

ON SOME GENERALIZED EINSTEIN METRIC CONDITIONS

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Communicated by Mileva Prvanović

Abstract. We present a construction of compact warped product manifolds realizing certain generalized Einstein metric conditions.

1. Introduction

A semi-Riemannian manifold (M, g) , $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$(1) \quad S = (\kappa/n)g$$

holds on M , where S and κ denote the Ricci tensor and the scalar curvature of (M, g) , respectively. According to [1], p. 432], (1) is called the Einstein metric condition. Einstein manifolds form a natural subclass of various classes of semi-Riemannian manifolds determined by a curvature condition imposed on their Ricci tensor [1, Table, pp. 432–433]. For instance, every Einstein manifold belongs to the class of semi-Riemannian manifolds (M, g) realizing the following relation

$$(2) \quad (\nabla(S - (\kappa/(2(n-1)))g))(X, Y; Z) = (\nabla(S - (\kappa/(2(n-1)))g))(X, Z; Y),$$

for all $X, Y, Z \in \Xi(M)$, which means that $S - (\kappa/(2(n-1)))g$ is a Codazzi tensor on M . In the above formula ∇ denotes the Levi-Civita connection of (M, g) , $\Xi(M)$ being the Lie algebra of vector fields on M . Manifolds of dimensions ≥ 4 fulfilling (2) are called manifolds with harmonic Weyl tensor [1, p. 440]. In the following we will denote by S^n a sphere of radius 1 in the Euclidean space \mathbf{E}^{n+1} , $n \geq 1$. It is known that every warped product $S^1 \times_F M$ of the sphere S^1 , with a positive smooth function F , and an Einstein manifold (M, g) , $\dim M \geq 2$, realizes (2) [1, p. 433]. Such warped product is a non-Einstein manifold, in general. We say that (2) is a generalized Einstein metric condition [1, Chapter XVI]. On the other hand, such warped product realize too another curvature condition, so called

a condition of pseudosymmetry type (see section 2 of this paper). Namely, the warped product $S^1 \times_F M$ of the sphere S^1 , with a positive smooth function F , and an Einstein manifold (M, g) , $\dim M \geq 2$, is a Ricci-pseudosymmetric manifold [15, Corollary 3.2]. Thus, in particular, the warped product $S^1 \times_F \mathbf{C}P^n$ of S^1 , with a positive smooth function F , and the complex projective space $\mathbf{C}P^n$ (considered with its standard Riemannian locally symmetric metric) is a Ricci-pseudosymmetric manifold. A semi-Riemannian manifold (M, g) , $\dim M \geq 3$, is said to be Ricci-pseudosymmetric [6], [15] if at every point of M the following condition is satisfied:

(*) the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The tensors $R \cdot S$ and $Q(g, S)$ are defined by

$$\begin{aligned} (R \cdot S)(X_1, X_2; X, Y) &= (\tilde{\mathcal{R}}(X, Y) \cdot S)(X_1, X_2) \\ &= -S(\tilde{\mathcal{R}}(X, Y)X_1, X_2) - S(X_1, \tilde{\mathcal{R}}(X, Y)X_2), \\ Q(g, S)(X_1, X_2; X, Y) &= ((X \wedge Y) \cdot S)(X_1, X_2) \\ &= -S((X \wedge Y)X_1, X_2) - S(X_1, (X \wedge Y)X_2), \end{aligned}$$

respectively. In the above formulas $\tilde{\mathcal{R}}(X, Y) \cdot$ and $X \wedge Y \cdot$ denote the derivations of the algebra of the tensor fields on M , determined by the curvature operator $\tilde{\mathcal{R}}(X, Y)$ and the wedge product $X \wedge Y$, respectively. The pseudosymmetric manifolds form very important subclass of the class of Ricci-pseudosymmetric manifolds (see section 2). Evidently, any Einstein manifold is Ricci-pseudosymmetric. Thus we see that (*) is a generalized Einstein metric condition. Recently, some examples of compact and non-Einstein Ricci-pseudosymmetric manifolds were found in [19]. Namely, in [19, Theorem 1] it was shown that the Cartan hypersurfaces M in the spheres S^7 , S^{17} or S^{25} are non-pseudosymmetric, Ricci-pseudosymmetric manifolds with non-pseudosymmetric Weyl tensor. The Cartan hypersurfaces M in S^4 are non-semisymmetric, pseudosymmetric manifolds. It is known that Cartan hypersurfaces are manifolds, with non-parallel Ricci tensor, satisfying a generalized Einstein metric condition of the following form [20, Theorem 4.1]

$$(3) \quad (\nabla S)(X, Y; Z) + (\nabla S)(Y, Z; X) + (\nabla S)(Z, X; Y) = 0,$$

for all $X, Y, Z \in \Xi(M)$. Evidently, the Cartan hypersurfaces do not satisfy (2).

Let f be a non-constant function satisfying on S^p , $p \geq 2$, the equality [20]

$$(4) \quad \bar{\nabla}(df) + f\bar{g} = 0,$$

where \bar{g} is the standard metric on S^p and $\bar{\nabla}$ is the Levi-Civita connection of \bar{g} . We put

$$(5) \quad F = (f + c)^2,$$

where c is a non-zero constant such that $f+c$ is a positive or negative function on S^p . In this paper we shall describe a family of compact Ricci-pseudosymmetric warped product manifolds. Our main result states that the warped product $S^p \times_F M$ of S^p , $p \geq 2$, with the function F defined by (5), and an Einstein manifold M , $\dim M \geq 2$, is a Ricci-pseudosymmetric manifold. From this it follows immediately that the warped product $S^p \times_F M$ of S^p , $p \geq 2$, with the function F defined by (5), and a compact irreducible symmetric space M is a compact Ricci-pseudosymmetric manifold. In particular, the warped product S^2 (resp., S^3), with the function F defined by (5), and the quaternion projective space $\mathbf{H}P^n$ (considered with its standard Riemannian locally symmetric metric), is a non-pseudosymmetric, Ricci-pseudosymmetric manifold with non-pseudosymmetric Weyl tensor. Applying this construction of the warped product manifolds to the generalized Hopf fibrations [1, p. 258]: $S^2 \rightarrow \mathbf{C}P^{2n+1} \rightarrow \mathbf{H}P^n$ and $S^3 \rightarrow S^{4n+3} \rightarrow \mathbf{H}P^n$, we obtain a local curvature property of the projective complex space $\mathbf{C}P^{2n+1}$ and of the sphere S^{4n+3} , respectively. More precisely, at the end of section 4, we shall state that every point of $\mathbf{C}P^{2n+1}$ (resp., S^{4n+3}) has a neighbourhood on which is defined a Riemannian metric isometric with the Ricci-pseudosymmetric warped product metric of an open submanifold of the manifold $S^2 \times_F \mathbf{H}P^n$, (resp., $S^3 \times_F \mathbf{H}P^n$). We note that by an application of Theorem 4.1 of [11] to the generalized Hopf fibration [1, p. 258]: $S^7 \rightarrow S^{15} \rightarrow S^8$, we can obtain the following local curvature property of S^{15} . Namely, every point of S^{15} has a neighbourhood on which is defined a Riemannian metric isometric with the non-conformally flat, pseudosymmetric warped product metric of an open submanifold of the manifold $S^7 \times_F S^8$. Furthermore, applying Corollary 3.2 of [15] to the Hopf fibration [1, p. 257]: $S^1 \rightarrow S^{2n+1} \rightarrow \mathbf{C}P^n$ we see that every point of S^{2n+1} , $n \geq 2$, has a neighbourhood on which is defined a Riemannian metric isometric with a non-conformally flat and non-pseudosymmetric, Ricci-pseudosymmetric warped product metric of an open submanifold of the manifold $S^1 \times_F \mathbf{C}P^n$. In section 3 we will consider warped products manifolds. We compute the local components of some tensors defined on the warped product $S^p \times_F M$ of the sphere S^p , with the function F defined by (5), and an Einstein manifold (M, \tilde{g}) . The main results of this paper are presented in section 4. Throughout this paper all manifolds are assumed to be connected, paracompact manifolds of class C^∞ .

2. Curvature conditions of pseudosymmetry type

Let (M, g) be a connected n -dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞ . We define on M the endomorphisms $\tilde{\mathcal{R}}(X, Y)$ and $X \wedge Y$ by

$$\tilde{\mathcal{R}}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad (X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

respectively, where $X, Y, Z \in \Xi(M)$. We define the Riemann-Christoffel curvature tensor R and the concircular tensor $Z(R)$ of (M, g) by $R(X_1, \dots, X_4) = g(\tilde{\mathcal{R}}(X_1, X_2)X_3, X_4)$, $Z(R) = R - (\kappa/(n(n-1)))G$, respectively, where G is defined by $G(X_1, \dots, X_4) = g((X_1 \wedge X_2)X_3, X_4)$. For a $(0, k)$ -tensor field T , $k \geq 1$,

we define the $(0, k + 2)$ -tensors $R \cdot T$ and $Q(g, T)$ by

$$\begin{aligned} (R \cdot T)(X_1, \dots, X_k; X, Y) &= (\tilde{\mathcal{R}}(X, Y) \cdot T)(X_1, \dots, X_k) \\ &= -T(\tilde{\mathcal{R}}(X, Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, \tilde{\mathcal{R}}(X, Y)X_k), \\ Q(g, T)(X_1, \dots, X_k; X, Y) &= ((X \wedge Y) \cdot T)(X_1, \dots, X_k) \\ &= -T((X \wedge Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, (X \wedge Y)X_k). \end{aligned}$$

The semi-Riemannian manifold (M, g) is said to be pseudosymmetric [13] if at every point of M the following condition is satisfied:

(**) the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The manifold (M, g) is pseudosymmetric if and only if

$$(6) \quad R \cdot R = LQ(g, R)$$

holds on the set $U_R = \{x \in M | Z(R) \neq 0 \text{ at } x\}$, where L is some function on U_R . It is clear that any semisymmetric manifold ($R \cdot R = 0$) is pseudosymmetric. The condition (**) arose on the study on totally umbilical submanifolds of semisymmetric manifolds as well as when considering of geodesic mappings of semisymmetric manifolds [10], [23]. There exist many examples of pseudosymmetric manifolds which are not semisymmetric (see, e.g. [11], [12], [13]). Among these examples we can distinguish also compact pseudosymmetric manifolds. For instance, in [11] (see Example 3.1 and Theorem 4.1) it was proved that the warped product $S^p \times_F S^{n-p}$, $p \geq 2$, $n-p \geq 1$, with the function F defined by (5), is a pseudosymmetric manifold. Another example of a compact pseudosymmetric manifold is the warped product $S^1 \times_F S^{n-1}$, with a positive smooth function F , as well as n -dimensional tori T^n with a certain metric (see [11, Examples 4.1 and 4.2]).

The manifold (M, g) is Ricci-pseudosymmetric if and only if

$$(7) \quad R \cdot S = L_S Q(g, S)$$

holds on the set $U_S = \{x \in M | S - (\kappa/n)g \neq 0 \text{ at } x\}$, where L_S is some function on U_S . It is clear that if at a point x of a manifold (M, g) (**) is satisfied then also (*) holds at x . The converse statement is not true. E.g. every warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = n - 1 \geq 3$, of a manifold (M_1, \bar{g}) and a non-pseudosymmetric, Einstein manifold (M_2, \tilde{g}) is a non-pseudosymmetric, Ricci-pseudosymmetric manifold (cf. [15, Remark 3.4] and [13, Theorem 4.1]). Warped products realizing (*) were considered in [6] and [15]. Recently, Ricci-pseudosymmetric hypersurfaces immersed isometrically in a semi-Riemannian manifolds of constant curvature were investigated in [4].

For any $X, Y \in \Xi(M)$ we define the endomorphism $\tilde{\mathcal{C}}(X, Y)$ by

$$\tilde{\mathcal{C}}(X, Y) = \tilde{\mathcal{R}}(X, Y) - (1/(n - 2))(X \wedge \tilde{S}Y + \tilde{S}X \wedge Y - (\kappa/(n - 1))X \wedge Y),$$

where the Ricci operator \tilde{S} of (M, g) is defined by $g(\tilde{S}X, Y) = S(X, Y)$. The Weyl curvature tensor C of (M, g) is defined by $C(X_1, \dots, X_4) = g(\tilde{C}(X_1, X_2)X_3, X_4)$. Now we define on M the $(0, 6)$ -tensor $C \cdot C$ by

$$\begin{aligned} (C \cdot C)(X_1, \dots, X_4; X, Y) &= (\tilde{C}(X, Y) \cdot C)(X_1, \dots, X_4) \\ &= -C(\tilde{C}(X, Y)X_1, \dots, X_4) - \dots - C(X_1, X_2, X_3, \tilde{C}(X, Y)X_4). \end{aligned}$$

A semi-Riemannian manifold (M, g) , $\dim M \geq 4$, is said to be a manifold with pseudosymmetric Weyl tensor [18] if at every point of M the following condition is satisfied:

(***) the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent.

The manifold (M, g) is a manifold with pseudosymmetric Weyl tensor if and only if

$$(8) \quad C \cdot C = L_C Q(g, C)$$

holds on the set $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on U_C . The condition (***) arose in the study of 4-dimensional warped products [9]. Namely, in [9, Theorem 2] it was proved that at every point of a warped product $M_1 \times_F M_2$, with $\dim M_1 = \dim M_2 = 2$, (***) is fulfilled. Many examples of manifolds satisfying (***) are present in [3]. For instance, the Cartesian product of two manifolds of constant curvature is a manifold realizing (***). Warped products satisfying (***) were considered in [18]. In [3] it was shown that the classes of manifolds realizing (**) and (***) do not coincide. However, there exist pseudosymmetric manifolds fulfilling (8), e.g. Einsteinian pseudosymmetric manifolds [3, Theorem 3.1].

Remark 2.1. The above mentioned warped product $S^p \times_F S^{n-p}$, $p \geq 2$, $n - p \geq 2$, is a pseudosymmetric manifold with pseudosymmetric Weyl tensor which cannot be realized as a hypersurface isometrically immersed in a space of constant curvature $M^{n+1}(c)$, $n \geq 4$ [11, Theorem 4.1].

Further, we define on M the $(0, 6)$ -tensor $Q(S, R)$ by

$$\begin{aligned} Q(S, R)(X_1, \dots, X_4; X, Y) &= ((X \wedge_S Y) \cdot R)(X_1, \dots, X_4) \\ &= -R((X \wedge_S Y)X_1, X_2, X_3, X_4) - \dots - R(X_1, X_2, X_3, (X \wedge_S Y)X_4), \end{aligned}$$

where $X \wedge_S Y$ is the endomorphism defined by $(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y$. A semi-Riemannian manifold (M, g) is said to be Ricci-generalized pseudosymmetric [10] if at every point of M the following condition is satisfied:

(****) the tensors $R \cdot R$ and $Q(S, R)$ are linearly dependent.

A very important subclass of Ricci-generalized pseudosymmetric manifolds form manifolds fulfilling (see, e.g. [2]):

$$(9) \quad R \cdot R = Q(S, R).$$

Any 3-manifold (M, g) fulfils (9) [8, Theorem 3.1]. Moreover, any hypersurface M immersed isometrically in an $(n + 1)$ -dimensional Euclidean space \mathbf{E}^{n+1} , $n \geq 4$, satisfies (9) [16, Corollary 3.1].

Remark 2.3. It is easy to see that if (**) holds on a semi-Riemannian manifold (M, g) , $n \geq 4$, then at every point of M the following condition is satisfied:

(*****) the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

Manifolds fulfilling (*****) has been studied in [7], [9] and [14].

As it was proved in [16], at every point of a hypersurface M immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c)$, $n \geq 4$, the following condition is satisfied:

(*****) the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent.

Remark 2.3. In [16, Proposition 3.1] it was proved that every hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $M^{n+1}(c)$, $n \geq 4$, satisfies the following equality $R \cdot R - Q(S, R) = -(((n - 2)\tilde{\kappa})/(n(n + 1)))Q(g, C)$, where $\tilde{\kappa}$ is the scalar curvature of $M^{n+1}(c)$ and R, S and C is the Riemann-Christoffel curvature tensor, the Ricci tensor and the Weyl tensor of M , respectively.

Using Theorem 3.1 of [8, which was mentioned above, and the fact that the Weyl tensor of any 3-dimensional semi-Riemannian manifold vanishes identically, we conclude that (*****) is trivially satisfied on any 3-dimensional semi-Riemannian manifold. Recently, warped products realizing (*****) were considered in [5]. For instance, in [5] it was stated that every warped product $M_1 \times_F M_2$, with $\dim M_1 = 1$, $\dim M_2 = 3$, with an arbitrary positive smooth function F , satisfies (*****). The relations (*)-(*****) are called conditions of pseudosymmetry type. We refer to [10] and [23] as the review papers on semi-Riemannian manifolds satisfying such conditions.

Remark 2.4. Every Einsteinian, as well as every conformally flat hypersurface M in $M^{n+1}(c)$, $n \geq 4$, is a pseudosymmetric manifold [16, Proposition 3.2]. Thus every quasi-umbilical hypersurface M in $M^{n+1}(c)$, $n \geq 4$, is pseudosymmetric. We recall that an n -dimensional hypersurface M , in a Riemannian manifold N , $\dim N \geq 4$, is called quasi-umbilical if M has at every point a principal curvature of multiplicity $\geq n - 1$. In [17, Theorem 1] it was stated that every hypersurface M in $M^{n+1}(c)$, $n \geq 3$, having at every point at most two distinct principal curvatures is also pseudosymmetric. Necessary and sufficient conditions for hypersurfaces in $M^4(c)$, to be pseudosymmetric were found in [17]. In [17] it was shown that the Cartan hypersurface M in $S^4(c)$ is a non-semisymmetric pseudosymmetric manifold. The Cartan hypersurfaces in the sphere $S^{n+1}(c)$ are compact, minimal hypersurfaces with constant principal curvatures $-\sqrt{3c}$, 0 , $\sqrt{3c}$ having the same multiplicity. The Cartan hypersurfaces exist only for $n = 3, 6, 12, 24$. More precisely, the Cartan hypersurfaces are tubes of constant radius over the standard

Veronese embeddings $i : \mathbf{F}P^2 \rightarrow S^{3d+1}(c) \rightarrow \mathbf{E}^{3d+2}$, $d = 1, 2, 4, 8$, of the projective plane $\mathbf{F}P^2$ in the sphere $S^{3d+1}(c)$ in a Euclidean space \mathbf{E}^{3d+2} , where $\mathbf{F} = \mathbf{R}$ (real numbers), \mathbf{C} (complex numbers), \mathbf{Q} (quaternions) or \mathbf{O} (octonions), respectively. These hypersurfaces were discovered by E. Cartan in his work about isoparametric hypersurfaces. In [19, Theorem 1] it was proved that every Cartan hypersurface M in $S^{n+1}(c)$, $n = 6, 12, 14$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold with non-pseudosymmetric Weyl tensor.

3. Warped products

Let (M_1, \bar{g}) and (M_2, \tilde{g}) , $\dim M_1 = p$, $\dim M_2 = n - p$, $1 \leq p < n$, be Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Let F be a positive C^∞ function on M_1 . The warped product $M_1 \times_F M_2$ of (M_1, \bar{g}) and (M_2, \tilde{g}) is the product manifold $M_1 \times M_2$ with the metric $g = \bar{g} \times_F \tilde{g} = \Pi_1^* \bar{g} + (F \circ \Pi_1) \Pi_2^* \tilde{g}$, where $\Pi_i : M_1 \times M_2 \rightarrow M_i$ being the natural projections, $i = 1, 2$. Let $\{U \times V; x^1, \dots, x^p, x^{p+1} = y^1, \dots, x^n = y^{n-p}\}$ be a product chart for $M_1 \times M_2$. The local components of the metric $g = \bar{g} \times_F \tilde{g}$ with respect to this chart are the following $g_{rs} = \bar{g}_{ab}$ if $r = a$ and $s = b$, $g_{rs} = F \tilde{g}_{\alpha\beta}$ if $r = \alpha$ and $s = \beta$, $g_{rs} = 0$ otherwise, where $a, b, c, \dots \in \{1, \dots, p\}$, $\alpha, \beta, \gamma, \dots \in \{p+1, \dots, n\}$ and $r, s, t, \dots \in \{1, 2, \dots, n\}$. We shall denote by bars (resp., tildes) tensors formed from \bar{g} (resp., \tilde{g}). The local components of the tensors R and S of $M_1 \times_F M_2$ which may not vanish identically are the following []:

$$(10) \quad R_{abcd} = \bar{R}_{abcd},$$

$$(11) \quad R_{\alpha ab\beta} = -(1/(2F))T_{ab}g_{\alpha\beta},$$

$$(12) \quad R_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\delta} - ((\Delta_1 F)/(4F^2))G_{\alpha\beta\gamma\delta},$$

$$(13) \quad S_{ab} = \bar{S}_{ab} - ((n-p)/(2F))T_{ab},$$

$$(14) \quad S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - (1/(2F))(tr(T) + ((n-p-1)/(2F))\Delta_1 F)g_{\alpha\beta},$$

$$(15) \quad T_{ab} = \bar{\nabla}_b F_a - (1/(2F))F_a F_b, \quad tr(T) = \bar{g}^{ab}T_{ab}\Delta_1 F = \Delta_1 \bar{g} F = \bar{g}^{ab}F_a F_b,$$

where T is the $(0, 2)$ -tensor with the local components T_{ab} .

Example 3.1. (cf. [11, Example 2.1]) Let $(M_1, \bar{g}) = S^p$ be the sphere in a Euclidean \mathbf{E}^{p+1} , $p \geq 2$, defined by $(x^0)^2 + (x^1)^2 + \dots + (x^p)^2 = 1$, with the induced metric \bar{g} . Let f be a non-constant function on S^p satisfying (4). We put

$$(16) \quad L = 1 - c\tau, \quad \tau = 1/\sqrt{F},$$

where c is a non-zero constant such that $f + c$ is a positive or a negative function on S^p and F is defined by (5). Now, using (15), (4), (5) and (16), we can easily verify that the tensor $(1/2)T + FL\bar{g}$ vanishes on S^p . Furthermore, from (4) we get

$$(17) \quad \Delta_1 f = -f^2 + c_2, \quad c_2 \in \mathbf{R}.$$

Combining (17) with (5) we can state that

$$(18) \quad (1/(4F^2))\Delta_1 F = c_1 \tau^2 + 2c\tau - 1, \quad c_1 \in \mathbf{R},$$

holds on S^p . Now we prove that τ cannot be constant on a non-empty open subset of S^p . We suppose that $\tau = \text{const.}$ on a non-empty open subset $U \subset S^p$. Evidently, we have also $F = \text{textconst.}$ on U . From (5) it follows that $f = \text{const.}$ on U . Thus, by (4), we have

$$(19) \quad f = 0 \text{ on } U.$$

On the other hand, it is known [21], [22] that every solution of (4) is a function Φ , defined by $\Phi = \lambda_0 x^0 + \lambda_1 x^1 + \dots + \lambda_p x^p$, $\lambda_i = \text{const.}$, $i = 0, 1, \dots, p$, restricted to the sphere S^p . Now (19) implies that $\lambda_0 = \lambda_1 = \dots = \lambda_p = 0$, i.e. f vanishes identically on S^p , a contradiction with the fact that f is non-constant on S^p .

Example 3.2. Let (M, \tilde{g}) , $\dim M = n - p \geq 2$, $p \geq 2$, be a semi-Riemannian Einstein manifold. We consider the warped product $S^p \times_F M$, where F is defined by (5). By (16), (18) and the fact that the tensor $(1/2)T + FL\tilde{g}$, defined in Example 3.1, is a zero tensor, (10)–(14) turns into

$$(20) \quad R_{abcd} = G_{abcd},$$

$$(21) \quad R_{a\alpha\beta b} = (1 - c\tau)G_{a\alpha\beta b},$$

$$(22) \quad R_{\alpha\beta\gamma\delta} = FZ(\tilde{R})_{\alpha\beta\gamma\delta} + ((l - c_1)\tau^2 - 2c\tau + 1)G_{\alpha\beta\gamma\delta},$$

$$(23) \quad S_{ab} = (n - 1 - (n - p)c\tau)g_{ab},$$

$$(24) \quad S_{\alpha\beta} = ((n - p - 1)(l - c_1)\tau^2 - (2n - p - 2)c\tau + n - 1)g_{\alpha\beta},$$

$$(25) \quad l = (\tilde{\kappa}/((n - p)(n - p - 1))).$$

Next, by using (19)–(23) and the relations

$$(26) \quad \begin{aligned} C_{rstu} &= R_{rstu} - (1/(n - 2))((g_{ru}S_{ts} + g_{ts}S_{ru} - g_{rt}S_{us} - g_{us}S_{rt}) \\ &\quad - (\kappa/(n - 1))G_{rstu}), \\ \kappa &= (n - p)(n - p - 1)(l - c_1)\tau^2 - 2(n - 1)(n - p)c\tau + n(n - 1), \end{aligned}$$

we find the non-zero components of C

$$(27) \quad C_{abcd} = (\rho/(p(p - 1)))G_{abcd},$$

$$(28) \quad C_{a\alpha\beta b} = -(\rho/(p(n - p)))G_{a\alpha\beta b},$$

$$(29) \quad C_{\alpha\beta\gamma\delta} = FZ(\tilde{R})_{\alpha\beta\gamma\delta} + (\rho/((n - p)(n - p - 1)))G_{\alpha\beta\gamma\delta},$$

$$(30) \quad \rho = ((p(p - 1)(n - p)(n - p - 1))/((n - 1)(n - 2)))(l - c_1)\tau^2.$$

Furthermore, applying (18), (20)–(24), (27)–(29), we can easily verify that only components of $R \cdot R$, $Q(g, R)$, $R \cdot S$, $Q(g, S)$, $C \cdot C$, $Q(g, C)$ and $Q(S, R)$, which are not identically equal to zero are those related to

$$\begin{aligned}
(31) \quad & (R \cdot R)_{\alpha a b c d \beta} = -c\tau(1 - c\tau)G_{dabc}g_{\alpha\beta}, \\
(32) \quad & (R \cdot R)_{\alpha\alpha\beta\gamma d\delta} = F(1 - c\tau)g_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} - \tau(c + (c_1 - c^2 - l)\tau \\
& \quad + (l - c_1)c\tau^2)g_{ad}G_{\delta\alpha\beta\gamma}, \\
(33) \quad & (R \cdot R)_{\alpha\beta\gamma\delta\lambda\mu} = F(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} - F(c_1\tau^2 + 2c\tau - 1)Q(g, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}, \\
(34) \quad & Q(g, R)_{\alpha a b c d \beta} = -c\tau G_{dabc}g_{\alpha\beta}, \\
(35) \quad & Q(g, R)_{\alpha\alpha\beta\gamma d\delta} = Fg_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} - (c + (c_1 - l)\tau)\tau g_{ad}G_{\delta\alpha\beta\gamma}, \\
(36) \quad & Q(g, R)_{\alpha\beta\gamma\delta\lambda\mu} = FQ(g, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}, \\
(37) \quad & (R \cdot S)_{\alpha\alpha\beta b} = \tau(1 - c\tau)((n - p - 1)(c_1 - l)\tau + (n - 2)c)g_{ab}g_{\alpha\beta}, \\
(38) \quad & Q(g, S)_{\alpha\alpha\beta b} = \tau((n - p - 1)(c_1 - l)\tau + (n - 2)c)g_{ab}g_{\alpha\beta}, \\
(39) \quad & (C \cdot C)_{\alpha a b c d \beta} = (((n - 1)\rho^2)/(p^2(n - p)^2(p - 1)))G_{dabc}g_{\alpha\beta}, \\
(40) \quad & (C \cdot C)_{\alpha\alpha\beta\gamma d\delta} = -(\rho/(p(n - p)))Fg_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} \\
& \quad - (((n - 1)\rho^2)/(p^2(n - p)^2(n - p - 1)))g_{ad}G_{\delta\alpha\beta\gamma}, \\
(41) \quad & (C \cdot C)_{\alpha\beta\gamma\delta\lambda\mu} = F^2(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} \\
& \quad + F(\rho/((n - p)(n - p - 1)) - l)Q(g, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}, \\
(42) \quad & Q(g, C)_{\alpha a b c d \beta} = -(((n - 1)\rho)/(p(p - 1)(n - p)))G_{dabc}g_{\alpha\beta}, \\
(43) \quad & Q(g, C)_{\alpha\alpha\beta\gamma d\delta} = Fg_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} \\
& \quad + (((n - 1)\rho)/(p(n - p)(n - p - 1)))g_{ad}G_{\delta\alpha\beta\gamma}, \\
(44) \quad & Q(g, C)_{\alpha\beta\gamma\delta\lambda\mu} = FQ(g, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}, \\
(45) \quad & Q(S, R)_{\alpha a b c d \beta} = -(-(n - p - 1) + (2n - 2p - 1)c\tau \\
& \quad + (n - p - 1)((l - c) - (n - p)c^2)\tau^2 - c_1\tau^3)G_{dabc}g_{\alpha\beta}, \\
(46) \quad & Q(S, R)_{\alpha\alpha\beta\gamma d\delta} = -F^2(n - 1 - (n - p)c\tau)g_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} \\
& \quad - \tau(c + ((p - 2)c^2 - p(l - c_1))\tau + (l - c_1)c\tau^2)g_{ad}G_{\delta\alpha\beta\gamma}, \\
(47) \quad & Q(S, R)_{\alpha\beta\gamma\delta\lambda\mu} = F((n - p - 1)(l - c_1)\tau^2 \\
& \quad - (2n - p - 2)c\tau + n - 1)Q(g, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}.
\end{aligned}$$

4. Main results

THEOREM 4.1. *Let $(N, g) = S^p \times_F M$ be the warped product of the sphere S^p , with the function F defined by (5), and an Einstein manifold (M, \tilde{g}) , $p \geq 2$, $\dim M = n - p \geq 3$. Then we have:*

(i) (N, g) is a non-Einstein Ricci-pseudosymmetric manifold.

(ii) (N, g) is a pseudosymmetric manifold if and only if (M, \tilde{g}) is a space of constant curvature.

(iii) If $l \neq c_1$ then (N, g) is a non-conformally flat manifold.

(iv) (N, g) is a manifold with pseudosymmetric Weyl tensor if and only if (M, \tilde{g}) is a space of constant curvature.

(v) The tensor $R \cdot R - Q(S, R)$ is a non-zero tensor on N .

(vi) (N, g) is a manifold do not satisfying (3).

(vii) If $l \neq c_1$ then the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are not linearly dependent on N .

(viii) If $l \neq c_1$ then (N, g) is a manifold with non-harmonic Weyl tensor.

The constants c_1 and l are defined by (18) and (25), respectively.

Proof. (i) From (23)–(25) it follows that (1) is not satisfied on N . Further, (37) and (38) lead to (7).

(ii) We assume that (N, g) is pseudosymmetric. Let x be a point of N . If the tensor $Z(R) = R - (\kappa/(n(n-1)))G$ vanishes at x , i.e. $x \in N - U_R$, then (12) implies that the tensor $Z(\tilde{R}) = \tilde{R} - (\tilde{\kappa}/((n-p)(n-p-1)))\tilde{G}$ vanishes at x . Let now $x \in U_R$. We prove that the tensor $Z(\tilde{R})$ must also vanishes at x . Since (6) is fulfilled at x , (31)–(33) together with (34)–(36) imply that (16) and

$$(48) \quad \tilde{R} \cdot \tilde{R} = (c_1\tau^2 + c\tau)Q(\tilde{g}, \tilde{R})$$

hold at x . We suppose that $Z(\tilde{R})$ is non-zero at x . Now, from (48) we can deduce that the function $c_1\tau^2 + c\tau$ is constant on a non-empty subset $U' \subset S^p$. From this fact it follows that τ is constant on U' . But, as we have stated in Example 3.1, the function τ cannot be constant on an open non-empty subset of S^p . So, $Z(\tilde{R})$ must vanishes at x . Thus, (M, \tilde{g}) is a space of constant curvature. Conversely, if (M, \tilde{g}) is a space of constant curvature, then pseudosymmetry of (N, g) was proved in [11, Theorem 4.1(i)].

(iii) This assertion follows immediately from (27)–(30).

(iv) We assume that (N, g) is a manifold with pseudosymmetric Weyl tensor. Let x be a point of N . Using (39)–(44) we can deduce, in the same way as in the proof of the assertion (ii), that (M, \tilde{g}) is a space of constant curvature. The converse statement was proved in [11, Theorem 4.1(ii)].

(v) (31) and (45) yield

$$\begin{aligned} & ((R \cdot R) - Q(S, R))_{\alpha b c d \beta} \\ &= -((n-p-1)(1-2c\tau + (c-l+(n-p)c^2)\tau^2 + c_1\tau^3) - c^2\tau^2)g_{\alpha\beta}G_{dabc}. \end{aligned}$$

Thus $R \cdot R - Q(S, R)$ is a non-zero tensor on N .

(vi) From (26) it follows that the scalar curvature κ of (N, g) is not constant. Since any semi-Riemannian manifold satisfying (3) must have constant scalar curvature, (3) can not be fulfilled on N .

(vii) If at a point x of N the tensor $Z(\tilde{R})$ vanishes then our assertion was proved in [11, Theorem 4.1(v)]. If at a point x of N the tensor $Z(\tilde{R})$ is non-zero then our assertion follows from (32), (46) and (43).

(viii) Using (23), (24) and (26) we get

$$\begin{aligned} \nabla_c S_{ab} - \nabla_b S_{ac} - (1/(2(n-1)))((\nabla_c \kappa)g_{ab} - (\nabla_b \kappa)g_{ac}) \\ = -(((n-p)(n-p-1))/(n-1))(l - c_1)\tau((\nabla_c \tau)g_{ab} - (\nabla_b \tau)g_{ac}). \end{aligned}$$

Thus (2) can not be satisfied on (N, g) . Our theorem is thus proved.

Using the above theorem and Remark 2.4 we obtain the following statement.

THEOREM 4.3. *Let $(N, g) = S^p \times_F M$, be the warped product of the sphere S^p , with the function F defined in Example 3.1, and an Einstein manifold (M, \tilde{g}) , $p \geq 2$, $\dim M = n - p \geq 3$. If (M, \tilde{g}) is not of constant curvature then (N, g) is a non-conformally flat, non-pseudosymmetric and non-Einstein, Ricci-pseudosymmetric manifold. If the constants c_1 and l defined by (18) and (25), respectively, are not equal then the manifold (N, g) cannot be realized as a hypersurface isometrically immersed in a space of constant curvature.*

Remark 4.1. In [6, Corollary 2] an example of a Ricci-pseudosymmetric warped product of the sphere S^p and a compact Einstein manifold (M, g) , $\dim M \geq 2$, is presented. To construct this example we have needed Lemma 3.1(ii) of [12]. Let now (\tilde{M}, \tilde{g}) be a Riemannian manifold isometric with the Cartan hypersurface M in the unit sphere S^{n-p+1} , $n - p \in \{3, 6, 12, 24\}$. We can apply the mentioned above construction of warped product manifolds to obtain a Ricci-pseudosymmetric warped product of the sphere S^p and the manifold (\tilde{M}, \tilde{g}) .

It is well known that every irreducible locally symmetric space (M, \tilde{g}) , $\dim M \geq 3$, is an Einstein manifold. Thus we have the following corollary.

COROLLARY 4.1. *Let $(N, g) = S^p \times_F M$ be the warped product of the sphere S^p , with the function F defined in Example 3.1, and irreducible locally symmetric space (M, \tilde{g}) , $p \geq 2$, $\dim M = n - p \geq 3$. If (M, \tilde{g}) is not of constant curvature then (N, g) is a non-conformally flat, non-pseudosymmetric and non-Einstein, Ricci-pseudosymmetric manifold.*

Using the above Corollary we can construct a family of Ricci-pseudosymmetric manifolds. This family of manifolds contains also compact manifolds.

Example 4.1. Let (M, \tilde{g}) , $\dim M = n - p \geq 3$, be an irreducible symmetric space of nonconstant curvature (see [1, Tables 1–4, pp. 201–202]). Then the warped product $(N, g) = S^p \times_F M$, of the sphere S^p with the function F defined in Example 3.1, and the manifold (M, \tilde{g}) , $p \geq 2$, is a non-conformally flat, non-pseudosymmetric, and non-Einstein, Ricci-pseudosymmetric manifold.

Using Theorem 4.2 and the fact that the projective space $\mathbf{H}P^n$, $n \geq 1$ (with its standard Riemannian locally symmetric metric), is an Einstein manifold which is not of constant curvature, we get the following corollary.

COROLLARY 4.2. *The warped product $S^{k+1} \times_F \mathbf{H}P^n$, of the sphere S^{k+1} , $k = 1, 2$, with the function F defined in Example 3.1, and the space $\mathbf{H}P^n$ (with its standard Riemannian locally symmetric metric), is a non-conformally flat, non-pseudosymmetric and non-Einstein, Ricci-pseudosymmetric manifold.*

Example 4.2. We consider the generalized Hopf fibrations: $S^2 \rightarrow \mathbf{C}P^{2n+1} \rightarrow \mathbf{H}P^n$ and $S^3 \rightarrow S^{4n+3} \rightarrow \mathbf{H}P^n$, with the projections $\pi_1 : \mathbf{C}P^{2n+1} \rightarrow \mathbf{H}P^n$ and $\pi_2 : S^{4n+3} \rightarrow \mathbf{H}P^n$, respectively. Let $\{U_\alpha\}_{\alpha \in A}$, be an open covering of the manifold $\mathbf{H}P^n$. Thus we have two families of diffeomorphisms $\Phi_{\alpha,k} : \pi_k^{-1}U_\alpha \rightarrow U_\alpha \times S^{k+1}$, $k = 1, 2$. Further, we denote by $i_{\alpha,k}$, $k = 1, 2$, the natural diffeomorphisms $i_{k,\alpha} : U_\alpha \times S^{k+1} \rightarrow S^{k+1} \times U_\alpha$. Thus on the open submanifolds: $\pi_1^{-1}U_\alpha$ in $\mathbf{C}P^{2n+1}$ and $\pi_2^{-1}U_\alpha$ in S^{4n+3} are given metric tensors $h_{\alpha,k}$, defined by $h_{\alpha,k} = (i_{k,\alpha} \circ \Phi_{k,\alpha})^*g_k$, where g_k is the warped product metric on $S^{k+1} \times_F \mathbf{H}P^n$, $k = 1, 2$, defined in Corollary 4.2.

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(Received 05 02 1996)

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