline{\lim}_{\substack{\lambda\to 1\\ x\to+\infty}} \frac{F(\lambda x)}{F(x)} \leq \overline{\lim}_{\substack{\lambda\to 1\\ x\to+\infty}} \frac{F(\lambda x)}{F(x)} \leq 1 + \epsilon,$$

and consequently $\lim_{\substack{\lambda\to 1\\ x\to+\infty}} \big(F(\lambda x)/F(x)\big) = 1$. This means that $F \in CRV$. $\square$

COROLLARY 2. *If F is a measurable, positive function on the interval I_a, then the following is true:*

(a) $\displaystyle \overline{\lim_{\lambda\to 1}}\ \overline{\lim_{x\to+\infty}} \frac{F(\lambda x)}{F(x)} = 1 \implies \lim_{\substack{\lambda\to 1\\ x\to+\infty}} \frac{F(\lambda x)}{F(x)} = 1;$

(b) $\displaystyle \lim_{\substack{\lambda\to 1\\ x\to+\infty}} \frac{F(\lambda x)}{F(x)} = 1 \implies \lim_{\lambda\to p}\ \overline{\lim_{x\to+\infty}} \frac{F(\lambda x)}{F(x)} = K_F(p),\ p>0$
$(K_F(1) = 1)$.

For implication (a), one can consult [2] .

COROLLARY 3. *If F is any continuous positive function on the interval I_A, then the following conditions are equivalent:*

(a) $\displaystyle \overline{\lim}_{\lambda\to 1}\, \overline{\lim}_{x\to+\infty} \frac{F(\lambda x)}{F(x)} = 1;$

(b) *The function $h(x) = \log F(e^x)$ is uniformly continuous on the interval $I_{\log A}$.*

COROLLARY 4. *Every function from the Matuszewska class ERV has the continuous index function $K_F(\lambda)$ on the interval I.*

By the next example, we prove that $ERV \subsetneq CRV$.

Example VI. Notice that the function $F(x) = \exp\big(|\sin(\log x)|\big)^{1/2}$ has the following properties:

(a) $F\colon I \mapsto I;\quad F(1) = 1;$

(b) For arbitrary $s, t \in I$,

$$F(st) = \exp\big(|\sin(\log s + \log t)|\big)^{1/2} \le \exp\big(|\sin\log s| + |\sin\log t|\big)^{1/2} \le$$
$$\le \exp\big((\sin\log s|)^{1/2} + (|\sin\log t|)^{1/2}\big) = F(s)\cdot F(t).$$

(c) If $p = e^{2\pi} > 1$, then $F(pt) = \exp\big(|\sin(\log t + 2\pi)|\big)^{1/2} = F(t)$.

Hence, by a result from [2], we have $\overline{\lim}_{x\to+\infty}\big(F(\lambda x)/F(x)\big) = F(\lambda)$ for every $\lambda \in I$, which gives $F \in CRV$. On the other side, assuming $F \in ERV$, we find a real number c such that $\overline{\lim}_{x\to+\infty}\big(F(\lambda x)/F(x)\big) = K_F(\lambda) \le \lambda^c$ $(\lambda \ge 1)$, that is $\big(\log K_F(\lambda)\big)/(\log\lambda) \le c$, for every $\lambda > 1$. Then

$$\frac{\log K_F(\lambda)}{\log\lambda} = \frac{\log F(\lambda)}{\log\lambda} = \frac{(|\sin(\log\lambda)|)^{1/2}}{\log\lambda} \le c$$

for all λ close to 1. But since

$$\lim_{\lambda\to 1+} \frac{(|\sin\log\lambda|)^{1/2}}{\log\lambda} = \lim_{\lambda\to 1+} \frac{1}{(\log\lambda)^{1/2}} = +\infty,$$

we get the contradiction. This proves that $F \notin ERV$. $\square$

Since

$$SRV \subsetneq RV \subsetneq ERV \subsetneq CRV \subsetneq ORV.$$

using the corresponding representation theorems for classes ERV, CRV and ORV, we get the following conclusion: If ϵ is a bounded, measurable function on the interval I_B, and

$$F(x) = \exp\left\{\nu(x) + \int_B^x \frac{\epsilon(t)}{t}\,dt\right\} \qquad (x \in I_B),$$

then $F(x) \in ORV$ if and only if ν is a measurable and bounded function on I_B, while $F(x) \in CRV$ if and only if $\nu(x) = r(\log x) + \mu(x)$ $(x \in I_B)$ where $r \circ \log$, μ are bounded on I_B, r is uniformly continuous on $I_{\log B}$, μ is measurable on I_B, and $\mu(x) \to 0$ as $x \to +\infty$. We also recall that $F(x) \in ERV$ if and only if $\nu(x) = c + \mu(x)$ $(x \in I_B)$, where $c \in \mathbb{R}$ and μ is a measurable and bounded on I_B such that $\mu(x) \to 0$ as $x \to +\infty$.

3. Morphism theorem

Let $H = \{x | x\colon I_a \mapsto I\}$, $H_1 = \{x \in H | x(t) \to +\infty, t \to +\infty\}$. If $\simeq$ denotes the asymptotic similarity of functions, and ρ_1, ρ_2 are arbitrary relations from the set $\{\sim, \simeq, \asymp\}$, then let $\operatorname{Hom}\big((H_1, \rho_1); (H, \rho_2)\big)$ be the set of all measurable functions $F\colon I_a \mapsto I$ such that

$$x(t)\,\rho_1\,y(t) \quad (t \to +\infty) \implies F(x(t))\,\rho_2\,F(y(t)) \quad (t \to +\infty)$$

for any two functions $x, y \in H_1$.

THEOREM 3. *Let $F\colon I_a \mapsto I$ be a measurable function. Then:*

(a) $F \in ORV \implies F \in \operatorname{Hom}\big((H_1, \asymp); (H, \asymp)\big)$;

(b) $F \in \operatorname{Hom}\big((H_1, \sim); (H, \asymp)\big) \implies F \in ORV$;

(c) $F \in CRV \implies F \in \operatorname{Hom}\big((H_1, \sim); (H, \sim)\big)$;

(d) $F \in \operatorname{Hom}\big((H_1, \sim); (H, \simeq)\big) \implies F \in CRV$;

(e) $F \in RV \iff F \in \operatorname{Hom}\big((H_1, \simeq); (H, \simeq)\big)$;

(f) $F \in SRV \implies F \in \operatorname{Hom}\big((H_1, \asymp); (H, \sim)\big)$;

(g) $F \in \operatorname{Hom}\big((H_1, \asymp); (H, \simeq)\big) \implies F \in SRV$;

(h) $F \in \operatorname{Hom}\big((H_1, \simeq); (H, \sim)\big) \implies F \in SRV$.

Proof. (a) If a function $F \in ORV$, then using some results from [1] we have $\overline{\lim}_{x\to+\infty} \sup_{\lambda\in K} \big(F(\lambda x)/F(x)\big) < +\infty$ for any compact interval $K \subseteq I$. Hence, for any functions $x, y \in H_1$ such that $x(t) \asymp y(t)$ as $t \to +\infty$, there is an $M > 0$ such that

$$\frac{F(x(t))}{F(y(t))} = \frac{F\Big(\frac{x(t)}{y(t)}\, y(t)\Big)}{F(y(t))} \le M < +\infty \qquad (t \ge t_0).$$

Similarly, since $\underline{\lim}_{x\to+\infty} \big(F(\lambda x)/F(x)\big) > 0$ holds uniformly in $\lambda \in K$, we find

$$\frac{F(x(t))}{F(y(t))} = \frac{F\Big(\frac{x(t)}{y(t)}\, y(t)\Big)}{F(y(t))} \ge m > 0,$$

for some $m > 0$ and all $t \ge t_0$. Consequently, $F \in \mathrm{Hom}\,\big((H_1, \asymp); (H, \asymp)\big)$.

(b) If $F \in \mathrm{Hom}\,\big((H_1, \sim); (H, \asymp)\big)$, then similarly as in Lemmas 1 and 2 we obtain $F \in ORV$.

(c) If $F \in CRV$, then by Theorem 2 we have $F \in \mathrm{Hom}\,\big((H_1, \sim), (H, \sim)\big)$.

(d) Assume $F \in \mathrm{Hom}\,\big((H_1, \sim); (H, \simeq)\big)$. Then for arbitrary functions $x, y \in H_1$ such that $x(t) \sim y(t)$ as $t \to +\infty$ we have $F(x(t)) \simeq F(y(t))$ as $t \to +\infty$. Taking two arbitrary sequences $(a'_n), (b'_n)$ tending to $+\infty$ as $n \to \infty$, such that $a'_n \sim b'_n$ as $n \to \infty$, we have $\lim_{n\to\infty} \big(F(a'_n)/F(b'_n)\big) = c'$. Taking any other sequences $(a''_n), (b''_n)$ with the similar properties, we get $\lim_{n\to\infty} \big(F(a''_n)/F(b''_n)\big) = c''$. If a_n is the general term of the sequence $a'_1, a''_1, a'_2, a''_2, \ldots$, and b_n is the general term of the sequence $b'_1, b''_1, b'_2, b''_2, \ldots$, then we obviously have $\lim_{n\to\infty} \big(F(a_n)/F(b_n)\big) = c$. Hence we get $c = c' = c'' = 1$, thus $F \in CRV$.

(e) If $F \in RV$, then there is a function $L \in SRV$ and a number $\rho \in \mathbb{R}$ such that $F(t) = L(t)\, t^\rho$ for all $t \in I_a$. If x, y are arbitrary functions from the class H_1 such that $x(t)/y(t) \to c > 0$ as $t \to +\infty$, then

$$\lim_{t\to+\infty} \frac{F(x(t))}{F(y(t))} = \lim_{t\to+\infty} \left(\frac{x(t)}{y(t)}\right)^\rho \cdot \lim_{t\to+\infty} \frac{L\Big(\frac{x(t)}{y(t)}\, y(t)\Big)}{L(y(t))} = c^\rho,$$

which means that $F \in \mathrm{Hom}\,\big((H_1, \simeq); (H, \simeq)\big)$. Here we used the Uniform convergence theorem for slowly varying functions.

Conversely, suppose $F \in \mathrm{Hom}\,\big((H_1, \simeq); (H, \simeq)\big)$, $\lambda > 0$, $x \in H_1$ and $y = \lambda x$. Then

$$\lim_{t\to+\infty} \frac{F(y(t))}{F(x(t))} = \lim_{t\to+\infty} \frac{F(\lambda\, x(t))}{F(x(t))} = d(\lambda) < +\infty.$$

Using the characterization theorem for functions from the class RV, we have $d(\lambda) = \lambda^\rho$ for some $\rho \in \mathbb{R}$, which gives $F \in RV$.

(f) If $F \in SRV$, then by the Uniform convergence theorem for slowly varying functions, we obtain $F \in \mathrm{Hom}\,\big((H_1, \asymp); (H, \sim)\big)$.

(g) Let $F \in \operatorname{Hom}\big((H_1,\asymp);(H,\simeq)\big)$. Then for arbitrary sequences (a'_n), (b'_n) such that $a'_n, b'_n \to +\infty$ and $a'_n \asymp b'_n$, we have $\lim_{n\to+\infty}\big(F(a'_n)/F(b'_n)\big) = c'$. For any other sequences $(a''_n), (b''_n)$ with the similar properties, we obtain $\lim_{n\to\infty}\big(F(a''_n)/F(b''_n)\big) = c''$. Denoting the general term of the sequence $a'_1, a''_1 a'_2, a''_2, \dots$ by a_n, and the general term of the sequence $b'_1, b''_1, b'_2, b''_2, \dots$ by b_n, we have $a_n \asymp b_n$ as $n \to \infty$, whence $\lim_{n\to\infty}\big(F(a_n)/F(b_n)\big) = c$. Hence we get $c = c' = c'' = 1$. If next (a_n) is an arbitrary sequence such that $a_n \to +\infty$ as $n \to \infty$, $\lambda > 0$ and $b_n = \lambda\, a_n$, then we get $\lim_{n\to\infty}\big(F(\lambda\, a_n)/F(a_n)\big) = 1$. This means that $F \in SRV$.

(h) Finally, assume $F \in \operatorname{Hom}\big((H_1,\simeq);(H,\sim)\big)$, $x \in H_1$ and $y = \lambda\, x$ for some $\lambda > 0$. Then obviously

$$\lim_{t\to+\infty}\frac{F(y(t))}{F(x(t))} = \lim_{t\to+\infty}\frac{F(\lambda x(t))}{F(x(t))} = 1,$$

which means that $F \in SRV$. $\square$

Corollary 5. *Let $F\colon I_a \mapsto I$ be a measurable function. Then:*

(a) $F \in ORV \implies F \in \operatorname{Hom}\big((H_1,\simeq);(H,\asymp)\big) \cap \operatorname{Hom}\big((H_1,\sim);(H,\asymp)\big)$;

(b) $F \in \operatorname{Hom}\big((H_1,\asymp);(H,\asymp)\big) \cup \operatorname{Hom}\big((H_1,\simeq);(H,\asymp)\big) \implies F \in ORV$;

(c) $F \in CRV \implies F \in \operatorname{Hom}\big((H_1,\sim);(H,\simeq)\big)$;

(d) $F \in \operatorname{Hom}\big((H_1,\sim);(H,\sim)\big) \implies F \in CRV$;

(e) $F \in SRV \implies F \in \operatorname{Hom}\big((H_1,\asymp);(H,\simeq)\big) \cap \operatorname{Hom}\big((H_1,\simeq);(H,\sim)\big)$;

(f) $F \in \operatorname{Hom}\big((H_1,\asymp);(H,\sim)\big) \implies F \in SRV$.

Remark. The proposition "$F \in ORV$ if and only if $F \in \operatorname{Hom}\big((H_1,\simeq);(H,\asymp)\big)$" is closely related to the definition of the class ORV, which have been introduced by V. Avakumović 1935.

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