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WEAK CYLINDRIC PROBABILITY ALGEBRAS

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Abstract. We prove an analog of the Boolean representation theorem for locally finite-dimensional weak cylindric probability algebras. These algebras are designed to provide an apparatus for an algebraic study of the weak probability logic $L_{\mathcal{A}P\forall}$.

The notion of a weak cylindric probability algebra will be introduced as a common algebraic abstraction from the theory of deductive systems of the weak probability logic $L_{\mathcal{A}P\forall}$, and the geometry associated with basic set-theoretic notions.

The logic $L_{\mathcal{A}P\forall}$ is the minimal extension of the infinitary logic $L_{\mathcal{A}}$ (see [**3**]) and the probability logic $L_{\mathcal{A}P}$ (see [**4**]), where $\mathcal{A}$ is a countable admissible set such that $\omega \in \mathcal{A}$. Let L be a countable $\mathcal{A}$-recursive set of finitary relation, function and constant symbols. The set Form_L of all formulas of $L_{\mathcal{A}P\forall}$ is closed under countable disjunctions ($\vee$) and conjunctions ($\wedge$), negation ($\neg$), usual quantifiers ($\forall, \exists$) and probability quantifiers ($P\mathbf{v} \geq r$), where $\mathbf{v}$ is a finite tuple of distinct variables and $r \in \mathcal{A} \cap [0,1]$. This set contains as distinguished elements the expressions: false (F), true (T) and $v_p{=}v_q$ for any $p, q < \omega$. The structure

$$\mathfrak{Form}_L = \langle \mathrm{Form}_L, \vee, \wedge, \neg, F, T, \exists v_i, P\mathbf{v} \geq r, v_p{=}v_q \rangle$$

is the free algebra of formulas of $L_{\mathcal{A}P\forall}$.

Axioms and rules of inference for $L_{\mathcal{A}P\forall}$ are those for $L_{\mathcal{A}}$ and the weak $L_{\mathcal{A}P}$, as listed in [**3**] and [**4**], together with the following axioms (see [**6**]):

($AP\forall_1$) $(\forall x)\varphi \to (Px \geq 1)\varphi$;
($AP\forall_2$) $(Px_1 \dots x_n \geq r)\varphi \to (Px_{\pi 1} \dots x_{\pi n} \geq r)\varphi$,

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where π is a permutation of $\{1,\ldots,n\}$.

Let Σ be any set of sentences of $L_{\mathcal{A}P\forall}$. The notion of a deduction of a formula φ from Σ (denoted by $\Sigma\vdash\varphi$) is defined in the usual way. Let $\equiv_\Sigma$ be a relation on Form_L defined by

$$\varphi\equiv_\Sigma\psi \quad\text{iff}\quad \Sigma\vdash\varphi\leftrightarrow\psi.$$

If $\Sigma\vdash\varphi\leftrightarrow\psi$, then $\Sigma\vdash(\exists x)\varphi\leftrightarrow(\exists x)\psi$ and $\Sigma\vdash(P\mathbf{x}\geq r)\varphi\leftrightarrow(P\mathbf{x}\geq r)\psi$. Hence the relation $\equiv_\Sigma$ is a congruence relation on $\mathfrak{Form}_L$. Let φ^Σ be a set of all formulas $\equiv_\Sigma$-equivalent to φ, and let $\mathrm{Form}_L/{\equiv_\Sigma}$ be a set of all equivalence classes φ^Σ, $\varphi\in\mathrm{Form}_L$. Now, we construct the quotient algebra

$$\mathfrak{Form}_L/{\equiv_\Sigma}=\langle\mathrm{Form}_L/{\equiv_\Sigma},\vee^\Sigma,\wedge^\Sigma,\neg^\Sigma,F^\Sigma,T^\Sigma,(\exists v_i)^\Sigma,(P\mathbf{v}\geq r)^\Sigma,(v_p{=}v_q)^\Sigma\rangle,$$

which will be called a weak cylindric probability algebra of formulas.

Let $\mathfrak{A}=\langle A,R_i^{\mathfrak{A}},f_j^{\mathfrak{A}},c_k^{\mathfrak{A}},\mu_n\rangle_{n<\omega}$ be a weak probability structure for $L_{\mathcal{A}P\forall}$; i.e., $\langle A,R_i^{\mathfrak{A}},f_j^{\mathfrak{A}},c_k^{\mathfrak{A}}\rangle$ is a classical first-order structure and μ_n's are finitely additive probability measures defined on the set of all definable subsets of A^n. By using the natural definition of the satisfaction relation, we obtain the collection $\mathbb{A}$ of all sets of the form $\varphi^{\mathfrak{A}}=\{a\in A^\omega:\mathfrak{A}\models\varphi[a]\}$, $\varphi\in\mathrm{Form}_L$. Then

$$\big((\exists v_i)\varphi\big)^{\mathfrak{A}}=\{a\in A^\omega: a\restriction\omega\setminus\{\,i\,\}=b\restriction\omega\setminus\{\,i\,\}\ \text{ for some } b\in\varphi^{\mathfrak{A}}\},$$
$$\big((P\mathbf{v}\geq r)\varphi\big)^{\mathfrak{A}}=\big\{a\in A^\omega:\mu_n\{(b_{k_1},\ldots,b_{k_n}):b\in\varphi^{\mathfrak{A}},\ (j\notin K\rightarrow b_j=a_j)\}\geq r\big\},$$

where $\mathbf{v}=v_{k_1},\ldots,v_{k_n}$ and $K=\{k_1,\ldots,k_n\}$. Thus we get a weak cylindric probability set algebra. As usual, a unary cylindric set operation C_i is defined on the subsets of A^ω by setting, for any $X\subseteq A^\omega$,

$$C_i(X)=\{y\in A^\omega: y\restriction\omega\setminus\{\,i\,\}=x\restriction\omega\setminus\{\,i\,\}\ \text{ for some } x\in X\}.$$

Let $\langle K\rangle$ be a tuple of distinct integers corresponding to a finite subset $\{k_1,\ldots,k_n\}$ of ω. For each $\langle K\rangle$ and $r\in[0,1]$, we introduce a unary cylindric probability set operation $C^r_{\langle K\rangle}$ on the subsets of A^ω by setting, for any $X\subseteq A^\omega$,

$$C^r_{\langle K\rangle}(X)=\big\{y\in A^\omega:\mu_n\{(x_{k_1},\ldots,x_{k_n}):x\in X\ \ \&\ \ (j\notin K\rightarrow x_j=y_j)\}\geq r\big\}.$$

By means of $C^r_{\langle K\rangle}$ we obtain a cylinder generated by translating only the section of X whose measure is not less than r parallelly to the $(k_1,\ldots,k_n)$-axis of A^ω. If K is a singleton $\{k\}$, then we write C^r_k instead of $C^r_{\langle\{k\}\rangle}$. It follows from $C_i(\varphi^{\mathfrak{A}})=\big((\exists v_i)\varphi\big)^{\mathfrak{A}}$ and $C^r_{\langle K\rangle}(\varphi^{\mathfrak{A}})=\big((P\mathbf{v}\geq r)\varphi\big)^{\mathfrak{A}}$ that the function $f\colon\mathrm{Form}_L/{\equiv_\Sigma}\rightarrow\mathbb{A}$ defined by $f(\varphi^\Sigma)=\varphi^{\mathfrak{A}}$ is a "natural" homomorphic transformation from the weak cylindric probability algebra of formulas $\mathfrak{Form}_L/{\equiv_\Sigma}$ onto the weak cylindric probability set algebra

$$\langle\mathbb{A},\cup,\cap,\sim,\emptyset,A^\omega,C_i,C^r_{\langle K\rangle},D_{pq}\rangle,$$

where $D_{pq} = \{a \in A^\omega : a_p = a_q\}$ and, so, $D_{pq} = (v_p{=}v_q)^{\mathfrak{A}}$.

The abstract notion of a weak cylindric probability algebra is defined by equations which hold in both algebras mentioned above. We suppose in advance that a fixed indexation by hereditarily countable sets (from $\mathcal{A} \subseteq \mathrm{HC}$) is given. So let $A = \{x_i : i \in I\}$ and $I \subseteq \mathcal{A}$. We say that a Boolean algebra $\langle A, +, \cdot, -, 0, 1\rangle$ is $\mathcal{A}$-complete if for any $\{x_j : j \in J\} \subseteq A$, where $J \subseteq I$ and $J \in \mathcal{A}$, we have $\sum_{j\in J} x_j \in A$.

Definition 1. A weak cylindric probability algebra is a structure

$$\mathbf{A} = \langle A, +, \cdot, -, 0, 1, C_i, C^r_{\langle K\rangle}, d_{pq}\rangle,$$

such that $\langle A, +, \cdot, -, 0, 1\rangle$ is an $\mathcal{A}$-complete Boolean algebra, C_i and $C^r_{\langle K\rangle}$ are unary operations on A for each $i < \omega$ and each finite $K \subseteq \omega$, $d_{pq} \in A$ for all $p, q < \omega$, and the following postulates hold (by convention, let $C^r_{\langle K\rangle}x = C^1_{\langle K\rangle}x$ for $r \geq 1$, and $C^r_{\langle K\rangle}x = C^0_{\langle K\rangle}x$ for $r \leq 0$).

(WCP$_0$) $\langle A, +, \cdot, -, 0, 1, C_i, d_{pq}\rangle$ is a cylindric algebra of dimension ω.
(WCP$_1$) (i) $C^r_{\langle\emptyset\rangle}x = x$, (ii) $C^r_{\langle K\rangle}0 = 0$, where $r > 0$.
(WCP$_2$) $C^0_{\langle K\rangle}x = 1$.
(WCP$_3$) If $r \geq s$, then $C^r_{\langle K\rangle}x \leq C^s_{\langle K\rangle}x$.
(WCP$_4$) $C^r_{\langle K\rangle}(x + C^s_{\langle L\rangle}y) = C^r_{\langle K\rangle}x + C^s_{\langle L\rangle}y$, where $K \subseteq L$.
(WCP$_5$) (i) $C^r_{\langle K\rangle}x \cdot C^s_{\langle K\rangle}y \leq C^{r+s-1}_{\langle K\rangle}(x \cdot y)$,
(ii) $C^r_{\langle K\rangle}x \cdot C^s_{\langle K\rangle}y \cdot C^1_{\langle K\rangle} - (x \cdot y) \leq C^{r+s}_{\langle K\rangle}(x + y)$.
(WCP$_6$) $C^r_{\langle K\rangle} - x = -\sum_{m>0} C^{1-r+1/m}_{\langle K\rangle}x$.
(WCP$_7$) $C^r_{\langle K\rangle}x \leq C^r_{\langle\pi(K)\rangle}x$, where π is a permutation of $\{1, \ldots, n\}$ and $\langle\pi(K)\rangle$ is $k_{\pi 1}, \ldots, k_{\pi n}$.
(WCP$_8$) $-C^1_k - x \leq C_k x$.
(WCP$_9$) If $i \in K$, then: (i) $C_i C^r_{\langle K\rangle}x = C^r_{\langle K\rangle}x$, (ii) $C^r_{\langle K\rangle}C_i x = C^r_{\langle K\setminus\{i\}\rangle}C_i x$.
(WCP$_{10}$) If $i, j \notin K$, then: (i) $C^r_{\langle K\rangle}C_i(d_{ij} \cdot x) = C_i C^r_{\langle K\rangle}(d_{ij} \cdot x)$,
(ii) $C^r_{\langle K\cup\{i\}\rangle}C_j(d_{ij} \cdot x) = C^r_{\langle K\cup\{j\}\rangle}C_i(d_{ij} \cdot x)$.

We point out that the axioms WCP$_2$–WCP$_6$ express a well-known properties of finitely additive measures. The axioms WCP$_7$ and WCP$_8$ express the conditions $(AP\forall_2)$ and $(AP\forall_1)$ of $L_{\mathcal{A}P\forall}$, respectively.

Now we give some properties of the operations $C^r_{\langle K\rangle}$. The necessary properties of C_i and the substitution operation S^i_j defined by $S^i_j x = \begin{cases} x, & \text{if } i = j \\ C_i(d_{ij} \cdot x), & \text{if } i \neq j \end{cases}$, are well-known (see [**2**] and [**5**]).

THEOREM 1. *If $\langle A, +, \cdot, -, 0, 1, C_i, C^r_{\langle K\rangle}, d_{pq}\rangle$ is a weak cylindric probability algebra, then:*

(1) $C^r_{\langle K\rangle}1 = 1$.

(2) *If* $r > 0$ *and* $s > 0$, *then* $C^r_{\langle K\rangle}x = x$ *iff* $C^s_{\langle K\rangle} - x = -x$.
(3) *If* $r > 0$ *or* $r = s = 0$ *and* $K \subseteq L$, *then* $C^r_{\langle K\rangle}(x \cdot C^s_{\langle L\rangle}y) = C^r_{\langle K\rangle}x \cdot C^s_{\langle L\rangle}y$.
(4) $C^r_{\langle K\rangle}x \cdot -C^r_{\langle K\rangle}y \leq \sum_{m>0} C^{1/m}_{\langle K\rangle}(x \cdot -y)$.
(5) *If* $x \leq y$, *then* $C^r_{\langle K\rangle}x \leq C^r_{\langle K\rangle}y$.
(6) $C^r_{\langle K\rangle}x + C^r_{\langle K\rangle}y \leq C^r_{\langle K\rangle}(x+y)$.
(7) $C^r_{\langle K\rangle}(x \cdot y) \leq C^r_{\langle K\rangle}x \cdot C^r_{\langle K\rangle}y$.
(8) $C^1_{\langle K\rangle}x \cdot C^1_{\langle K\rangle}y = C^1_{\langle K\rangle}(x \cdot y)$.
(9) $C^1_{\langle K\rangle}x = x$ *iff* $C_{\langle K\rangle}x = x$.
(10) *If* $K = \{k_1, \ldots, k_n\}$ *and* $r > 0$, *then* $C^r_{\langle K\rangle}x \leq C_{k_1} \ldots C_{k_n}x$.
(11) $C^1_{\langle K\rangle}d_{pq} = d_{pq}$, *where* $p, q \notin K$.
(12) *If* $i \in K$, *then:* (a) $S^i_j C^r_{\langle K\rangle}x = C^r_{\langle K\rangle}x$, (b) $S^i_j S^m_i C^r_{\langle K\rangle}x = S^m_j C^r_{\langle K\rangle}x$.
(13) *If* $i, j \notin K$, *then:* (a) $S^i_j C^r_{\langle K\rangle}x = C^r_{\langle K\rangle}S^i_j x$,
(b) $C^r_{\langle K\cup\{i\}\rangle}S^j_i x = C^r_{\langle K\cup\{j\}\rangle}S^i_j x$.

Proof. (1) It follows from WCP$_1$ (ii) and WCP$_6$ that

$$C^r_{\langle K\rangle}1 = -\sum_{m>0} C^{1-r+1/m}_{\langle K\rangle}0 = 1.$$

(2) If $C^r_{\langle K\rangle}x = x$, then

$$\begin{aligned}
C^s_{\langle K\rangle} - x &= -\sum_{m>0} C^{1-s+1/m}_{\langle K\rangle}x && \text{by WCP}_6\\
&= -\sum_{m>0} C^{1-s+1/m}_{\langle K\rangle}C^r_{\langle K\rangle}x && \text{by assumption}\\
&= -\sum_{m>0} C^r_{\langle K\rangle}x && \text{by WCP}_4 \text{ (putting } x=0\text{) and WCP}_1\\
&= -x && \text{by WCP}_0.
\end{aligned}$$

The converse follows by symmetry.
(3) It follows from (2), WCP$_4$ and WCP$_6$ that, for $r > 0$, we have:

$$\begin{aligned}
C^r_{\langle K\rangle}(x \cdot C^s_{\langle L\rangle}y) &= C^r_{\langle K\rangle} - (-x + -C^s_{\langle L\rangle}y)\\
&= -\sum_{m>0} C^{1-r+1/m}_{\langle K\rangle}(-x + -C^s_{\langle L\rangle}y)\\
&= -\Big(\Big(\sum_{m>0} C^{1-r+1/m}_{\langle K\rangle} - x\Big) + -C^s_{\langle L\rangle}y\Big)\\
&= \Big(-\sum_{m>0} C^{1-r+1/m}_{\langle K\rangle} - x\Big) \cdot C^s_{\langle L\rangle}y\\
&= C^r_{\langle K\rangle}x \cdot C^s_{\langle L\rangle}y.
\end{aligned}$$

(4) We have:

$$\begin{aligned}
C^r_{\langle K\rangle}x \cdot -C^r_{\langle K\rangle}y &= C^r_{\langle K\rangle}x \cdot \sum_{m>0} C^{1-r+1/m}_{\langle K\rangle} - y && \text{by WCP}_6\\
&= \sum_{m>0} C^r_{\langle K\rangle}x \cdot C^{1-r+1/m}_{\langle K\rangle} - y && \text{by WCP}_0\\
&\leq \sum_{m>0} C^{1/m}_{\langle K\rangle}(x \cdot -y) && \text{by WCP}_5 \text{ (i).}
\end{aligned}$$

(5) If $x \leq y$, then $x \cdot -y = 0$. So, $C^r_{\langle K\rangle} x \cdot -C^r_{\langle K\rangle} y = 0$ from (4) and WCP$_1$; i.e., $C^r_{\langle K\rangle} x \leq C^r_{\langle K\rangle} y$.

(6),(7) Immediate by (5) and $x \leq x+y,\ y \leq x+y,\ x \cdot y \leq x,\ x \cdot y \leq y$.

(8) By WCP$_5$ (i) we have $C^1_{\langle K\rangle} x \cdot C^1_{\langle K\rangle} y \leq C^1_{\langle K\rangle}(x \cdot y)$. The reverse inequality is an instance of (7).

(9) If $C^1_k x = x$, then $C_k x = C_k C^1_k x = C^1_k x = x$ from WCP$_9$ (i). It follows from WCP$_3$, WCP$_6$ and WCP$_8$ that $C^1_k x \leq \sum_{m>0} C^{1/m}_k x = -C^1_k - x \leq C_k x$. Hence, if $C_k x = x$, then $C^1_k x \leq x$ and $x = -C_k - x \leq C^1_k x$ by WCP$_0$ and WCP$_8$; i.e., $x = C^1_k x$. Now, by induction, it follows from WCP$_9$ that $C^1_{\langle K\rangle} x = x$ if and only if $C_{\langle K\rangle} x = x$.

(10) First, we prove $-C^1_{\langle K\rangle} - x \leq C_{k_1} \ldots C_{k_n} x$ by induction on $|K|$. The inequality is clear if $K = \emptyset$. Suppose that $K = \{k_1, \ldots, k_{n+1}\}$. Now $x \leq C_{k_{n+1}} x$, so $-C_{k_{n+1}} x \leq -x$, and hence $C^1_{\langle K\rangle} - C_{k_{n+1}} x \leq C^1_{\langle K\rangle} - x$, and so

$$\begin{aligned} -C^1_{\langle K\rangle} - x &\leq -C^1_{\langle K\rangle} - C_{k_{n+1}} x = \sum\nolimits_{m>0} C^{1/m}_{\langle K\rangle} C_{k_{n+1}} x \\ &= \sum\nolimits_{m>0} C^{1/m}_{\langle K\setminus\{k_{n+1}\}\rangle} C_{k_{n+1}} x = -C^1_{\langle K\setminus\{k_{n+1}\}\rangle} - C_{k_{n+1}} x \\ &\leq C_{k_1} \ldots C_{k_{n+1}} x. \end{aligned}$$

Finally, choose $p > 0$ so that $1/p < r$. Then

$$C^r_{\langle K\rangle} x \leq C^{1/p}_{\langle K\rangle} x \leq \sum\nolimits_{m>0} C^{1/m}_{\langle K\rangle} x = -C^1_{\langle K\rangle} - x \leq C_{k_1} \ldots C_{k_n} x.$$

(11) Immediate by (9) and $C_{\langle K\rangle} d_{pq} = d_{pq}$, where $p, q \notin K$.

(12) Assuming $i \neq j$ and $i \in K$, we have:

$$\begin{aligned} S^i_j C^r_{\langle K\rangle} x &= C_i(d_{ij} \cdot C^r_{\langle K\rangle} x) \\ &= C_i(d_{ij} \cdot C_i C^r_{\langle K\rangle} x) && \text{by WCP}_9 \text{ (i)} \\ &= C^r_{\langle K\rangle} x && \text{by WCP}_0, \end{aligned}$$

and $S^i_j S^m_i C^r_{\langle K\rangle} x = S^i_j S^m_j C^r_{\langle K\rangle} x = S^m_j S^i_j C^r_{\langle K\rangle} x = S^m_j C^r_{\langle K\rangle} x$ by WCP$_0$.

(13) Assuming $i \neq j$ and $i, j \notin K$, we have:

$$\begin{aligned} S^i_j C^r_{\langle K\rangle} x &= C_i(d_{ij} \cdot C^r_{\langle K\rangle} x) = C_i C^r_{\langle K\rangle}(d_{ij} \cdot x) && \text{by (3)} \\ &= C^r_{\langle K\rangle} C_i(d_{ij} \cdot x) = C^r_{\langle K\rangle} S^i_j x, && \text{by WCP}_{10} \text{ (i)}, \end{aligned}$$

and

$$\begin{aligned} C^r_{\langle K\cup\{i\}\rangle} S^j_i x &= C^r_{\langle K\cup\{i\}\rangle} C_j(d_{ij} \cdot x) \\ &= C^r_{\langle K\cup\{j\}\rangle} C_i(d_{ij} \cdot x) && \text{by WCP}_{10} \text{ (ii)} \\ &= C^r_{\langle K\cup\{j\}\rangle} S^i_j x. \quad \square \end{aligned}$$

The algebraic notion of an ideal in a weak cylindric probability algebra can be modified using specific properties of these algebras.

Definition 2. An ideal in a cylindric probability algebra $\mathbf{A}$ is a nonempty set $\mathcal{I} \subseteq A$ such that the following conditions hold:

(1) $\mathcal{I}$ is a Boolean ideal of $\mathbf{A}$; i.e.,

(a) $0 \in \mathcal{I}$,

(b) If $\{a_j : j \in J\} \subseteq \mathcal{I}$ and $J \in \mathcal{A}$, then $\sum_{j\in J} a_j \in \mathcal{I}$,

(c) If $x \in \mathcal{I}$ and $y \leq x$, then $y \in \mathcal{I}$;

(2) For all $i < \omega$, if $x \in \mathcal{I}$, then $C_i x \in \mathcal{I}$.

It follows from Definition 2 and (10) of Theorem 1 that, for any finite $K \subseteq \omega$ and $r \in (0,1]$, if $x \in \mathcal{I}$, then $C^r_{\langle K\rangle} x \in \mathcal{I}$. An ideal $\mathcal{I}$ determines the relation $\sim\, = \{(x,y) : x \cdot -y + y \cdot -x \in \mathcal{I}\}$. As usual, if $x \sim y$, then $C_i x \sim C_i y$. For $r > 0$ and $x, y \in A$, we have

$$C^r_{\langle K\rangle} x \cdot -C^r_{\langle K\rangle} y + C^r_{\langle K\rangle} y \cdot -C^r_{\langle K\rangle} x \leq \sum\nolimits_{m>0} C^{1/m}_{\langle K\rangle}(x \cdot -y) + \sum\nolimits_{m>0} C^{1/m}_{\langle K\rangle}(y \cdot -x)$$

by (4) of Theorem 1. So, if $x \sim y$, then $C^r_{\langle K\rangle} x \sim C^r_{\langle K\rangle} y$. Hence, $\sim$ is a congruence relation of $\mathbf{A}$. We define a new algebra $\mathbf{A}/\mathcal{I} = \langle A/\mathcal{I}, \widehat{+}, \widehat{\cdot}, \widehat{-}, \widehat{0}, \widehat{1}, \widehat{C}_i, \widehat{C^r_{\langle K\rangle}}, \widehat{d}_{pq}\rangle$ as usual. It is not difficult to see that $\mathbf{A}/\mathcal{I}$ is a weak cylindric probability algebra, and that there is a "natural" homomorphism from $\mathbf{A}$ onto $\mathbf{A}/\mathcal{I}$.

The dimension set Δx of an element $x \in A$ is introduced by $\Delta x = \{k : C_k x \neq x\}$. It follows from the clause (9) of Theorem 1 that $\Delta x = \{k : C^1_k x \neq x\}$, i.e., the coordinates in which x is not a cylinder can be obtained also by applying probability cylindrifications of the form C^1_k.

Definition 3. A weak cylindric probability algebra $\mathbf{A}$ is locally finite-dimensional if Δx is finite for all $x \in A$.

Every formula φ of $L_{\mathcal{A}P\forall}$ has only finitely many free variables. If v_i is a variable not occurring in φ, then $\models (\exists v_i)\varphi \leftrightarrow \varphi$ and $\models (Pv_i > 0)\varphi \leftrightarrow \varphi$. So, for any given set Σ of sentences of $L_{\mathcal{A}P\forall}$, there are at most finitely many indices $i < \omega$ such that φ is not equivalent under Σ neither to $(\exists v_i)\varphi$ nor to $(Pv_i > 0)\varphi$; hence, $\mathfrak{Form}_L/{\equiv_\Sigma}$ is locally finite-dimensional weak cylindric probability algebra.

The following theorem gives some elementary properties of Δ.

THEOREM 2. *If* $\langle A, +, \cdot, -, 0, 1, C_i, C^r_{\langle K\rangle}, d_{pq}\rangle$ *is a weak cylindric probability algebra, then:*

(1) $\Delta 0 = \Delta 1 = \emptyset$;

(2) $\Delta\big(\sum_{j\in J} x_j\big) \subseteq \bigcup_{j\in J} \Delta x_j$, $J \in \mathcal{A}$;

(3) $\Delta\big(\prod_{j\in J} x_j\big) \subseteq \bigcup_{j\in J} \Delta x_j$, $J \in \mathcal{A}$;

(4) $\Delta - x = \Delta x$;

(5) $\Delta d_{pq} = \{p, q\}$;

(6) $\Delta C_i x \subseteq \Delta x \setminus \{i\}$;

(7) $\Delta S^i_j x \subseteq (\Delta x \setminus \{i\}) \cup \{j\}$;

(8) $\Delta C^r_{\langle K\rangle} x \subseteq \Delta x \setminus K$.

Proof. The clauses (1)–(7) are well-known properties of Δ from the classical theory of cylindric algebras.

(8) Let i be any integer such that $i \notin \Delta x \setminus K$. If $i \in K$, then $C_i C^r_{\langle K\rangle} x = C^r_{\langle K\rangle} x$ by WCP$_9$ (i). If $i \notin \Delta x \cup K$, then

$$\begin{aligned} C_i C^r_{\langle K\rangle} x &= C_i C^r_{\langle K\rangle} C_i x = C_i C^r_{\langle K\cup\{i\}\rangle} C_i x && \text{by WCP}_9 \text{ (ii)} \\ &= C^r_{\langle K\cup\{i\}\rangle} C_i x = C^r_{\langle K\rangle} x && \text{by WCP}_9 \text{ (i).} \end{aligned}$$

So, $i \notin \Delta C^r_{\langle K\rangle} x$. □

The main result of this paper is the following analog of the Boolean representation theorem from the classical theory of cylindric algebras.

THEOREM 3. *If* $\mathbf{A}$ *is a locally finite-dimensional weak cylindric probability algebra and* $|A| > 1$*, then there is a homomorphism from* $\mathbf{A}$ *onto a weak cylindric probability set algebra.*

Proof. We prove that $\mathbf{A}$ is isomorphic to a weak cylindric probability algebra of formulas $\mathfrak{Form}_L/{\equiv_\Sigma}$ for some L and Σ.

Let R_a be an n-ary relation symbol corresponding to a for each $a \in A$, where the integer n is obtained from $\Delta a \subseteq \{1,\dots,n\}$. Fix the language $L = \{R_a : a \in A\}$. By induction on the complexity of formulas of the logic $L_{\mathcal{A}P\forall}$ we define a function $f\colon \mathrm{Form}_L \to A$ satisfying: if $\vdash \varphi$, then $f(\varphi) = 1$ as follows:

(1) Let φ be an atomic formula $R_a(v_{k_1},\dots,v_{k_n})$ and let $j_1,\dots,j_n$ be the first n integers in $\omega \setminus \{1,\dots,n,k_1,\dots,k_n\}$. Then

$$f(\varphi) = S^{j_1}_{k_1} \cdots S^{j_n}_{k_n} S^{1}_{j_1} \cdots S^{n}_{j_n}\, a\ ;$$

(2) $f(v_p{=}v_q) = d_{pq}$;

(3) $f(\neg\varphi) = -f(\varphi)$;

(4) $f(\bigvee \Phi) = \sum_{\varphi\in\Phi} f(\varphi)$, $\Phi \in \mathcal{A}$;

(5) $f(\bigwedge \Phi) = \prod_{\varphi\in\Phi} f(\varphi)$, $\Phi \in \mathcal{A}$;

(6) $f\big((\exists v_i)\varphi\big) = C_i f(\varphi)$;

(7) $f\big((P\mathbf{v} \geq r)\varphi\big) = C^r_{\langle K\rangle} h(\varphi)$,

where $\mathbf{v} = v_{k_1},\dots,v_{k_m}$ and $K = \{k_1,\dots,k_m\}$.

Let φ be a formula of $L_{\mathcal{A}P\forall}$ and let φ^* be a formula obtained by the substitution of some free variables $v_{k_1},\dots,v_{k_n}$ of φ with $v_{m_1},\dots,v_{m_n}$, respectively. By induction on complexity of formulas of $L_{\mathcal{A}P\forall}$, we prove the following *substitution property*:

$$\text{(S)} \qquad f(\varphi) = S^{j_1}_{k_1} \cdots S^{j_n}_{k_n} S^{m_1}_{j_1} \cdots S^{m_n}_{j_n} f(\varphi^*),$$

where $j_1,\dots,j_n$ are some distinct integers in $\omega \setminus \{1,\dots,n,k_1,\dots,k_n,m_1,\dots,m_n\}$.

Suppose φ is $R_a(v_{k_1},\dots,v_{k_n})$. Let $p_1,\dots,p_n,q_1,\dots,q_n$ be distinct integers in $\omega \setminus \{1,\dots,n,k_1,\dots,k_n,m_1,\dots,m_n\}$. For some distinct integers $j_1,\dots,j_n$ in the set $\omega \setminus \{1,\ldots,n,k_1,\ldots,k_n,m_1,\ldots,m_n\}$, we have:

$$\begin{aligned}
S^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}f(\varphi^*) &= S^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}S^{q_1}_{m_1}\cdots S^{q_n}_{m_n}S^{1}_{q_1}\cdots S^{n}_{q_n}a\\
&= S^{p_1}_{k_1}\cdots S^{p_n}_{k_n}S^{m_1}_{p_1}\cdots S^{m_n}_{p_n}S^{q_1}_{m_1}\cdots S^{q_n}_{m_n}S^{1}_{q_1}\cdots S^{n}_{q_n}a\\
&= S^{p_1}_{k_1}\cdots S^{p_n}_{k_n}S^{k_1}_{p_1}\cdots S^{k_n}_{p_n}S^{q_1}_{k_1}\cdots S^{q_n}_{k_n}S^{1}_{q_1}\cdots S^{n}_{q_n}a\\
&= S^{p_1}_{k_1}\cdots S^{p_n}_{k_n}S^{q_1}_{k_1}\cdots S^{q_n}_{k_n}S^{1}_{q_1}\cdots S^{n}_{q_n}a\\
&= S^{p_1}_{k_1}\cdots S^{p_n}_{k_n}S^{p_1}_{k_1}\cdots S^{p_n}_{k_n}S^{1}_{p_1}\cdots S^{n}_{p_n}a\\
&= S^{p_1}_{k_1}\cdots S^{p_n}_{k_n}S^{1}_{p_1}\cdots S^{n}_{p_n}a\\
&= f(\varphi)
\end{aligned}$$

by WCP_0 (see [**2**] or [**5**]).

Let φ be $v_{k_1}{=}v_{k_2}$. We may suppose $k_1 \neq k_2$. It follows from WCP_0 that

$$f(\varphi) = d_{k_1k_2} = S^{j_1}_{k_1}S^{j_2}_{k_2}S^{m_1}_{j_1}S^{m_2}_{j_2}d_{m_1m_2} = S^{j_1}_{k_1}S^{j_2}_{k_2}S^{m_1}_{j_1}S^{m_2}_{j_2}f(\varphi^*).$$

The steps $\neg\psi$, $\bigvee\Phi$ and $\bigwedge\Phi$ in the inductive proof of (S) are easy using appropriate properties of S^i_j (see [**2**] and [**5**]).

Let φ be $(\exists v_i)\psi(v_{k_1},\ldots,v_{k_n},v_i)$ and $i \notin \{k_1,\ldots,k_n,m_1,\ldots,m_n\}$. For some distinct integers $j_1,\ldots,j_n$ in $\omega \setminus \{1,\ldots,n,k_1,\ldots,k_n,m_1,\ldots,m_n,i\}$ we have:

$$\begin{aligned}
f(\varphi) &= C_iS^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}f(\psi^*) && \text{by induction assumption}\\
&= S^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}C_if(\psi^*) && \text{by } \mathrm{WCP}_0\\
&= S^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}f(\varphi^*).
\end{aligned}$$

Suppose φ is $(Pv_{l_1},\ldots,v_{l_m} \geq r)\psi(v_{k_1},\ldots,v_{k_n},v_{l_1},\ldots,v_{l_m})$, $L = \{l_1,\ldots,l_m\}$ and $L \cap \{m_1,\ldots,m_n,k_1,\ldots,k_n\} = \emptyset$. For some distinct integers $j_1,\ldots,j_n$ in $\omega \setminus \{1,\ldots,n,k_1,\ldots,k_n,m_1,\ldots,m_n,l_1,\ldots,l_n\}$ we have:

$$\begin{aligned}
f(\varphi) &= C^r_{\langle L\rangle}S^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}f(\psi^*) && \text{by induction assumption}\\
&= S^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}C^r_{\langle L\rangle}f(\psi^*) && \text{by (13) (a) of Theorem 1}\\
&= S^{j_1}_{k_1}\cdots S^{j_n}_{k_n}S^{m_1}_{j_1}\cdots S^{m_n}_{j_n}f(\varphi^*).
\end{aligned}$$

Next, by induction on the complexity of formulas of the logic $L_{\mathcal{A}P\forall}$, we prove the following *dimension property*:

(D) if v_i does not occur free in φ, then $i \notin \Delta f(\varphi)$.

We point out only the case of the probability quantification, because other cases are easy using appropriate parts of Theorem 2. So, let φ be the formula

$(Pv_{l_1},\dots,v_{l_m} \geq r)\psi(v_{k_1},\dots,v_{k_n},v_{l_1},\dots,v_{l_m})$ such that v_i does not occur free in φ, i.e., $i \notin \{k_1 \dots, k_n\}$. Then

$$\begin{aligned} \Delta f(\varphi) &\subseteq \Delta f(\psi) \setminus \{l_1,\dots,l_m\} && \text{by (8) of Theorem 2} \\ &\subseteq \{k_1,\dots,k_n\} && \text{by induction hypothesis,} \end{aligned}$$

i.e., $i \notin \Delta f(\varphi)$.

Now we shall prove that each logical axiom of $L_{\mathcal{A}P\forall}$ is in the set

$$\Gamma = \{\varphi \in \mathrm{Form}_L : f(\varphi) = 1\}.$$

(A) *All axioms of* $L_{\mathcal{A}}$ (see [**3**]):

It follows from the classical theory of cylindric algebras that each logical axiom of $\mathcal{A} \cap L_{\omega\omega}$ is in Γ. Suppose φ is $\bigwedge \Psi \to \psi$, where $\psi \in \Psi$. Then

$$f(\varphi) = -\prod\nolimits_{\xi\in\Psi} f(\xi) + f(\psi) \geq -f(\psi) + f(\psi) = 1.$$

Similarly, if φ is $\neg \bigwedge \Psi \leftrightarrow \bigvee_{\psi\in\Psi} \neg\psi$, then $f(\varphi) = 1$.

(AP) *All axioms of the weak logic* $L_{\mathcal{A}P}$ (see [**4**]):

Monotonicity: Let φ be $(P\mathbf{v} \geq r)\psi \to (P\mathbf{v} \geq s)\psi$, where $r \geq s$. Then for $\mathbf{v} = v_{k_1},\dots,v_{k_n}$ and $K = \{k_1,\dots,k_n\}$ we have $f(\varphi) = -C^r_{\langle K\rangle} f(\psi) + C^s_{\langle K\rangle} f(\psi) = 1$ by WCP_3.

Non-negativity: If φ is $(P\mathbf{v} \geq 0)\psi$, then $f(\varphi) = C^0_{\langle K\rangle} f(\psi) = 1$ by WCP_2.

Let φ be $\theta_1 \leftrightarrow \theta_2$, where θ_1 is $(Pv_{k_1},\dots,v_{k_n} \geq r)\psi(v_{k_1},\dots,v_{k_n})$ and θ_2 is $(Pv_{l_1},\dots,v_{l_n} \geq r)\psi(v_{l_1},\dots,v_{l_n})$. Let $K = \{k_1,\dots,k_n\}$ and $L = \{l_1,\dots,l_n\}$. We may assume that $L \cap K = \emptyset$. Let $m_1,\dots,m_n$ be distinct integers in the set $\omega \setminus \{k_1,\dots,k_n,l_1,\dots,l_n\}$. For some distinct integers $j_1,\dots,j_n$ taken from the set $\omega \setminus \{1,\dots,n,k_1,\dots,k_n,l_1,\dots,l_n,m_1,\dots,m_n\}$ we have:

$$\begin{aligned} f(\theta_1) &= C^r_{\langle K\rangle} S^{j_1}_{k_1}\cdots S^{j_n}_{k_n} S^{m_1}_{j_1}\cdots S^{m_n}_{j_n} f(\psi^*) && \text{by (S)} \\ &= C^r_{\langle K\rangle} S^{l_1}_{k_1}\cdots S^{l_n}_{k_n} S^{m_1}_{l_1}\cdots S^{m_n}_{l_n} f(\psi^*) && \text{by WCP}_0 \\ &= C^r_{\langle L\rangle} S^{k_1}_{l_1}\cdots S^{k_n}_{l_n} S^{m_1}_{l_1}\cdots S^{m_n}_{l_n} f(\psi^*) && \text{by (13) (b) of Theorem 1} \\ &= C^r_{\langle L\rangle} S^{k_1}_{l_1}\cdots S^{k_n}_{l_n} S^{m_1}_{k_1}\cdots S^{m_n}_{k_n} f(\psi^*) && \text{by WCP}_0 \\ &= C^r_{\langle L\rangle} S^{j_1}_{l_1}\cdots S^{j_n}_{l_n} S^{m_1}_{j_1}\cdots S^{m_n}_{j_n} f(\psi^*) && \text{by WCP}_0 \\ &= f(\theta_2) && \text{by (S);} \end{aligned}$$

so, $f(\varphi) = 1$.

Finite additivity: (i) If φ is $(P\mathbf{v} \leq r)\psi \wedge (P\mathbf{v} \leq s)\theta \to (P\mathbf{v} \leq r+s)(\psi \vee \theta)$, then

$$\begin{aligned} f(\varphi) &= -\big(C^{1-r}_{\langle K\rangle} - f(\psi) \cdot C^{1-s}_{\langle K\rangle} - f(\theta)\big) + C^{1-(r+s)}_{\langle K\rangle} - \big(f(\psi) + f(\theta)\big) \\ &\geq -C^{1-r-s}_{\langle K\rangle}\big(- f(\psi) \cdot -f(\theta)\big) + C^{1-(r+s)}_{\langle K\rangle} - \big(f(\psi) + f(\theta)\big) \qquad \text{by WCP}_5\text{ (i)} \\ &= 1. \end{aligned}$$

(ii) If φ is $(P\mathbf{v} \geq r)\psi \wedge (P\mathbf{v} \geq s)\theta \wedge (P\mathbf{v} \leq 0)(\psi \wedge \theta) \to (P\mathbf{v} \geq r+s)(\psi \vee \theta)$, then

$$\begin{aligned} f(\varphi) &= -\Big(C^r_{\langle K\rangle} f(\psi) \cdot C^s_{\langle K\rangle} f(\theta) \cdot C^1_{\langle K\rangle} - \big(f(\psi) \cdot f(\theta)\big)\Big) + C^{r+s}_{\langle K\rangle}\big(f(\psi) + f(\theta)\big) \\ &\geq -C^{r+s}_{\langle K\rangle}\big(f(\psi) + f(\theta)\big) + C^{r+s}_{\langle K\rangle}\big(f(\psi) + f(\theta)\big) \qquad \text{by WCP}_5 \text{ (ii)} \\ &= 1. \end{aligned}$$

The Archimedean property: If φ is $(P\mathbf{v} > r)\psi \leftrightarrow \bigvee_{m>0}(P\mathbf{v} \geq r + 1/m)\psi$, then

$$\begin{aligned} f\big((P\mathbf{v} > r)\psi\big) &= -C^{1-r}_{\langle K\rangle} - f(\psi) = \sum\nolimits_{m>0} C^{r+1/m}_{\langle K\rangle} f(\psi) \quad \text{by WCP}_6 \\ &= f\Big(\bigvee\nolimits_{m>0}(P\mathbf{v} \geq r + 1/m)\psi\Big); \end{aligned}$$

so, $f(\varphi) = 1$.

$(AP\forall_1)$ Let φ be $(\forall v_i)\psi \to (Pv_i \geq 1)\psi$. Then

$$\begin{aligned} f(\varphi) &= - - C_i - f(\psi) + C^1_i f(\psi) \\ &\geq C_i - f(\psi) + - C_i - f(\psi) \quad \text{by WCP}_8 \\ &= 1. \end{aligned}$$

$(AP\forall_2)$ Let φ be $(Pv_{k_1} \cdots v_{k_n} \geq r)\psi \to (Pv_{k_{\pi 1}} \cdots v_{k_{\pi n}} \geq r)\psi$. Then

$$\begin{aligned} f(\varphi) &= -C^r_{\langle K\rangle} f(\psi) + C^r_{\langle \pi(K)\rangle} f(\psi) \\ &\geq -C^r_{\langle \pi(K)\rangle} f(\psi) + C^r_{\langle \pi(K)\rangle} f(\psi) \quad \text{by WCP}_7 \\ &= 1. \end{aligned}$$

Finally, we shall prove that each logical theorem of $L_{AP\forall}$ is in Γ. Obviously Γ is closed under Modus Ponens and under Conjunction rule. We have two Generalization rules.

If $\varphi \to \psi(v_i) \in \Gamma$ and v_i is not free in φ, then

$$\begin{aligned} f\big(\varphi \to (\forall v_i)\psi\big) &= -f(\varphi) + - C_i - f(\psi) \\ &= -\big(C_i f(\varphi) \cdot C_i - f(\psi)\big) \quad \text{by (D)} \\ &= -C_i\big(C_i f(\varphi) \cdot - f(\psi)\big) \quad \text{by WCP}_0 \\ &= 1 \quad \text{by assumption.} \end{aligned}$$

So, $\varphi \to (\forall v_i)\psi \in \Gamma$.

If $\varphi \to \psi(v_{k_1}, \ldots, v_{k_n}) \in \Gamma$ and $v_{k_1}, \ldots, v_{k_n}$ are not free in φ, then

$$\begin{aligned} f\big(\varphi \to (P\mathbf{v} \geq 1)\psi\big) &= -f(\varphi) + C^1_{\langle K\rangle} f(\psi) \\ &= -C^1_{\langle K\rangle} f(\varphi) + C^1_{\langle K\rangle} f(\psi) \quad \text{by (D) and (11) of Theorem 1.} \\ &= C^1_{\langle K\rangle}\big(-f(\varphi) + f(\psi)\big) \quad \text{by WCP}_4 \text{ and (2) of Theorem 1.} \\ &= 1 \quad \text{by assumption.} \end{aligned}$$

So, $\varphi \to (P\mathbf{v} \geq 1)\psi \in \Gamma$.

It follows that $\vdash \varphi \leftrightarrow \psi$ implies $f(\varphi) = f(\psi)$. So, we introduce a well-defined function $g\colon \mathrm{Form}_L /{\equiv_\emptyset} \to A$ by $g(\varphi^\emptyset) = f(\varphi)$. It is easy to see that g is a homomorphism from $\mathfrak{Form}_L /{\equiv_\emptyset}$ onto $\mathbf{A}$ such that $g\big(R_a(v_1, \dots, v_n)^\emptyset\big) = a$. Let $\mathcal{I} = \{\varphi^\emptyset : g(\varphi^\emptyset) = 0\}$ be a subset of $\mathrm{Form}_L /{\equiv_\emptyset}$, and let Σ be a set of all sentences φ of $L_{\mathcal{A}P\forall}$ such that $(\neg\varphi)^\emptyset \in \mathcal{I}$. Then $\mathcal{I}$ is an ideal in $\mathfrak{Form}_L /{\equiv_\emptyset}$ and

$$\mathbf{A} \cong (\mathfrak{Form}_L /{\equiv_\emptyset}) / \mathcal{I} \cong \mathfrak{Form}_L /{\equiv_\Sigma} .$$

Moreover, Σ is consistent, since $|A| > 1$. Let $\mathfrak{A}$ be a weak probability model of Σ (see [**6**]). Then we have a "natural" homomorphism from $\mathfrak{Form}_L /{\equiv_\Sigma}$ onto the weak cylindric probability set algebra

$$\big\langle \{\, \varphi^{\mathfrak{A}} : \varphi \in \mathrm{Form}_L \,\}, \cup, \cap, \sim, \emptyset, A^\omega, C_i, C^r_{\langle K \rangle}, D_{pq} \big\rangle.$$

This completes the proof. □

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