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IT IS CONSISTENT THAT THERE EXISTS AN η_1-ORDERED REAL CLOSED FIELD WHICH IS NOT HYPER-REAL

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Abstract. We provide an example of a model of ZFC in which there exists an η_1-ordered real closed field which is not a hyper-real field.

Introduction

In [**2**] and [**7**], was asked to find an example of an η_1-ordered real closed field which is not a hyper-real field. We will construct such an example under some additional set-theoretic assumptions. We will use the characterization of η_1-ordered real closed fields obtained in [**1**], and the related results in [**2**].

1. Background

In this section we recall some definitions and facts, mostly following [**1**].

Hausdorff generalization of rationals is

$$\mathcal{Q}_1 = \{(x_\alpha)_{\alpha<\omega_1} | x_\alpha \in \{0,1\}, \{\alpha \mid x_\alpha = 1\} \text{ has maximum}\}$$

ordered lexicographicaly.

Let $\mathcal{G} = (G, +, 0, \leq)$ be a totally ordered, divisible Abelian group. Absolute value of $a \in G$ is defined as $|a| = \max\{a, -a\}$. An o-subgroup $\mathcal{H}$ of $\mathcal{G}$ is convex if for every $b \in H, a \in G, |a| \leq |b| \Rightarrow a \in H$. The principal convex subgroup generated by $a \in G$ is $V(a) = \{b \in G \mid |b| \leq n|a| \text{ for some } n \in \omega\}$. Group $\mathcal{G}$ is Archimedean if all the principal convex subgroups generated by nonzero elements coincide. The maximal convex subgroup not containing a is $V^-(a) = \{b \in G \mid n|b| < |a| \text{ for every } n \in \omega\}$. Obviously, $V^-(a)$ is a subgroup of $V(a)$ and $V(a)/V^-(a)$ is an

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Archimedean group. $\{V(a) \mid a \in G\}$ ordered by inclusion is the value set of $\mathcal{G}$. Let $\mathcal{F} = (F, +, \cdot, 0, 1, -, ^{-1}, \leq)$ be a totally ordered field and Π be the value set of its additive group. We define addition in Π as $V(a) + V(b) = V(ab)$. The ordered group $(\Pi, +, V(1), \leq)$ is the value group of the field $\mathcal{F}$. For every $a \in F$, $V(a)/V(a)^- \cong V(1)/V(1)^-$ is the residue field of $\mathcal{F}$.

Let T be a totally ordered set, G a totally ordered Abelian group, and $f \in G^T$. The set $s(f) = \{t \in T \mid f(t) \neq 0\}$ is the support of f. $G\{T\} = \{f \in G^T \mid s(f)$ is well ordered$\}$ is an Abelian group under pointwise addition and totally ordered if we define that an element is positive if its minimal nonzero coordinate is positive. If G is an Archimedean group, then T is the value set of $G\{T\}$.

Let F be a totally ordered field and G a totally ordered Abelian group. $K = F\{G\}$ becomes a totally ordered field under multiplication: For $a, b \in K$, $ab(g) = \sum_{x \in G} a(x)b(g - x)$. This is so called the field of formal power series.

$F\{G\}_\alpha$ will denote the o-subfield of $F\{G\}$ consisting of the sequences whose support is of cardinality $< \aleph_\alpha$.

Let α be an ordinal. An ordered field $\mathcal{F}$ is η_α-ordered if $(F, \leq)$ is an η_α-set i.e., for every two subsets $H, K \subset F$ such that $|H| + |K| < \aleph_\alpha$ and $H < K$, there exists $a \in F$ such that $H < a < K$. We will now recall the definition of α-maximal fields introduced by I. Kaplansky.

Definition. Let $\mathcal{F}$ be an ordered field with the valuation V, $\mathcal{G}$ be its value group and α be an ordinal. A sequence $\{a_\beta \mid \beta < \alpha\}$ is pseudo-convergent if for every $\delta < \gamma < \beta < \alpha$ we have $V(a_\beta - a_\gamma) < V(a_\gamma - a_\delta)$. An $a \in F$ is a pseudo-limit of that pseudo-convergent sequence if $V(a - a_\beta) = V(a_{\beta+1} - a_\beta)$, for all $\beta < \alpha$.

Definition. $\mathcal{F}$ is α-maximal if every pseudo-convergent sequence of length less then $\aleph_\alpha$ has a pseudo-limit. $\mathcal{F}$ is maximal if it is α-maximal for $|F| = \aleph_\alpha$.

We will now recall a few facts from [**1**]:

THEOREM 1.1. (called Main Theorem in [**1**]) *Let G be a totally ordered Abelian group and α a nonzero ordinal number. G is an η_α-set iff* (1) *its factors are conditionally complete,* (2) *its value set is an η_α-set, and* (3) *it is α-maximal.*

THEOREM 1.2. (called Main Corollary in [**1**]) *Let $\mathcal{F}$ be an ordered field with valuation V and the value group $\mathcal{G}$. $\mathcal{F}$ is η_α-ordered if and only if:*

(i) *Its residue field is isomorphic to $\mathcal{R}$.*

(ii) *Its value group $\mathcal{G}$ is an η_α-ordered group.*

(iii) *It is α-maximal.*

Real-closed fields are fields where every positive element is a square, and every polinomial (of one variable) of odd degree has a zero.

The following theorem appears in [**1**] as Corollary 3.2:

THEOREM 1.3. *Let F be a real-closed field, H a totally ordered Abelian divisible group, and α a nonzero ordinal number. Then $F\{H\}_\alpha$ is a real-closed field.*

Hyper-real fields are the non-Archimedean fields of the form $C(X)/M$, where X is a completely regular space, $C(X)$ the ring of continuous functions from X to R, and M a maximal ideal in $C(X)$. They are η_1-ordered and real closed fields. For the basic facts on hyper-real fields one could see [**5**]. Here we just mention that for a maximal ideal M of $C(X)$, $Z[M]$ denotes the z-ultrafilter consisting of all the zero sets of functions from M. Sometimes we write X_U for $C(X)/M$ and $U = Z[M]$; U is called a hyper-real ultrafilter. It is free i.e., it has empty intersection.

2. Construction of the example

THEOREM 2.1. *Let L be an η_1-ordered set which does not contain an increasing sequence of type ω_2. Then $G = R\{L\}_1$ is an η_1-ordered divisible Abelian group which does not contain an increasing well ordered sequence of size ω_2.*

Proof. By Theorem 1.1, G is an η_1-ordered divisible Abelian group. We are left with the proof that it does not contain well ordered sequences of type ω_2. Suppose, to the contrary, that G contains such a sequence e.g. g^α, $\alpha < \omega_2$. Let for $\alpha < \omega_2$, $g^\alpha = \sum_{\beta<\gamma_\alpha} t^\alpha_\beta l^\alpha_\beta$, where γ_α is a countable ordinal, and $(l^\alpha_\beta)_{\beta<\gamma_\alpha}$ is a well ordered sequence in L. Since γ_α are countable ordinals, there exists $S \subset \omega_2$ of cardinality $\aleph_2$ such that g^α is constant for $\alpha \in S$. Without loss of generality we can suppose that $\gamma_\alpha = \gamma$, $\alpha < \omega_2$. We will recursively define a sequence k_δ of ordinals below ω_2.

Consider the sequence $l_0 = (l^\alpha_0)_{\alpha<\omega_2}$. It is nondecreasing. Since L does not contain an increasing sequence of type ω_2, l_0 is ultimately constant. Let k_0 be the minimal index after which l_0 is constant. If $\beta = \delta + 1$, then k_β is the minimal ordinal after k_δ such that t^α_β is constant for $\alpha > k_\beta$. On limit steps we define $k_\beta = \sup\{k_\delta | \delta < \beta\}$. This finishes the construction of the sequence. Obviously $k_\beta < \omega_2$, for $\beta < \gamma$. Let $k = \sup\{k_\beta | \beta < \gamma\}$. Since γ is countable, and all the k_β, for $\beta < \gamma$, are less than ω_2, we have $k < \omega_2$. It is evident from the construction that for every $\beta < \gamma$, and all $\alpha > \kappa$, l^α_β does not depend on α. Obviously, without loss of generality we can suppose that $g^\alpha = \sum_{\beta<\gamma} t^\alpha_\beta l_\beta$ for all $\alpha < \omega_2$.

Since $(g^\alpha)_{\alpha<\omega_2}$ is well ordered, and the ordering of G is lexicographic, $(t^\alpha_0)_{\alpha<\omega_2}$ is a well ordered sequence in R. Since R allows just countable well ordered subsets, there exists δ_0, such that t^α_0 is constant for $\alpha > \delta_0$. Repeating this construction we get a sequence $(\delta_\beta)_{\beta<\gamma}$, of ordinals smaller than ω_2, such that for $\alpha > \delta_\beta$, t^α_ξ does not depend on α, for every $\xi < \beta$. Let $\delta = \sup\{\delta_\beta | \beta < \gamma\}$. Since γ is countable, $\delta < \omega_2$. Hence, the sequence g^α is constant for $\alpha > \delta$, contradicting the assumption that it is strictly increasing. This contradiction proves the statement. $\square$

Let G be any group from the theorem above and $F = R\{G\}_1$. Then we can prove the following statement.

THEOREM 2.2. *The field F is an η_1-ordered real closed field which does not contain a well ordered increasing sequence of type ω_2.*

Proof. F is an η_1-ordered real closed field by Theorem 1.3. It does not contain a well ordered sequence of type ω_2, by the theorem above, taking G for L, and F for G. $\square$

Now we specify the ordering L from Theorem 2.1, to be $\mathcal{Q}_1$. Following the ideas from [**2**], we use the result from [**6**], that there exists a model of ZFC in which $2^{\aleph_0} = \aleph_2$, $2^\kappa = \kappa^+$, for $\kappa > \aleph_0$, and in which there exists $F \subset \omega^\omega$ of size $\aleph_2$, such that F is well ordered under relation $\prec$, defined as "smaller almost ewerywhere".

THEOREM 2.3. *There exists a model of ZFC in which every hyper-real field contains a sequence of type* ω_2.

Proof. Let $\mathcal{M}$ be the model mentioned just before the theorem. Let $C(X)/M = X_U$ be any hyper-real field. By Theorem 8.2 in [**2**], there exists a free ultrafilter p on N such that N_p elementarily embeds in X_U. Since p is free, it contains Freschet filter. Henceforth, for $[F] = \{[f]_p : f \in F\} \subset N_p$, we have that $[F]$ is a well ordered sequence of type ω_2 in N_p. Since N_p embeds in X_U, we have that X_U contains a well ordered sequence of type ω_2. $\square$

From the two theorems above we get the assertion.

THEOREM 2.4. *There exists a model of ZFC in which there exists an* η_1-*ordered real closed field which is not isomorphic to any hyper-real feld.*

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