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TOTALLY UMBILICAL DEGENERATE MONGE HYPERSURFACES OF R_1^4

Constantin Călin

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Abstract. We determine all the totally umbilical lightlike Monge hypersurfaces of R_1^4. This is done by using the Bejancu-Duggal method [**1**] of studying lightlike hypersurfaces and then integrating a system of partial differential equations.

Bejancu-Duggal [**1**] proved that a lightlike cone of the semi-Euclidian space R_s^{m+1} is a totally umbilical degenerate hypersurface. We determine all totally umbilical degenerate Monge hypersurfaces of R_1^4. To this end we recall the terminology and a few results from general theory of degenerate hypersurfaces of semi-Riemannian manifolds (see [**1**]).

Let $(\tilde{M}, \tilde{g})$ be an $(m+1)$-dimensional semi-Riemannian manifold and let M be a hypersurface of $\tilde{M}$. Denote by g the induced tensor field on M by $\tilde{g}$. We say that M is a degenerate hypersurface of $\tilde{M}$ if $\operatorname{rank} g = m-1$ on M. Thus, both the tangent space T_xM and the normal space $T_xM^\perp$ are degenerate for each $x \in M$. It is easy to see that M is a degenerate hypersurface of $\tilde{M}$ iff the vector bundle

$$TM^\perp = \bigcup_{x\in M} T_xM^\perp, \quad T_xM^\perp = \{X_x \in T_x\tilde{M} \mid \tilde{g}(X_x, Y_x) = 0, \ \forall\, Y_x \in T_xM\},$$

becomes a distribution of rank 1 on M.

Throughout the paper we suppose all manifolds to be paracompact and smooth. We denote by $F(M)$ the algebra of differentiable functions on M and by $\Gamma(E)$ the $F(M)$-module of differentiable sections of a vector bundle E over M.

The *screen distribution* SM on M is a complementary orthogonal distribution of $TM^\perp$ in TM, that is, we have $TM = SM \perp TM^\perp$, where $\perp$ between vector bundles means orthogonal direct sum.

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From [**1**] we recall

Theorem 1. *Let M be a degenerate hypersurface of $(\tilde{M}, \tilde{g})$ and SM be a screen distribution on M. Then there exists a unique vector bundle NM of rank 1 over M, such that, for any non-null section ξ of $TM^{\perp}$ on a coordinate neighborhood $U \subset M$, there exists a unique local section N of NM satisfying*

$$\tilde{g}(N, \xi) = 1, \tag{1}$$

$$\tilde{g}(N, N) = \tilde{g}(N, X) = 0, \ \ \forall X \in \Gamma(SM). \tag{2}$$

From (1) and (2) it follows that NM is a lightlike vector bundle on M, and we have the next decompositions

$$T\tilde{M}_{|M} = SM \perp (TM^{\perp} \oplus NM) = TM \oplus NM, \tag{3}$$

where $\oplus$ between vector bundles means direct sum but not orthogonal. The vector bundle NM is called the *lightlike transversal vector bundle* of M. Next, suppose $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$ with respect to $\tilde{g}$ and by using the last decomposition in (3) we infer

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \forall X, Y \in \Gamma(TM). \tag{4}$$

It is easy to see that ∇ is a torsion-free linear connection on M, but it is not a metric connection, because

$$(\nabla_X g)(Y, Z) = B(X, Y)\tilde{g}(Z, N) + B(X, Z)\tilde{g}(Y, N), \quad \forall X, Y, Z \in \Gamma(TM).$$

The 2-form B is symmetric on U and it is called the local second fundamental form of M. By using (1) and (4) we infer

$$B(X, Y) = \tilde{g}(\tilde{\nabla}_X Y, \xi), \ \ \forall X, Y \in \Gamma(TM), \tag{5}$$

which proves that B *does not depend on the screen distribution* SM (cf. [**1**]).

Next, we say that M is a totally umbilical hypersurface, if locally, on each $U \subset M$, there exists a smooth function λ such that

$$B(X, Y) = \lambda g(X, Y), \ \ \forall X, Y \in \Gamma(TM|_U). \tag{6}$$

It is proved in[**1**] that a lightlike cone of the semi-Euclidian space R_s^{m+1} is a totally umbilical degenerate hypersurface with $\lambda = -1$. Here we shall determine all degenerate Monge hypersurfaces of R_1^4 with the semi-Euclidian metric

$$\tilde{g} = -x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4.$$

Suppose that the Monge hypersurface M is given by the explicit equation

$$x^4 = F(x^1, x^2, x^3) \tag{7}$$

where F is a smooth function on a domain $D \subset R^3$. In this case $TM^{\perp}$ is globally spanned by

$$\xi = \frac{\partial F}{\partial x^1}\frac{\partial}{\partial x^1} - \frac{\partial F}{\partial x^2}\frac{\partial}{\partial x^2} - \frac{\partial F}{\partial x^3}\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4}.$$

and we infer

THEOREM 2. *Let M be a Monge hypersurface of R_1^4. Then M is degenerate iff F satisfies the equation*

$$\left(\frac{\partial F}{\partial x^1}\right)^2 = \left(\frac{\partial F}{\partial x^2}\right)^2 + \left(\frac{\partial F}{\partial x^3}\right)^2 + 1. \tag{8}$$

Now we can prove a characterization theorem for all degenerate Monge hypersurfaces of R_1^4.

THEOREM 3. *Let M be a Monge hypersurface of R_1^4 given by (7). Then the hypersurface M is degenerate iff F is given by*

$$\begin{aligned} F(x^1,x^2,x^3) = \int_{x_0^1}^{x^1} \frac{1}{\cos v(t,x_0^2,x_0^3)}dt + \int_{x_0^2}^{x^2} \cos u(x^1,t,x_0^3)\tan v(x^1,t,x_0^3)dt \\ + \int_{x_0^3}^{x^3} \sinh u(x^1,x^2,t)\tan v(x^1,x^2,t)dt + \alpha, \end{aligned} \tag{9}$$

where α is a real constant, (x_0^1, x_0^2, x_0^3) are the cartesian coordinates of a fixed point x_0 from D and u, v are two smooth functions on D satisfying the partial differential equations

$$\begin{aligned} \cos u\frac{\partial v}{\partial x^3} - \sin u\frac{\partial v}{\partial x^2} &= \cos v\frac{\partial u}{\partial x^1}, \\ (\sin u\frac{\partial v}{\partial x^3} + \cos u\frac{\partial v}{\partial x^2})\sin v &= \frac{\partial v}{\partial x^1}, \\ (\cos u\frac{\partial u}{\partial x^2} + \sin u\frac{\partial u}{\partial x^3})\sin v &= \frac{\partial u}{\partial x^1}. \end{aligned} \tag{10}$$

Proof. Suppose M is a Monge degenerate hypersurface of R_1^4. By using (8) it follows that there exist two smooth functions u and v such that

$$\frac{\partial F}{\partial x^1} = \frac{1}{\cos v}, \quad \frac{\partial F}{\partial x^2} = \cos u\tan v, \quad \frac{\partial F}{\partial x^3} = \sin u\tan v, \tag{11}$$

There exists a smooth function F on a domain $D \subset R^3$, iff

$$\begin{aligned} \frac{\partial}{\partial x^2}(\frac{1}{\cos v}) &= \frac{\partial}{\partial x^1}(\cos u\tan v) \\ \frac{\partial}{\partial x^2}(\sin u\tan v) &= \frac{\partial}{\partial x^3}(\cos u\tan v) \\ \frac{\partial}{\partial x^1}(\sin u\tan v) &= \frac{\partial}{\partial x^3}(\frac{1}{\cos v}) \end{aligned}$$

which is equivalent with

$$
(12)\qquad
\begin{aligned}
\sin v\frac{\partial v}{\partial x^2} &= \frac{\partial v}{\partial x^1}\cos u - \frac{1}{2}\frac{\partial u}{\partial x^1}\sin u\sin 2v\\
\sin v\frac{\partial v}{\partial x^3} &= \frac{\partial v}{\partial x^1}\sin u + \frac{1}{2}\frac{\partial u}{\partial x^1}\cos u\sin 2v\\
\sin v\frac{\partial v}{\partial x^2} + \frac{1}{2}\frac{\partial u}{\partial x^2}\cos u\sin 2u &= \frac{\partial v}{\partial x^3}\cos u - \frac{1}{2}\frac{\partial u}{\partial x^3}\sin u\sin 2v
\end{aligned}
$$

By eliminating $\frac{\partial v}{\partial x^1}$ from the first equation from (12) we obtain the first equation of (10), and by eliminating $\frac{\partial u}{\partial x^1}$ from the same equation we obtain the second equation of (10). Finally the last equation of (10) is obtained by eliminating $\frac{\partial v}{\partial x^2}$ and $\frac{\partial v}{\partial x^3}$ from all equations of (12). Then with the help of (11) we deduce

$$
(13)\qquad dF(x^1,x^2,x^3) = \frac{1}{\cos v}dx^1 + \cos u\tan v dx^2 + \sin u\tan v dx^3.
$$

Because (10) holds, then (9) follows from (13). Conversely, suppose F is given by (9) and u and v satisfy (10). By direct calculation and by using (10) we deduce that F satisfies (11) and consequently (8) is verified. The proof is complete.

Next we consider a particular screen distribution on M. Let $V = \frac{\partial F}{\partial x^1}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}$ be the vector field defined on M such that $g(V,\xi)\neq 0$, and consequently V is not tangent to M. Now, take SM such that it is orthogonal to $\mathrm{span}\{V,\xi\}$ and obtain

$$
SM = \mathrm{span}\{X_1 = -\frac{\partial F}{\partial x^3}\frac{\partial}{\partial x^2} + \frac{\partial F}{\partial x^2}\frac{\partial}{\partial x^3};\ X_2 = \frac{\partial}{\partial x^1} + \frac{\partial F}{\partial x^1}\frac{\partial}{\partial x^4}\}.
$$

It is easy to check that SM is a complementary distribution to $TM^{\perp}$ in TM. According to [**1**] we call it the *canonical screen distribution* on M. Next we have

THEOREM 4. *Let M be a Monge hypersurface of R_1^4 given by* (7). *M is totally umbilical iff F satisfies the partial differential equations*

$$
-\frac{\partial^2 F}{\partial (x^1)^2} + \left(\frac{\partial F}{\partial x^2}\right)^2\frac{\partial^2 F}{\partial (x^3)^2} + \left(\frac{\partial F}{\partial x^3}\right)^2\frac{\partial^2 F}{\partial (x^2)^2} - 2\frac{\partial F}{\partial x^2}\frac{\partial F}{\partial x^3}\frac{\partial^2 F}{\partial x^2\partial x^3} = 0,
$$
$$
\frac{\partial F}{\partial x^2}\frac{\partial^2 F}{\partial x^1\partial x^3} = \frac{\partial F}{\partial x^3}\frac{\partial^2 F}{\partial x^1\partial x^2}.
$$

Proof. Because $B(X,\xi)=0$ for any $X\in\Gamma(TM)$, we have to calculate $B(X,Y)$ for any $X,Y\in\Gamma(SM)$. Choose SM as a canonical screen distribution on M and by direct calculation, using (5) we deduce

$$
(14)\qquad
\begin{aligned}
B(X_1,X_1) &= -2\frac{\partial F}{\partial x^2}\frac{\partial F}{\partial x^3}\frac{\partial^2 F}{\partial x^2\partial x^3} + \left(\frac{\partial F}{\partial x^2}\right)^2\frac{\partial^2 F}{\partial (x^3)^2} + \left(\frac{\partial F}{\partial x^3}\right)^2\frac{\partial^2 F}{\partial (x^2)^2},\\
B(X_1,X_2) &= \frac{\partial F}{\partial x^2}\frac{\partial^2 F}{\partial x^1\partial x^3} - \frac{\partial F}{\partial x^3}\frac{\partial^2 F}{\partial x^1\partial x^2},\quad B(X_2,X_2) = \frac{\partial^2 F}{\partial (x^1)^2}.
\end{aligned}
$$

By straightforward calculation, and using (5) we obtain

(15)
$$g(X_1,X_1)=\left(\frac{\partial F}{\partial x^2}\right)^2+\left(\frac{\partial F}{\partial x^3}\right)^2,\quad g(X_1,X_2)=0,\quad g(X_2,X_2)=\left(\frac{\partial F}{\partial x^1}\right)^2-1.$$

Finally, our assertion follows from (6), (14) and (15).

From Theorem 4, by using (11) and the last relation of (10) we obtain

Corollary 1. *A degenerate Monge hypersurface M of R^4_1 is totally umbilical if u and v from Theorem 3 satisfy the partial differential equations*

(16)
$$\begin{gathered}\frac{\partial v}{\partial x^1}\cos v+\left(\frac{\partial u}{\partial x^2}\sin u-\frac{\partial u}{\partial x^3}\cos u\right)\sin^2 v=0,\\ \cos u\frac{\partial v}{\partial x^3}=\sin u\frac{\partial v}{\partial x^2}.\end{gathered}$$

Next we determine all functions u and v which satisfy the relations (10) and (16) and consequently all totally umbilical degenerate Monge hypersurfaces of R^4_1. From the first equation of (10) and the last equation of (16) we deduce $\frac{\partial u}{\partial x^1}=0$ and this introduced in the last equation of (10) implies

(17)
$$\cos u\frac{\partial u}{\partial x^2}+\sin u\frac{\partial u}{\partial x^3}=0.$$

Therefore, u must be given by an implicit equation of the form

(18)
$$x^2\sin u-x^3\cos u=\epsilon(u),$$

where ϵ is an arbitrary smooth function. By using the fact that $\frac{\partial u}{\partial x^1}$=0, and the first equation (16) we obtain

(19)
$$\frac{\partial v}{\partial x^1}\cos v=-\frac{\sin^2 v}{\cos u}\frac{\partial u}{\partial x^2}.$$

Next from the last equation of (10), (17), (19) and the second equations of (16) we infer

(20)
$$\begin{gathered}\frac{\cos v}{\sin v}\frac{\partial v}{\partial x^2}=-\frac{\cos u}{\sin u}\frac{\partial u}{\partial x^2}\\ \frac{\cos v}{\sin v}\frac{\partial v}{\partial x^3}=-\frac{\sin u}{\cos u}\frac{\partial u}{\partial x^3}.\end{gathered}$$

Integrating (20) we derive

(21)
$$\sin v=\frac{\alpha(x^1,x^3)}{\sin u}=\frac{\beta(x^1,x^2)}{\cos u}.$$

From (19) and (21) we deduce

$$\alpha(x^1,x^3) = h(x^3) - \frac{x^1}{\sin u}\frac{\partial u}{\partial x^2}, \quad \beta(x^1,x^2) = k(x^2) - x^1\frac{\cos u}{\sin u}\frac{\partial u}{\partial x^2}, \tag{22}$$

where h and k are smooth functions satisfying

$$h(x^3)\cos u = k(x^2)\sin u. \tag{23}$$

Finally we deduce

$$v = \arcsin\left(\frac{h(x^3)}{\sin u} - \frac{x^1}{\sin u}\frac{\partial u}{\partial x^2}\right) = \arcsin\left(\frac{k(x^2)}{\cos u} - \frac{x^1}{\sin u}\frac{\partial u}{\partial x^2}\right),$$

with h and k satisfying (23). Thus, we can state

THEOREM 5. *A degenerate Monge hypersurface of* (R^4_1, g) *given by the equation* (7) *is totally umbilical if and only if* F *is given by* (9) *and* u *and* v *are expressed as in* (18), (22) *and* (23).

Remark. If instead of (7) we consider one of the next equations $x^2 = F(x^1,x^3,x^4)$, $x^3 = F(x^1,x^2,x^4)$, $x^4 = F(x^1,x^2,x^3)$, we obtain the same results. If $\epsilon(u) = 0$ and $h(x^3) = k(x^2) = 0$, we obtain the lightlike cone of R^4_1. By other choices of functions ϵ, h and k we obtain other totally umbilical degenerate Monge hypersurfaces of R^4_1.

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Department of Mathematics
Technical University
6600 Iasi, Romania
e-address: calin.@math.tuiasi.ro

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