

PARAQUATERNIONIC PROJECTIVE SPACE AND PSEUDO-RIEMANNIAN GEOMETRY

Novica Blažić

Communicated by Mileva Prvanović

Abstract. A natural and geometrical definition of projective space $(P_n(\mathbb{B}), g_0)$, based on the algebra of paraquaternionic numbers \mathbb{B} , is given. Using the technique of pseudo-Riemannian submersions, we determine the curvature of the paraquaternionic space $(P_n(\mathbb{B}), g_0)$. Moreover, the properties of this curvatures are studied.

0. Introduction. Studying differential geometry of spaces related to the algebra of paracomplex numbers was initiated by Rasevskii [R] and Libermann [L] who defined paracomplex and para-Kählerian manifolds. This topic was discussed later by several authors (see [B,CFG] for details).

Also, the geometry based on the paraquaternionic numbers \mathbb{B} is interesting. Etayo [E] studied $P_{n,n}(\mathbb{C})$, the space of paraquaternionic projective type, as a para-Hermitian symmetric space related to Kaneyuki-Kozai classification. He studied the principal bundle $\pi : GL(n+1, \mathbb{C})/GL(1, \mathbb{C}) \rightarrow P_{n,n}$ whose structure group is $GL(1, \mathbb{C}) = \mathbb{C}^*$. An integrable paraquaternionic structure appeared naturally in the study of Osserman pseudo-Riemannian manifolds (see [BBR, Rk]).

Here we consider a pseudo-Riemannian submersion $\pi : S_{2n+1}^{4n+3} \rightarrow P_n(\mathbb{B}) = S_{2n+1}^{4n+3}/S_1^3$ with totally geodesic fibres S_1^3 . The submersion π is used to give a natural and geometrically oriented definition of paraquaternionic projective space $P_n(\mathbb{B})$. It is simply connected and complete with respect to the metric induced by submersion. Using the well known result of O'Neill we determine the sectional curvature of a paraquaternionic projective space \mathbb{B} . It has constant paraquaternionic sectional curvature and unbounded sectional curvature. Some other properties of the paraquaternionic projective spaces are also studied.

1. Paraquaternionic numbers and projective spaces. Let \mathbb{B} denote the algebra of paraquaternionic numbers generated by $\{1, i, e, f = ie\}$ over \mathbb{R} where

AMS Subject Classification (1991): Primary 53C15; Secondary 53C50

Research partially supported by Science Foundation of Serbia, project # 0401.

$e^2 = 1, i^2 = -1, ie = -ei$. This is an associative, noncommutative and unitary algebra over \mathbb{R} of rank 3. To an arbitrary paraquaternionic number $q \in \mathbb{B}$, $q = x_1 + x_2i + x_3e + x_4f$, $x_i \in \mathbb{R}$, $1 \leq i \leq 4$, we can associate its conjugate $\bar{q} = x_1 - x_2i - x_3e - x_4f$, real part $\operatorname{Re} q = x_1$, and the norm

$$\|q\|^2 = q\bar{q} = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

Then we have $\|qp\|^2 = \|q\|^2\|p\|^2$ and clearly q is invertible if and only if $\|q\| \neq 0$.

Let $\tilde{\mathbb{B}} = \{q \in \mathbb{B} \mid \|q\| \neq 0\}$ be the multiplicative group of invertible paraquaternionic numbers. Now we can define the corresponding projective space. The set

$$\mathbb{B}^{n+1} = \{(q_0, \dots, q_n) \mid q_i \in \mathbb{B}, 0 \leq i \leq n\}$$

is a paraquaternionic unitary module. On \mathbb{B}^{n+1} we have the endomorphisms I, E, F defined by $I(q) = iq$, $E(q) = eq$, $F(q) = fq$ respectively. Then $E^2 = F^2 = -I^2 = \operatorname{id}$ and $IE = -EI = F$. A vector $q = (q_0, \dots, q_n) \in \mathbb{B}^{n+1}$ is a singular vector if and only if there exists a $\lambda \in \mathbb{B}, \lambda \neq 0$, such that $\lambda q = 0$. Clearly, then $\|q_0\|^2 = \dots = \|q_n\|^2 = 0$ and q_0, \dots, q_n are mutually proportional with same real factors. Let $\tilde{\mathbb{B}}^n$ denote the nonsingular vectors in \mathbb{B}^n . On the set $\mathbb{B}^{n+1} \setminus \{0\}$ the equivalence relation \sim is defined as usual $u \sim v$ if and only if $v = zu$ for some $z \in \tilde{\mathbb{B}}$. The corresponding quotient space:

$$P(\mathbb{B}^{n+1}) = (\mathbb{B}^{n+1} \setminus \{0\}) / \tilde{\mathbb{B}}$$

is called the algebraic paraquaternionic projective space associated with \mathbb{B}^{n+1} .

This definition is appropriate from an algebraic point of view. But difficulties are coming from the equivalence classes of singular elements of \mathbb{B}^{n+1} . From a geometrical point of view it is better to consider only equivalence classes of nonsingular paraquaternionic lines and we define the paraquaternionic projective space as

$$P(\tilde{\mathbb{B}}^{n+1}) = P_n(\mathbb{B}) = \tilde{\mathbb{B}}^{n+1} / \tilde{\mathbb{B}}.$$

For the algebra \mathbb{A} of paracomplex numbers, $\mathbb{A} = \mathbb{R}\langle 1, i \rangle$, the situation is similar. A broad survey of geometries based on paracomplex numbers is given in [CFG]. We are using some notations and notions from there, modified to the paraquaternionic numbers. The algebraic paracomplex projective space defined in [CFG] coincides with Rosenfeld's paracomplex projective spaces P_n^0 (see [Ro]). The geometrical definition of a paracomplex projective space $P_n(\mathbb{A})$ was given by Libermann [Lb]. In some sense similar to paracomplex projective spaces are the paracomplex projective models $P_n(B)$ in [GM] (open subsets of $\tilde{\mathbb{A}}^{n+1} / \tilde{\mathbb{A}}_{++}$, the four-fold covering space of the Liberman projective space $P_n(\mathbb{A})$, where $\tilde{\mathbb{A}}_{++}$ is the identity component of the group $\tilde{\mathbb{A}}$).

Etayo Gordejuela [E] defined and studied the spaces $P_{n,n}(\mathbb{C})$, similar to paracomplex projective models, and called them paraquaternionic projective spaces.

2. Pseudo-Riemannian geometry and submersions. In $\mathbb{R}^{4n+4} = \mathbb{B}^{n+1}$ we have the following natural scalar product $\langle \cdot, \cdot \rangle$ defined as

$$\langle u, v \rangle = \operatorname{Re} \sum_{i=0}^n u_i \bar{v}_i$$

where $u = (u_0, \dots, u_n)$, $v = (v_0, \dots, v_n)$, $u_i, v_i \in \mathbb{B}$. Clearly, the scalar product, of the signature $(2n+2, 2n+2)$, is anti-invariant with respect to endomorphisms E and F , and invariant with respect to I , i.e.,

$$\langle Eu, Ev \rangle = -\langle u, v \rangle, \quad \langle Fu, Fv \rangle = -\langle u, v \rangle, \quad \text{and} \quad \langle Iu, Iv \rangle = \langle u, v \rangle.$$

The pseudosphere and the pseudohyperbolic space of radius $r > 0$ in \mathbb{R}^{4n+4} , are the hyperquadrics

$$\begin{aligned} S_{2n+1}^{4n+3}(r) &= \left\{ u \in \mathbb{B}^{n+1} \mid \langle u, u \rangle = \sum \|u_i\|^2 = r^2 \right\}, \\ H_{2n+1}^{4n+3}(r) &= \{ u \in \mathbb{B}^{n+1} \mid \langle u, u \rangle = -r^2 \} \end{aligned}$$

respectively. On the tangent spaces $T_p S_{2n+1}^{4n+3}(r)$ and $T_p H_{2n+1}^{4n+3}(r)$ the induced metrics are of the signature $(2n+1, 2n+2)$ and $(2n+2, 2n+1)$. The pseudosphere and the pseudohyperbolic space are anti-isometric [O]. For $r = 1$ we set $S_{2n+1}^{4n+3}(r) = S_{2n+1}^{4n+3}$ and $H_{2n+1}^{4n+3}(r) = H_{2n+1}^{4n+3}$. The pseudosphere S_{2n+1}^{4n+3} is diffeomorphic to $\mathbb{R}^{2n+2} \times S^{2n+1}$ and the pseudohyperbolic space H_{2n+1}^{4n+3} is also diffeomorphic to $\mathbb{R}^{2n+2} \times S^{2n+1}$ (see [O, p. 110]).

In $\mathbb{R}^4 = \mathbb{B}$, we have $S_1^3 = \{\beta \in \mathbb{B} \mid \langle \beta, \beta \rangle = \|\beta\|^2 = 1\}$. The pseudosphere S_1^3 is the group of the unit paraquaternions. Then there is an action of S_1^3 on S_{2n+1}^{4n+3} given by

$$T_\beta(u_0, \dots, u_n) = (\beta u_0, \dots, \beta u_n),$$

which are obviously isometries with respect to $\langle \cdot, \cdot \rangle$. Then, the paraquaternionic projective space can be represented as

$$P_n(\mathbb{B}) = S_{2n+1}^{4n+3} / S_1^3 = H_{2n+1}^{4n+3} / S_1^3.$$

This means that the projection $\pi : S_{2n+1}^{4n+3} \rightarrow P_n(\mathbb{B})$ is a submersion with fiber S_1^3 . Moreover, $P_n(\mathbb{B})$ is homotopically equivalent to the complex projective space $P_n(\mathbb{C})$. Therefore, $P_n(\mathbb{B})$ is simply connected.

The action of these isometries on S_{2n+1}^{4n+3} provides for $P_n(\mathbb{B}) = S_{2n+1}^{4n+3} / S_1^3$ a unique pseudo-Riemannian metric of signature $(2n, 2n)$. More precisely, similarly as in the Riemannian case, the following lemma can be proved.

LEMMA 2.1. *Let M be a pseudo-Riemannian manifold of signature (p, q) and G closed group of isometries such that $\pi : M \rightarrow B = M/G$ is a submersion. If G admits a metric of signature (p_1, q_1) , $p_1 \leq p$, $q_1 \leq q$, then there is a unique*

pseudo-Riemannian metric on $B = M/G$ of signature $(p - p_1, q - q_1)$ such that π is a pseudo-Riemannian submersion. If M and G are complete, then the metric induced by the submersion, is also complete on M/G . \square

We fix a metric g_0 of signature $(2n, 2n)$ on $P_n(\mathbb{B}) = S_{2n+1}^{4n+3}/S_1^3$ such that $\pi : S_{2n+1}^{4n+3} \rightarrow P_n(\mathbb{B})$ is a pseudo-Riemannian submersion. The endomorphisms I , E and F induce corresponding endomorphisms on the tangent space $T_p P_n(\mathbb{B})$. Now we want to determine sectional curvature and the curvature tensor of the paraquaternionic projective space equipped with this metric g_0 .

Let us recall some basic definitions concerning the Levi-Civita connection of pseudo-Riemannian manifold (P, h) of signature (p, q) . We will also establish important notions for paraquaternionic manifolds.

For a vector $X \in T_p P$, $\|X\|^2 = h(X, X)$, X is isotropic if $\|X\|^2 = 0$ and $\epsilon(X) = \text{sign}(h(X, X))$. Let TP be the tangent bundle of P and X, Y, Z , etc., arbitrary vector fields. If ∇ is the corresponding Levi-Civita connection, $R(X, Y) : T_p P \rightarrow T_p P$ is a curvature operator defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z.$$

If Π is a nondegenerate 2-dimensional plane, $\Pi = \langle X, Y \rangle$, the sectional curvature of Π , $K(\Pi)$, is defined as

$$K(\Pi) = K(X, Y) = \frac{h(R(X, Y)Y, X)}{Q(X, Y)},$$

where $Q(X, Y) = h(X, X)h(Y, Y) - h^2(X, Y)$. Π is nondegenerate if and only if $Q(X, Y) \neq 0$. The sectional curvature does not depend on the choice of the base $\{X, Y\}$ for Π . For an orthonormal base, $K(\Pi) = h(R(X, Y)Y, X)/\epsilon(X)\epsilon(Y)$.

P is a paraquaternionic manifold if we have locally defined endomorphisms I , E and F of the tangent bundle such that $E^2 = F^2 = -I^2 = \text{id}$ and $IE = -EI = F$. (P, h) is a paraquaternionic hermitian manifold if (P, h) is a paraquaternionic manifold and I is an isometry and endomorphisms E, F are antiisometries, i.e.,

$$h(IX, IY) = h(X, Y), \quad h(EX, EY) = -h(X, Y), \quad h(FX, FY) = -h(X, Y),$$

for tangent vectors X and Y . Then we say that a paraquaternionic hermitian manifold P is of constant paraquaternionic sectional curvature if $K(\Pi) = \text{const}$ for all nondegenerate 2 planes contained in the vector subspace generated by vectors X, IX, EX, FX for arbitrary nonisotropic vector X . For notational convenience, from now on, we will use the following notation: $J_1 = I$, $J_2 = E$ and $J_3 = F$.

3. Curvature of the paraquaternionic projective space. We will determine now the sectional curvature operator of $P_n(\mathbb{B}) = S_{2n+1}^{4n+3}/S_1^3$ with respect to the induced metric g_0 (by submersion).

Let $\pi : P \rightarrow B$ be a pseudo-Riemannian submersion. At a point $p \in \pi^{-1}(b)$, \mathcal{H} and \mathcal{V} denote the orthogonal projections of $T_p M$ on its horizontal and vertical subspaces

$$\mathcal{H}_p = T_p(\pi^{-1}b)^\perp \quad \text{and} \quad \mathcal{V}_p = T_p(\pi^{-1}b)$$

respectively. The following well known theorem of O'Neill is important for us (see [O]):

THEOREM 3.1. *Let $\pi : P \rightarrow B$ be a pseudo-Riemannian submersion. If the horizontal vector fields X, Y on P span nondegenerate planes, then for their sectional curvatures the following holds*

$$(3.1) \quad K_B(d\pi X, d\pi Y) = K_p(X, Y) + \frac{3}{4} \frac{\langle \mathcal{V}[X, Y], \mathcal{V}[X, Y] \rangle}{Q(X, Y)}. \quad \square$$

We can now prove:

THEOREM 3.2. *The sectional curvature of some nondegenerate 2-plane in the paraquaternionic projective space $(P_n(\mathbb{B}) = S_{2n+1}^{4n+3}/S_1^3, g_0)$ is determined by*

$$(3.2) \quad K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = 1 + 3(\langle X, J_1 Y \rangle^2 - \langle X, J_2 Y \rangle^2 - \langle X, J_3 Y \rangle^2) / Q(X, Y).$$

The paraquaternionic projective space is of constant paraquaternionic sectional curvature 4.

Proof. We will apply O'Neill's theorem to the submersion $\pi : S_{2n+1}^{4n+3} \rightarrow P_n(\mathbb{B})$ with fiber S_1^3 . That means

$$(3.3) \quad K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = K_{S_{2n+1}^{4n+3}}(X, Y) + \frac{3}{4} \frac{\|\mathcal{V}[X, Y]\|^2}{Q(X, Y)}$$

for horizontal vector fields X and Y on S_{2n+1}^{4n+3} . To continue computation we need to determine the vertical subspace at the point N of S_{2n+1}^{4n+3} . Using the hyperbolic rotation we find the curve $\alpha(\theta) = \cosh \theta \cdot 1 + \sinh \theta \cdot e$ on S_1^3 , and the corresponding orbit curve is $\alpha(\theta)N$. Its tangent vector at N is $J_2 N$. Also, $J_3 N$ lies in the vertical subspace of $T_N S_{2n+1}^{4n+3}$. Finally we can see that $J_1 N$ is a vertical vector in a similar way, using euclidean rotations. Hence

$$(3.4) \quad \mathcal{V}[X, Y] = \langle [X, Y], J_1 N \rangle J_1 N - \langle [X, Y], J_2 N \rangle J_2 N - \langle [X, Y], J_3 N \rangle J_3 N.$$

Since: ∇ is a torsion-free connection, $[X, Y] = \nabla_X Y - \nabla_Y X$, and J_1, J_2 and J_3 are parallel with respect to ∇ , we have

$$\mathcal{V}[X, Y] = 2(\langle X, J_1 Y \rangle J_1 N - \langle X, J_2 Y \rangle J_2 N - \langle X, J_3 Y \rangle J_3 N)$$

and

$$(3.5) \quad \|\mathcal{V}[X, Y]\|^2 = 4(\langle X, J_1 Y \rangle^2 - \langle X, J_2 Y \rangle^2 - \langle X, J_3 Y \rangle^2).$$

Since the pseudosphere S_{2n+1}^{4n+3} is of constant sectional curvature 1, (3.2) follows from (3.3) and (3.5).

For horizontal vectors $X, Y \in T_p S_{2n+1}^{4n+3}$, the vectors $J_1 Y, J_2 Y, J_3 Y$ are also horizontal and

$$\begin{aligned} \text{Pr}_{\langle Y, J_1 Y, J_2 Y, J_3 Y \rangle} X &= \epsilon(Y) \langle X, Y \rangle Y + \epsilon(J_1 Y) \langle X, J_1 Y \rangle J_1 Y \\ &\quad + \epsilon(J_2 Y) \langle X, J_2 Y \rangle J_2 Y + \epsilon(J_3 Y) \langle X, J_3 Y \rangle J_3 Y \end{aligned}$$

where $\text{Pr}_{\langle Y, J_1 Y, J_2 Y, J_3 Y \rangle} X$ is normal projection of the vector X on the subspace $\langle Y, J_1 Y, J_2 Y, J_3 Y \rangle$. If X and Y are orthonormal vectors which generate the plane Π , then

$$\text{Pr}_{\langle Y, J_1 Y, J_2 Y, J_3 Y \rangle} X = \epsilon(Y) (\langle X, J_1 Y \rangle J_1 Y - \langle X, J_2 Y \rangle J_2 Y - \langle X, J_3 Y \rangle J_3 Y)$$

and (3.2) becomes

$$(3.6) \quad K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = 1 + 3 \|\text{Pr}_{\langle Y, J_1 Y, J_2 Y, J_3 Y \rangle} X\|^2 / \epsilon(X).$$

Hence, the paraquaternionic sectional curvature of a paraquaternionic projective space is 4. \square

COROLLARY 3.3. *The paraquaternionic projective space for $n = 1$, $P_1(\mathbb{B})$, is of constant sectional curvature 4. It is diffeomorphic to S_2^4 and we have a fiber bundle of Hopf type, $\pi : S_4^7 \rightarrow S_2^4$ with totally geodesic fibre S_1^3 . \square*

COROLLARY 3.4. *The curvature tensor of $(P_n(\mathbb{B}), g_0)$ is*

$$(3.8) \quad \begin{aligned} R(X, Y)Z &= c\{g_0(Y, Z)X - g_0(X, Z)Y \\ &\quad + \sum_r \epsilon_r g_0(J_r Y, Z)J_r X - \sum_r \epsilon_r g_0(J_r X, Z)J_r Y \\ &\quad + 2 \sum_r \epsilon_r g_0(X, J_r Y)J_r Z\}, \end{aligned}$$

where $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$ and c is a constant.

Proof. Using the submersion $\pi : S_{2n+1}^{4n+3} \rightarrow P_n(\mathbb{B})$, from (3.2) it follows that

$$g_0(R(X, Y)Y, X) = c\{Q(X, Y) + 3 \sum_r \epsilon_r g_0(X, J_r Y)^2\}.$$

By polarization, this formula implies (3.8). \square

Remark 1. For the paraquaternionic projective space corresponding to the pseudosphere of radius r , sectional curvature of some plane Π with respect to pseudoorthogonal base $\{X, Y\}$ is

$$K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = \frac{1}{r^2} (1 + 3 \|\text{Pr}_{\langle Y, J_1 Y, J_2 Y, J_3 Y \rangle} X\|^2 / \epsilon(X)).$$

It is well known that the sectional curvature of $P_n(\mathbb{B})$ is unbounded for $n > 1$ (see [O, p. 229]).

Remark 2. If we consider the metric $-\langle \cdot, \cdot \rangle$ on \mathbb{B}^{n+1} , then the paraquaternionic projective space $P_n(\mathbb{B})$ is of constant negative paraquaternionic sectional curvature -4 with respect to the induced metric $-g_0$. This is equivalent with the fact that the pseudosphere S_{2n+1}^{4n+3} and the pseudohyperbolic space H_{2n+2}^{4n+3} are anti-isometric.

Remark 3. Using the algebra of paracomplex numbers, $\mathbb{A} = \mathbb{R}\langle 1, i \rangle$, instead of the paraquaternionic numbers \mathbb{B} we have the submersion

$$\pi : S_n^{2n+1} \longrightarrow P_n(\mathbb{A}) = S_n^{2n+1}(r)/S_1^1$$

with totally geodesic fibres $S_1^1 = \{u \in \mathbb{A} \mid \langle u, u \rangle = 1\}$. Its sectional curvature is

$$K_{P_n(\mathbb{A})}(d\pi X, d\pi Y) = \pm r^{-2}(1 + 3\|\text{Pr}_{\langle Y, J_2 Y \rangle} X\|^2/\epsilon(X)).$$

In [GM] and [E], the projective model $P_n(B)$, diffeomorphic to TS^n , was studied using the appropriate submersion.

Acknowledgement. I would like to thank N. Bokan for valuable and motivating discussions.

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Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

(Received 25 12 1995)